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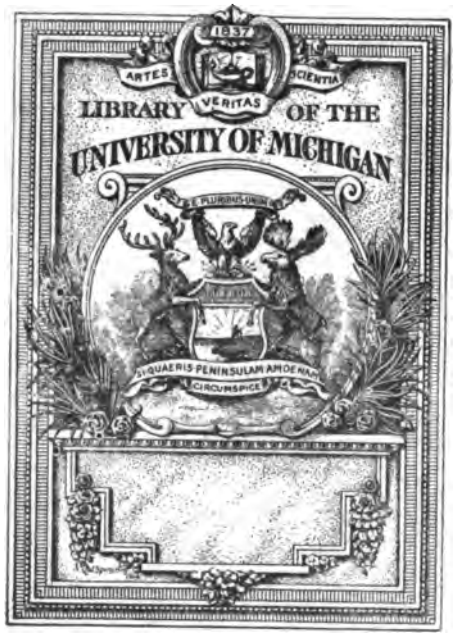

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I. *On the expression of the Double Zeta-Function and Double Gamma-Function in terms of Elliptic Functions.* By G. H. HARDY, M.A., Fellow of Trinity College, Cambridge.

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INTRODUCTION.

§ 1. THIS paper is a contribution to the theory of certain functions recently introduced into analysis by Mr E. W. Barnes. Just as the ordinary trigonometrical functions may be expressed as combinations of Gamma-functions, the elliptic functions may be expressed as combinations of Double Gamma-functions. On the other hand the Gamma (and Zeta) functions may be expressed in terms of trigonometrical functions by means of definite integrals which contain only trigonometrical and algebraic functions. This naturally suggests that Mr Barnes' functions must be capable of being expressed in terms of elliptic functions by means of definite integrals which contain only elliptic and algebraic functions. To show in detail that this is the case is the object of the following pages. It will be found that the fundamental formulae from which all the others are deduced are comparatively easy to obtain. But when we attempt to apply them to the most interesting special cases, by assigning particular values to the constants involved, the integrals which occur in the formulae almost invariably become divergent. The formulae have therefore to be modified in a way which often requires analysis of considerable intricacy, leading in some cases to investigations which seem to me interesting in themselves. It is this fact which explains the length of the paper. The formulae which I obtain generally hold only for a restricted range of values of the variables which appear in them; and so it seems to me unlikely that they are of much importance in the general theory of Mr Barnes' functions. But many of them certainly seem to me very elegant, and the relations between those which hold under different conditions very curious; and it is more because of this that I have investigated them in such detail than because of any intrinsic importance attaching to them.

If there are any formulae for the Double Gamma-function analogous to the familiar formulae

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1}\, dx, \qquad \Gamma(a) = e^{-\frac{1}{2}a\pi i} \int_0^\infty e^{ix} x^{a-1}\, dx,$$

I have not at present succeeded in finding them. The formulae which I shall prove express not the double Gamma-function itself, but its logarithm and logarithmic derivatives,

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and are derived from formulae for the double Zeta-function. There is of course a corresponding set of formulae for the ordinary Zeta- and Gamma-functions, which, so far as I am aware, have never been investigated systematically. I have worked out some of these formulae in the first few sections of this paper, but I have not thought it worth while to enter into any very great detail concerning them, except in so far as they are needed later. They are much less elegant and symmetrical than the corresponding formulae for the double functions.

PART I.

THE SIMPLE ZETA- AND GAMMA-FUNCTIONS.

Formulae for the simple Zeta-function.

§ 2. If ω is any quantity whose real part is positive, and a is not zero or a negative multiple of ω, the generalised simple Riemann Zeta-function* $\zeta_1(s, a, \omega)$ may be defined as the term independent of n in the asymptotic expansion of $\sum_{m=0}^{n} \frac{1}{(a+m\omega)^s}$ in powers of n. It is supposed that $(a+m\omega)^{-s}$ has its principal value; *i.e.* the value assumed by $u^{-s} = e^{-s\log u}$ when u moves from a real positive value, for which $\log u$ is taken real, in a straight line to $a+m\omega$.

Now let us define the function $\lambda_1(u)$ by the equations

$$\lambda_1(u) = \left(\frac{\pi}{\omega}\right)^2 \operatorname{cosec}^2 \frac{\pi u}{\omega} = \sum_{-\infty}^{\infty} \frac{1}{(u+m\omega)^2} \quad \text{.....(1)},$$

and consider the integral

$$\int \lambda_1(u-a) \frac{du}{u^{s-1}}, \qquad (\Re(s) < 0) \quad \text{.....(2)},$$

taken round the contour formed by:

(i) the straight line from $re^{-i\chi}$ to $Re^{-i\chi}$, r being very small and R very large, and

$$am.\omega - \pi < -\chi < am.\omega\dagger;$$

(ii) a corresponding line from $Re^{i\psi}$ to $re^{i\psi}$, ψ being so chosen as to satisfy the inequalities

$$am.\omega < \psi < am.\omega + \pi;$$

(iii) an arc of a circle whose centre is O, from $re^{i\psi}$ to $re^{-i\chi}$;

(iv), (v), (vi), (vii) the straight lines from $Re^{-i\chi}$ to $Re^{-i\chi}+(n+1)\omega$, then to $(n+1)\omega$, $Re^{i\psi}+(n+1)\omega$, and $Re^{i\psi}$.

* E. W. Barnes, 'Theory of the Gamma-function,' *Messenger*, Vol. XXIX. p. 92.

† If $x = re^{i\theta}$, $am.x = \theta$.

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We suppose that u^{-s} has its principal value, and that a lies between (i) or (ii) and the parallels through ω downwards to (i) or upwards to (ii). It is obvious that we can always choose χ and ψ so that this is the case, unless $\frac{a}{\omega}$ is real.

Then it is easy to see that when r and R are made to tend to 0 and ∞ respectively, the contributions of (iii), (iv) and (vii) tend to 0, and the value of the contour integral to

$$-\int_0^{\infty e^{i\psi}} \left\{\frac{1}{u^{s-1}} - \frac{1}{(u+N)^{s-1}}\right\} \lambda_1(u-a)\,du + \int_0^{\infty e^{-i\chi}} \left\{\frac{1}{u^{s-1}} - \frac{1}{(u+N)^{s-1}}\right\} \lambda_1(u-a)\,du \quad \text{.....(3)},$$

where N is written for $(n+1)\omega$, and a notation is adopted for the complex rectilinear integrals which hardly requires explanation.

By Cauchy's theorem this is equal to

$$-2\pi i\,(s-1) \sum_{m=0}^{n} \frac{1}{(a+m\omega)^s} \quad \text{.....(4)},$$

where $(a+m\omega)^{-s}$ has its principal value.

We have now to pick out the term independent of n in the asymptotic expansions of (3) and (4).

§ 3. For this end we define a series of functions

$$\left.\begin{aligned}
\lambda_{-k}(u-a) &= \left(\frac{d}{du}\right)^{k+1} \lambda_1(u-a),\\
\lambda_0(u-a) &= \frac{d}{du}\lambda_1(u-a),\\
\lambda_2(u-a) &= -\int_u^{\infty e^{i\psi}} \lambda_1(u-a)\,du\\
&= -\frac{\pi}{\omega}\left\{\cot\frac{\pi(u-a)}{\omega} + i\right\},\\
\lambda_3(u-a) &= -\int_u^{\infty e^{i\psi}} \lambda_1(u-a)\,du,\\
&\text{.....}
\end{aligned}\right\} \text{.....(5)},$$

the path of integration being in each case the straight line (ii). We also define a similar set of functions

$$\mu_{-k}(u-a),\ \ldots,\ \mu_0(u-a),\ \mu_1(u-a),\ \ldots$$

in the same way, using the straight line (i). Evidently all the μ's with suffixes $\leqq 1$ are identical with the corresponding λ's; but

$$\mu_1(u-a) = -\frac{\pi}{\omega}\left\{\cot\frac{\pi(u-a)}{\omega} - i\right\} \quad \text{.....(6)},$$

and so on. It is easy to see that every λ or μ vanishes *exponentially* for $u = \infty e^{i\psi}$ or $u = \infty e^{-i\chi}$.

Suppose now $\qquad \sigma = -s,$

and $\qquad 0 \leqq r < \mathfrak{R}(\sigma) < r+1,$

where r is an integer.

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$$\left.\begin{aligned}
\text{Then}\quad & \int_0^{\infty e^{i\psi}} (u+N)^{\sigma+1}\lambda_1(u-a)\,du \\
&= -N^{\sigma+1}\lambda_2(-a) - (\sigma+1)\int_0^{\infty e^{i\psi}} (u+N)^{\sigma}\lambda_2(u-a)\,du \\
&= -N^{\sigma+1}\lambda_2(-a) + (\sigma+1)N^{\sigma}\lambda_2(-a) \\
&\qquad + (\sigma+1)\sigma\int_0^{\infty e^{i\psi}} (u+N)^{\sigma-1}\lambda_2(u-a)\,du \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= -N^{\sigma+1}\lambda_2(-a) + (\sigma+1)N^{\sigma}\lambda_3(-a) - \ldots\ldots \\
&\qquad + (-)^r(\sigma+1)\sigma(\sigma-1)\ldots(\sigma-r+1)N^{\sigma-r}\lambda_{r+3}(-a) \\
&\qquad + (-)^r(\sigma+1)\sigma(\sigma-1)\ldots(\sigma-r)\int_0^{\infty e^{i\psi}} (u+N)^{\sigma-r-1}\lambda_{r+3}(u-a)\,du
\end{aligned}\right\}\ \ldots\ldots(7).$$

The asymptotic expansions of the integrated terms do not contain any terms independent of n. And the limit of the last term for $n=\infty$ is 0. This is most easily seen by the help of one more integration by parts. Evidently

$$\int_0^{\infty e^{i\psi}} (u+N)^{\sigma-r-1}\lambda_{r+3}(u-a)\,du = N^{\sigma-r-1}\lambda_{r+4}(-a) - (\sigma-r-1)\int_0^{\infty e^{i\psi}} (u+N)^{\sigma-r-2}\lambda_{r+4}(u-a)\,du.$$

The first term obviously vanishes for $n=\infty$, and the modulus of the second is less than a constant multiple of

$$\int_0^{\infty e^{i\psi}} |u+N|^{\sigma-r-2}\,du,$$

which vanishes for $n=\infty$.

Thus the constant term in the expansion of

$$\int_0^{\infty e^{i\psi}} (u+N)^{\sigma+1}\lambda_1(u-a)\,du,$$

is zero. The same of course holds of the second integral of the same type.

§ 4. Hence

$$2\pi i(s-1)\zeta_1(s,a,\omega) = \int_0^{\infty e^{i\psi}} \lambda_1(u-a)\,u^{-s+1}\,du - \int_0^{\infty e^{-i\chi}} \mu_1(u-a)\,u^{-s+1}\,du \quad \ldots\ldots(8),$$

where
$$\lambda_1(u-a) \equiv \mu_1(u-a) \equiv \left(\frac{\pi}{\omega}\right)^2 \operatorname{cosec}^2\frac{\pi(u-a)}{\omega},$$

and $\mathfrak{R}(s)$ is negative and not integral. The last restriction may be removed at once, since the right-hand side evidently represents an analytic function of s for all values of $s \leqq 2-\epsilon$, where ϵ is any positive quantity. The restriction placed upon the range of values of a is however essential.

The formula may be verified by observing that if $s=1$ the left-hand side reduces to $\frac{2\pi i}{\omega}$ and the two integrals on the right hand to

$$\frac{\pi i}{\omega} - \frac{\pi}{\omega}\cot\frac{\pi a}{\omega} \quad\text{and}\quad -\frac{\pi i}{\omega} - \frac{\pi}{\omega}\cot\frac{\pi a}{\omega},$$

respectively.

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By equating, not the constant terms, but the coefficients of the leading powers of n in our asymptotic expansions, we obtain relations between the Bernoullian functions and the functions λ, μ. I have not thought it worth while to set these out in detail.

§ 5. From (8) we can deduce a variety of formulae for the Zeta-function for all values of s other than positive integral values.

In the first place, by differentiating (8) κ times with respect to a, and observing that

$$\frac{\partial}{\partial a}\zeta_1(s, a, \omega) = -s\zeta_1(s+1, a, \omega),$$

$$\frac{\partial}{\partial a}\lambda_i(u-a) = -\lambda_{i-1}(u-a),$$

we obtain*

$$\left.\begin{aligned} &2\pi i(s-1)s\ldots(s+\kappa-1)\zeta_1(s+\kappa, a, \omega) \\ &\quad = \int_0^{\infty e^{i\psi}} \lambda_{1-\kappa}(u-a)u^{-s+1}\,du - \int_0^{\infty e^{-i\chi}} \mu_{1-\kappa}(u-a)u^{-s+1}\,du \end{aligned}\right\}\ldots\ldots(9).$$

These two formulae (8) and (9) express $\zeta_1(t, a, \omega)$ for all values of t save positive integral values. For these values of t the Zeta-function reduces to a derivative of the logarithm of the Gamma-function.

We can obtain more formulae by integration by parts. If we integrate (8) $\nu+2$ times by parts $(-2<\nu\leqq r)$ we find that

$$\int_0^{\infty e^{i\psi}} \lambda_1(u-a)u^{-s+1}\,du = (s-1)s\ldots(s+\nu)\int_0^{\infty e^{i\psi}} \lambda_{\nu+2}(u-a)u^{-s-\nu-1}\,du,$$

and hence that

$$\left.\begin{aligned} 2\pi i\zeta_1(s, a, \omega) = s(s+1)\ldots(s+\nu)\Bigg[&\int_0^{\infty e^{i\psi}} \lambda_{\nu+2}(u-a)u^{-s-\nu-1}\,du \\ &- \int_0^{\infty e^{-i\chi}} \mu_{\nu+2}(u-a)u^{-s-\nu-1}\,du\Bigg] \end{aligned}\right\}\ldots\ldots(10).$$

Finally by differentiating κ times with respect to a we obtain the formula

$$\left.\begin{aligned} 2\pi i\frac{\Gamma(s+\kappa)}{\Gamma(s+\nu+1)}\zeta_1(s+\kappa, a, \omega) = &\int_0^{\infty e^{i\psi}} \lambda_{\nu-\kappa+2}(u-a)u^{-s-\nu-1}\,du \\ &- \int_0^{\infty e^{-i\chi}} \mu_{\nu-\kappa+2}(u-a)u^{-s-\nu-1}\,du \end{aligned}\right\}\ldots\ldots(11).$$

This holds for all values of s and all positive integral values of ν and κ such that (i) $\mathfrak{R}(s)<0$, (ii) $\nu<\mathfrak{R}(-s)$, and (iii) $s+\kappa$ is not zero or a positive integer.

These formulae may be further extended by the use of Cauchy's notation for '*intégrales extraordinaires.*' But I shall not do this now, as the extension is exactly parallel to that which I shall carry out when I come to the formula for the Double Zeta-function.

* Here (as frequently in what follows) I omit the proof that this inversion of limits is permissible; it is not difficult to supply.

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The Simple Gamma-function and its Logarithmic Derivates.

§ 6. The Gamma function $\Gamma_1(a, \omega)$ may be derived from the Zeta-function by means of the equation

$$\log \frac{\Gamma_1(a, \omega)}{\rho_1(\omega)} = \left[\frac{\partial}{\partial s} \zeta_1(s, a, \omega)\right]_{s=0} \quad \text{.....(12)},$$

where, in Mr Barnes' notation, which I shall follow throughout,

$$\rho_1(\omega) = \sqrt{\frac{2\pi}{\omega}}.$$

Now suppose $-1 < \Re(s) < 0$ and in (11) make $\nu = 0$ and $\kappa = 1$. We obtain on writing s for $s+1$

$$2\pi i\, \zeta_1(s, a, \omega) = \int_0^\infty e^{i\psi} \lambda_2(u-a)\, u^{-s}\, du - \int_0^\infty e^{-i\chi} \mu_2(u-a)\, u^{-s}\, du \quad \text{.....(13)}.$$

Differentiating with respect to s and making $s = 0$ we find that

$$-2\pi i \log \frac{\Gamma_1(a, \omega)}{\rho_1(\omega)} = \int_0^\infty e^{i\psi} \lambda_2(u-a) \log u\, du - \int_0^\infty e^{-i\chi} \mu_2(u-a) \log u\, du \quad \text{.....(14)}.$$

Differentiating s times with respect to a

$$\left.\begin{aligned}(-)^{s+1} 2\pi i \left(\frac{d}{da}\right)^s \log \Gamma_1(a, \omega) &= (-)^{s+1} 2\pi i \psi_1^{(s)}(a, \omega) = -2\pi i (s-1)!\, \zeta_1(s, a, \omega) \\ &= \int_0^\infty e^{i\psi} \lambda_{2-s}(u-a) \log u\, du - \int_0^\infty e^{-i\chi} \mu_{2-s}(u-a) \log u\, du\end{aligned}\right\} \text{.....(15)}.$$

This holds for all integral values of s except that, when $s=1$, $\psi_1^{(1)}(a, \omega)$ is not $-\zeta_1(1, a, \omega)$—which is infinite—but the finite term in the expansion of $-\zeta_1(s, a, \omega)$ in powers of $(s-1)$.

From (14) and the equation

$$\lim_{a=0} \{\log a + \log \Gamma_1(a, \omega)\} = 0 \quad \text{.....(16)},$$

we can find an integral expression for $\log \rho_1(\omega)$; and from (15) and the equations

$$\gamma_{11}(\omega) = -\lim_{a=0} \left[\psi_1^{(1)}(a, \omega) + \frac{1}{a}\right] \quad \text{.....(17)},$$

$$\gamma_{11}(\omega) = -\frac{\gamma}{\omega} + \frac{\log \omega}{\omega} \quad \text{.....(18)},$$

we can find one for γ_{11} or γ (Euler's constant). As similar investigations have to be undertaken in connection with the Double Gamma-function I shall not enter into the details of this at present.

It is interesting to throw (14) into a real form in the special case in which a and ω are real. In this case $0 < a < \omega$. We take $\psi = \chi = \frac{1}{2}\pi$. Under these circumstances the first integral in (14) becomes

$$\int_0^\infty \left[-\frac{\pi}{\omega}\left\{\cot \frac{\pi(iy-a)}{\omega} + i\right\}\right] (\log y + \tfrac{1}{2} i\pi)\, i\, dy.$$

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Now
$$\cot\frac{\pi(iy-a)}{\omega}+i=\frac{i\left(e^{-\frac{2\pi y}{\omega}}-\cos\frac{2\pi a}{\omega}\right)-\sin\frac{2\pi a}{\omega}}{\cosh\frac{2\pi y}{\omega}-\cos\frac{2\pi a}{\omega}}.$$

As the second integral differs from the first only in the sign of i, we obtain

$$-\log\frac{\Gamma_1(a,\omega)}{\rho_1(\omega)}=\frac{1}{2\omega}\int_0^\infty\frac{\pi\left(e^{-\frac{2\pi y}{\omega}}-\cos\frac{2\pi a}{\omega}\right)+2\sin\frac{2\pi a}{\omega}\log y}{\cosh\frac{2\pi y}{\omega}-\cos\frac{2\pi a}{\omega}}\,dy\text{.....}(19).$$

Suppose, *e.g.*, $\omega=1$, $a=\frac{1}{2}$. Then the left side reduces to
$$-\log\Gamma(\tfrac{1}{2})+\tfrac{1}{2}\log 2\pi,$$
and the right side to
$$\frac{1}{2}\pi\int_0^\infty\frac{1+e^{-2\pi y}}{1+\cosh 2\pi y}\,dy=\frac{1}{2}\int_0^1\frac{d\xi}{1+\xi}=\frac{1}{2}\log 2.$$
Thus we find
$$\log\Gamma(\tfrac{1}{2})=\tfrac{1}{2}\log\pi,$$
$$\Gamma(\tfrac{1}{2})=\sqrt{\pi},$$
which is known to be correct.

§ 7. All the formulae obtained so far are valid only for a restricted range of values of a. I shall not now discuss the modifications which have to be made in them when a lies outside this domain, as it is not this set of formulae but another rather different and less symmetrical set which I shall require later.

A Second Set of Formulae for the Simple Zeta-function.

§ 8. It will be convenient to adopt at this point a notation more natural when we are dealing with functions connected with *two* periods.

I shall denote by ω_1 the quantity up to now called ω, and by ω_2 another quantity satisfying the conditions
$$\mathfrak{R}(\omega_2)>0,\quad \mathfrak{R}\left(\frac{\omega_2}{i\omega_1}\right)>0.$$

I suppose that a lies within the strip bounded by lines from 0 and ω_1 parallel to the positive direction of ω_2, and I consider the complex integral
$$\int\lambda_1(u-a)\frac{du}{u^{s-1}},$$
the contour of integration being now the rectangle
$$0,\quad (n+1)\omega_1,\quad (n+1)\omega_1+k\omega_2,\quad k\omega_2$$
(k being a positive integer) modified as before by a small arc round 0. The value of the integral when k is made very great and the arc very small is
$$\int_0^N\lambda_1(u-a)\frac{du}{u^{s-1}}-\int_0^{\infty\omega_2}\left\{\frac{1}{u^{s-1}}-\frac{1}{(u+N)^{s-1}}\right\}\lambda_1(u-a)\,du,$$

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N being equal to $(n+1)\omega_1$ as before. As in § 3 we can prove that the term independent of n in the asymptotic expansion of the second integral is

$$\int_0^{\infty\omega_1} \lambda_1(u-a)\frac{du}{u^{s-1}};$$

it remains to discuss the first integral.

§ 9. For this end we define a new series of functions

$$\left.\begin{aligned}
\phi_{1,-k}(u-a) &= \left(\frac{d}{du}\right)^{k+1}\lambda_1(u-a)\\
\phi_{1,0}(u-a) &= \left(\frac{d}{du}\right)\lambda_1(u-a)\\
\phi_{1,1}(u-a) &= \lambda_1(u-a)+C_{1,1}\\
\phi_{1,2}(u-a) &= \int^u \lambda_1(u-a)\,du + C_{1,1}(u-a)+C_{1,2}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(20),$$

characterised by the following properties:

(i) each is the derivative of its successor;

(ii) each is periodic in ω_1;

(iii) each satisfies the equation

$$\int_0^{\omega_1}\phi(u)\,du = 0,$$

in which the path of integration is supposed rectilinear.

This is certainly possible. For suppose that functions up to $\phi_{1,\nu}(u-a)$ have been defined in such a way as to satisfy these conditions. Then

$$\begin{aligned}
\phi_{1,\nu+1}(u+\omega_1-a)-\phi_{1,\nu+1}(u-a) &= \int_{u-a}^{u+\omega_1-a}\phi_{1,\nu}(u)\,du\\
&= \int_0^{\omega_1}\phi_{1,\nu}(u)\,du
\end{aligned}$$

(as $\phi_{1,\nu}(u)$ is periodic)

$$= 0,$$

and so $\phi_{1,\nu+1}(u)$ will be periodic, and by choosing $C_{1,\nu+1}$ suitably we can ensure that it also satisfies condition (iii).

It is easy to calculate the first few functions $\phi_{1,\nu}$. Thus $C_{1,1}$ is determined by the equation

$$\int_0^{\omega_1}\left\{\left(\frac{\pi}{\omega_1}\right)^2 \operatorname{cosec}^2\frac{\pi(u-a)}{\omega_1}+C_{1,1}\right\}du = 0,$$

so that $C_{1,1}=0$, and

$$\phi_{1,2}(u-a) = -\frac{\pi}{\omega_1}\cot\frac{\pi(u-a)}{\omega_1}+C_{1,2}\quad\cdots\cdots\cdots\cdots(21).$$

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Similarly
$$C_{1,2} = \frac{1}{\omega_1}\int_0^{\omega_1} \frac{\pi}{\omega_1} \cot \frac{\pi(u-a)}{\omega_1}\, du$$
$$= \frac{1}{\omega_1}\left[\log \sin \frac{\pi(u-a)}{\omega_1}\right]_0^{\omega_1}$$
$$= \frac{(2k+1)\pi i}{\omega_1},$$

where k is an integer. We can determine k as follows. Let
$$\frac{\pi a}{\omega_1} = \alpha + i\alpha', \qquad \alpha' > 0.$$

Then
$$\log \sin \frac{\pi(u-a)}{\omega_1} = \log \sin (x - \alpha - i\alpha'),$$

where x is real. The imaginary part of this is
$$i\left\{m\pi - \tan^{-1} \frac{\tanh \alpha'}{\tan (x-\alpha)}\right\},$$

where m is an integer, and
$$-\frac{\pi}{2} < \tan^{-1} \frac{\tanh \alpha'}{\tan (x-\alpha)} < \frac{\pi}{2}.$$

As x goes from 0 to π, $\tan^{-1} \dfrac{\tanh \alpha'}{\tan (x-\alpha)}$ increases or decreases by some multiple of π. This must be a negative multiple as
$$\frac{d}{dx} \tan^{-1} \frac{\tanh \alpha'}{\tan (x-\alpha)} = -\frac{\tanh \alpha'}{\sin^2 (x-\alpha) + \tanh^2 \alpha' \cos^2 (x-\alpha)} < 0,$$

and it must be $-\pi$ since $\tan (x-\alpha)$ can only vanish once. Hence
$$C_{1,2} = \frac{\pi i}{\omega_1} \quad \text{.....(22)},$$

and
$$\phi_{1,3}(u-a) = -\log \sin \frac{\pi(u-a)}{\omega_1} + \frac{\pi i}{\omega_1}(u-a) + C_{1,3} \quad \text{.....(23)}.$$

The branch of the logarithm to be chosen is at present at our disposal. Again
$$C_{1,3} = \frac{1}{\omega_1}\int_0^{\omega_1} \left\{\log \sin \frac{\pi(u-a)}{\omega_1} - \frac{\pi i}{\omega_1}(u-a)\right\} du$$
$$= -(\tfrac{1}{2}\omega_1 - a)\frac{\pi i}{\omega_1} + \frac{1}{\omega_1}\int_0^{\omega_1} \log \sin \frac{\pi(u-a)}{\omega_1}\, du.$$

To determine the value of this integral integrate
$$\int \log \sin \frac{\pi u}{\omega_1}\, du,$$

round the parallelogram
$$(0,\ -a,\ -a+\omega_1,\ \omega_1).$$

The points 0, ω_1 may be avoided by small arcs of circles, and we choose that branch of the logarithm which is real when u is on the line $(0, \omega_1)$. This side contributes
$$-\int_0^{\omega_1} \log \sin \frac{\pi u}{\omega_1}\, du = -\frac{\omega_1}{\pi}\int_0^{\pi} \log \sin x\, dx = \omega_1 \log 2.$$

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The first and third sides contribute $a\pi i$, and the last contributes

$$\int_0^{\omega_1} \log \sin \frac{\pi(u-a)}{\omega_1}\, du,$$

which is therefore

$$-\omega_1 \log 2 - a\pi i.$$

Hence

$$C_{1,3} = -(\tfrac{1}{2}\omega_1 - a)\frac{\pi i}{\omega_1} - \log 2 - \frac{a\pi i}{\omega_1} = -\log 2 - \tfrac{1}{2}\pi i \quad\text{.....(24)},$$

provided the proper branch of $\log \sin \frac{\pi(u-a)}{\omega_1}$ be chosen. This is the branch which is real along the straight line

$$(a, \quad a+\omega_1).$$

Thus finally

$$\phi_{1,3}(u-a) = -\log \sin \frac{\pi(u-a)}{\omega_1} + \frac{\pi i}{\omega_1}(u-a) - \log 2 - \tfrac{1}{2}\pi i \quad\text{.....(25)}.$$

§ 10. Suppose now, as before,

$$\sigma = -s,$$

$$0 \leqq r < \mathfrak{R}(\sigma) < r+1,$$

then

$$\begin{aligned}\int_0^N \lambda_1(u-a)\frac{du}{u^{s-1}} &= \int_0^N \phi_{1,1}(u-a)\, u^{\sigma+1}\, du \\ &= \phi_{1,2}(-a)N^{\sigma+1} - (\sigma+1)\int_0^N \phi_{1,2}(u-a)\, u^{\sigma} du \\ &= \phi_{1,2}(-a)N^{\sigma+1} - (\sigma+1)\phi_{1,3}(-a)N^{\sigma} + (\sigma+1)\sigma\phi_{1,4}(u-a)N^{\sigma-1} - \dots \\ &\quad -(-)^r(\sigma+1)\sigma\dots(\sigma-r+1)\phi_{1,r+3}(-a)N^{\sigma-r} \\ &\quad +(-)^r(\sigma+1)\sigma\dots(\sigma-r)\int_0^N \phi_{1,r+3}(u-a)\, u^{\sigma-r-1}\, du.\end{aligned}$$

The last term tends for $n=\infty$ to the finite limit*

$$(-)^r(\sigma+1)\sigma\dots(\sigma-r)\int_0^{\infty\omega_1} \phi_{1,r+3}(u-a)\, u^{\sigma-r-1}\, du.$$

In the asymptotic expansions of the other terms there is no term independent of n. Hence

$$\left.\begin{aligned}-2\pi i(s-1)\zeta_1(s, a, \omega_1) = (s-1)s(s+1)\dots(s+r)\int_0^{\infty\omega_1} \phi_{1,r+3}(u-a)\, u^{-s-r-1}\, du \\ -\int_0^{\infty\omega_2} \lambda_1(u-a)\, u^{-s+1}\, du\end{aligned}\right\} \quad (26);$$

or if we integrate the last term repeatedly by parts as in § 5,

$$\left.\begin{aligned}-2\pi i\,\zeta_1(s, a, \omega_1) = s(s+1)\dots(s+r)\Big[\int_0^{\infty\omega_1} \phi_{1,r+3}(u-a)\, u^{-s-r-1}\, du \\ -\int_0^{\infty\omega_2} \lambda_{r+3}(u-a)\, u^{-s-r-1}\, du\Big]\end{aligned}\right\} \quad\text{.....(27)}.$$

From this formula a variety of others may be deduced as in §§ 5—6.

* In the *Messenger of Mathematics*, Vol. XXXII., pp. 1—3, I proved that

$$\int^{\infty} \phi(x)\, \psi(x)\, dx$$

is convergent if (i) $\psi(x)$ is positive and tends steadily to zero for $x=\infty$, and (ii) $\phi(x)$ has a period ω and does not change sign infinitely often in a period, and

(iii) $$\int_0^{\omega} \phi(x)\, dx = 0.$$

Only real variables and functions were considered in this theorem, but the extension to complex variables and integrals such as those in the text presents no difficulty.

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§ 11. I shall not, however, enter into the details of this, but shall consider the question of the extension of (27) to values of a which lie outside the domain to which a has so far been restricted. Let us suppose now that a lies in the strip bounded by the lines through $k\omega_1$ and $(k+1)\,\omega_1$ parallel to ω_2. The only difference is that instead of dealing with

$$\sum_{m=0}^{n} \frac{1}{(a+m\omega_1)^s},$$

we have to deal with

$$\sum_{m=-k}^{n-k} \frac{1}{(a+m\omega_1)^s};$$

and the term independent of n in the asymptotic expansion of this is

$$\zeta_1(s, a, \omega_1) + \sum_{m=-k}^{-1} \frac{1}{(a+m\omega_1)^s}.$$

Hence (27) becomes

$$\left.\begin{aligned} -2\pi i\left\{\zeta_1(s, a, \omega_1) + \sum_{m=-k}^{-1} \frac{1}{(a+m\omega)^s}\right\} = s(s+1)\dots(s+r)\Bigg[\int_0^{\infty\omega_1} \phi_{1,r+3}(u-a)\,u^{-s-r-1}\,du \\ -\int_0^{\infty\omega_2} \lambda_{r+3}(u-a)\,u^{-s-r-1}\,du\Bigg] \end{aligned}\right\} \quad \dots(28).$$

It must be carefully observed that the hypothesis that

$$\mathfrak{R}\left(\frac{a}{i\omega_1}\right) > 0$$

is still essential; if $\mathfrak{R}\left(\frac{a}{i\omega_1}\right) < 0$ the values of the constants which occur in the functions $\phi_{1,\nu}$ are different; for instance $C_{1,2} = -\frac{\pi i}{\omega_1}$ instead of $\frac{\pi i}{\omega_1}$.

§ 12. We have still to consider the cases in which a is *on* one of the lines from the points $0, \omega_1, \dots$ parallel to ω_2, or on the axis of ω_1. In these cases some of the functions ϕ, λ become infinite in the range of integration.

I shall illustrate the method to be pursued in such cases by considering a particular case. It is evidently not worth while to attempt to work out a detailed set of formulae applicable to all of the many special cases which may arise.

Suppose in (27) $-1 < \mathfrak{R}(s) < 0$, so that $r = 0$, and suppose that a is very near some point α on the line $(0, \omega_1)$, with which it is ultimately to be made to coincide. No difficulty arises from the last term of (27), in which we may put $a = \alpha$ at once. We saw that

$$\phi_{1,3}(u-a) = -\log\sin\frac{\pi(u-a)}{\omega_1} + \frac{\pi i}{\omega_1}(u-a) - \log 2 - \tfrac{1}{2}\pi i.$$

When a approaches α this tends to a definite value for all values of u save $\alpha, \alpha+\omega_1, \dots$. And these values define a function $\phi_{1,3}(u-\alpha)$ of u which becomes logarithmically infinite at $\alpha, \alpha+\omega_1, \dots$, but possesses the fundamental properties of $\phi_{1,3}(u-a)$, in that it is periodic, and

$$\int_0^{\omega_1} \phi_{1,3}(u-\alpha)\,du = 0.$$

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Hence
$$\int_0^{\infty\omega_1} \phi_{1,3}(u-a)\, u^{-s-1}\, du$$
is convergent. Moreover this integral is *uniformly* convergent for an interval of values of a which includes a. This of course requires a detailed proof which I shall omit. Hence it is continuous, and as the left-hand side of (27) is obviously continuous the formula still holds. The same is true whatever be k. It is of course not the case that all the formulae which we have obtained continue to hold in cases such as these, many of the integrals become divergent. The most interesting special case is that in which a is nearly $=0$. When I come to the double Zeta- and Gamma-functions I shall work out in detail the limiting forms assumed by several formulae in this case.

PART II.

THE DOUBLE ZETA- AND GAMMA-FUNCTIONS.

The double Zeta-function for a non-integral Exponent.

§ 13. The double Zeta-function
$$\zeta_2(s, a, \omega_1, \omega_2)$$
may be defined* as the term independent of n in the asymptotic expansion of
$$\sum_{m_1=0}^{p_1 n} \sum_{m_2=0}^{p_2 n} \frac{1}{(a+m_1\omega_1+m_2\omega_2)^s} \quad \text{.....(29)},$$
in powers of n†. It is of course assumed that a is not equal to any of the quantities $-m_1\omega_1-m_2\omega_2$. Moreover $(a+m_1\omega_1+m_2\omega_2)^{-s}$ has its principal value; *i.e.* the value assumed by $u^{-s}=e^{-s\log u}$ when u moves from a real positive value, for which $\log u$ is real, in a straight line to $a+m_1\omega_1+m_2\omega_2$. Then the Weierstrassian elliptic function $\wp(u)$ is defined by the equation
$$\wp(u)=\frac{1}{u^2}+\sum\left\{\frac{1}{(u-\Omega)^2}-\frac{1}{\Omega^2}\right\} \quad \text{.....(30)},$$
where
$$\Omega=m_1\omega_1+m_2\omega_2,$$
and m_1, m_2 are not to be taken equal to zero simultaneously.

§ 14. Now suppose that a lies within the fundamental period parallelogram $(0, \omega_1, \omega_1+\omega_2, \omega_2)$, and consider the integral
$$\int \wp(u-a)\frac{du}{u^{s-1}} \qquad (\Re(s)<0)$$

* E. W. Barnes, *Phil. Trans. Roy. Soc.* (A.), Vol. 196, p. 265.

† I follow Mr Barnes in using $\omega_1, \omega_2, \eta_1, \eta_2$ for $2\omega_1, 2\omega_2, 2\eta_1, 2\eta_2$.

Various complications, interesting only from the point of view of theoretical completeness, arise if we abandon the hypotheses
$$\Re(\omega_1)>0, \quad \Re(\omega_2)>0, \quad \Re\left(\frac{\omega_2}{i\omega_1}\right)>0.$$
I shall adhere to these hypotheses throughout.

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taken round the parallelogram

$$0,\ P_1\,[=(p_1u+1)\,\omega_1],\ P_1+P_2\,[=(p_2u+1)\,\omega_2],\ P_2.$$

The origin must be avoided by a small circular arc whose contribution to the integral can be made as small as we please, and u^{-s} is to have its principal value.

It is easy to see that the residue of $\wp(u-a)\,u^{-s+1}$ at $a+m_1\omega_1+m_2\omega_2$ is

$$-\frac{(s-1)}{(a+m_1\omega_1+m_2\omega_2)^s},$$

and so, by Cauchy's theorem,

$$\left.\begin{aligned}\int_0^{P_1}\wp(u-a)\left\{\frac{1}{u^{s-1}}-\frac{1}{(u+P_2)^{s-1}}\right\}du-\int_0^{P_2}\wp(u-a)\left\{\frac{1}{u^{s-1}}-\frac{1}{(u+P_1)^{s-1}}\right\}du\\ =-2\pi i\,(s-1)\sum_{m_1=0}^{p_1n}\sum_{m_2=0}^{p_2n}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s}.\end{aligned}\right\}\text{.....(31)}.$$

§ 15. We now define a series of functions

$$\left.\begin{aligned}\phi_{-k}(u-a)&=\wp^{(k+1)}(u-a)\\ \phi_0(u-a)&=\wp'(u-a)\\ \phi_1(u-a)&=\wp(u-a)+C_1\\ \phi_2(u-a)&=-\zeta(u-a)+C_1(u-a)+C_2\\ \phi_3(u-a)&=-\log\sigma(u-a)+\tfrac{1}{2}C_1(u-a)^2+C_2(u-a)+C_3\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\text{.....(32)},$$

to satisfy the same conditions as were satisfied by the functions $\phi_{1,\nu}(u-a)$. The analysis of § 9 shows that this is certainly possible. Then if $0<\mathfrak{R}(t)<1$

$$\int_0^{\infty\omega_1}\phi_\nu(u-a)\frac{du}{u^t}$$

is convergent, by the general theorem already quoted.

§ 16. Now let us consider the integral

$$\int_0^{P_1}\wp(u-a)\frac{du}{u^{s-1}}.$$

Suppose that $-s=\sigma$, and

$$r<\mathfrak{R}(\sigma)<r+1,$$

as before. Then

$$\begin{aligned}\int_0^{P_1}\wp(u-a)\frac{du}{u^{s-1}}&=\int_0^{P_1}\{\phi_1(u-a)-C_1\}\,u^{\sigma+1}du=-\frac{C_1P_1^{\sigma+2}}{\sigma+2}+\int_0^{P_1}\frac{d}{du}\phi_2(u-a)\,u^{\sigma+1}du\\ &=-\frac{C_1P_1^{\sigma+2}}{\sigma+2}+\phi_2(-a)\,P_1^{\sigma+1}-(\sigma+1)\int_0^{P_1}\phi_2(u-a)\,u^\sigma du\\ &=-\frac{C_1P_1^{\sigma+2}}{\sigma+2}+\phi_2(-a)\,P_1^{\sigma+1}-(\sigma+1)\,\phi_3(-a)\,P_1^\sigma\\ &\quad+(\sigma+1)\,\sigma\phi_4(-a)\,P_1^{\sigma-1}-\ldots\\ &\quad+(-)^r(\sigma+1)\,\sigma\ldots(\sigma-r)\int_0^{P_1}\phi_{r+3}(u-a)\,u^{\sigma-r-1}du.\end{aligned}$$

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The last term tends, for $n=\infty$, to the finite limit

$$(-)^r(\sigma+1)\sigma\ldots(\sigma-r)\int_0^{\infty\omega_1}\phi_{r+3}(u-a)\,u^{\sigma-r-1}du,$$

while in the asymptotic expansions of the other terms there is no finite term.

Moreover there is no finite term in the expansion of

$$\int_0^{P_1}\frac{\varphi(u-a)}{(u+P_2)^{s-1}}\,du.$$

For the argument of the preceding paragraph shows that the finite term in this expansion is the same as that in the expansion of

$$(-)^r(\sigma+1)\sigma\ldots(\sigma-r)\int_0^{P_1}\phi_{r+3}(u-a)\,(u+P_2)^{\sigma-r-1}du,$$

and it may be shown as in § 3 that this tends to zero for $n=\infty$.

§ 17. We therefore find by equating the finite terms in the expansions of the two sides of (31),

$$\left.\begin{aligned}&-2\pi i\zeta_2(s,a,\omega_1,\omega_2)=s(s+1)\ldots(s+r)\\&\qquad\left[\int_0^{\infty\omega_1}\phi_{r+3}(u-a)\,u^{-s-r-1}du-\int_0^{\infty\omega_2}\psi_{r+3}(u-a)\,u^{-s-r-1}du\right]\end{aligned}\right\}\ldots\ldots\ldots\ldots(33),$$

where ψ_{r+3} is one of a set of functions ψ_ν formed in precisely the same way as the functions ϕ_ν, except that in their formation ω_2 plays the part of ω_1.

§ 18. This formula expresses the double Zeta-function when $\Re(s)<0$ and is not integral. To obtain formulae applicable to other cases we have only to differentiate with respect to a. Since

$$\frac{\partial}{\partial a}\zeta_2(s,a,\omega_1,\omega_2)=-s\zeta_2(s+1,a,\omega_1,\omega_2),$$

$$\frac{\partial}{\partial a}\phi_\nu(u-a)=-\phi_{\nu-1}(u-a),$$

we find by differentiating κ times

$$\left.\begin{aligned}&-2\pi i\frac{\Gamma(s+\kappa)}{\Gamma(s+r+1)}\zeta_2(s+\kappa,a,\omega_1,\omega_2)\\&=\int_0^{\infty\omega_1}\phi_{r+3-\kappa}(u-a)\,u^{-s-r-1}du-\int_0^{\infty\omega_2}\psi_{r+3-\kappa}(u-a)\,u^{-s-r-1}du\end{aligned}\right\}\ldots\ldots\ldots(34).$$

§ 19. The most interesting case, as being that from which we shall derive the formula for the double Gamma-function, is that in which $0<\Re(s)<1$.

If in (34) we suppose $-1<\Re(s)<0$, $r=0$, $\kappa=1$, $r+3-\kappa=2$, we find

$$-2\pi i\zeta_2(s+1,a,\omega_1,\omega_2)=\int_0^{\infty\omega_1}\phi_2(u-a)\,u^{-s-1}du-\int_0^{\infty\omega_2}\psi_2(u-a)\,u^{-s-1}du\ldots\ldots(35),$$

or, writing s for $s+1$, so that now $0<\Re(s)<1$,

$$-2\pi i\zeta_2(s,a,\omega_1,\omega_2)=\int_0^{\infty\omega_1}\phi_2(u-a)\frac{du}{u^s}-\int_0^{\infty\omega_2}\psi_2(u-a)\frac{du}{u^s}\ldots\ldots\ldots\ldots\ldots(36).$$

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Since
$$\phi_2(u-a)=\frac{d}{du}\{\phi_3(u-a)-\phi_3(-a)\},$$
we obtain on integrating by parts

$$\left.\begin{aligned}&-2\pi i\zeta_2(s,a,\omega_1,\omega_2)\qquad(0<\mathfrak{R}(s)<1)\\&=s\int_0^{\infty\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{du}{u^{s+1}}-s\int_0^{\infty\omega_2}\{\psi_3(u-a)-\psi_3(-a)\}\frac{du}{u^{s+1}}\end{aligned}\right\}\text{.....(37).}$$

§ 20. This last transformation may be generalised, and further formulae obtained, by the help of Cauchy's notion of '*intégrales extraordinaires*'.

Suppose that $f(x)$ is a real or complex function of the real variable x which is continuous with all its derivates for all finite positive values of x, and can be expanded in a Taylor's series
$$a_0+a_1x+a_2x^2+\dots$$
valid near $x=0$. Then if
$$m+1<t<m+2,$$
(m being a positive integer),
$$\frac{f(x)-a_0-a_1x-\dots-a_{m-1}x^{m-1}}{x^{t-1}}$$
is infinite for $x=0$ of order <1, and so integrable up to $x=0$. And if $f(x)$ remains less than some constant as x tends to ∞, in particular if, as I shall now suppose, it possesses properties similar to those of the functions ϕ_ν
$$\int_0^\infty\{f(x)-a_0-a_1x-\dots-a_{m-1}x^{m-1}\}\frac{dx}{x^{t-1}}$$
is convergent. I shall denote this by*
$$E\int_0^\infty\frac{f(x)\,dx}{x^{t-1}}\text{ .....(38).}$$

Now by integration by parts
$$E\int_0^\infty\frac{f(x)\,dx}{x^{t-1}}=\frac{1}{t-2}\int_0^\infty\{f'(x)-a_1-\dots-(m-1)\,a_{m-1}x^{m-2}\}\frac{dx}{x^{t-2}}=\frac{1}{t-2}E\int_0^\infty\frac{f'(x)\,dx}{x^{t-2}}.$$
If $m=1$, $2<t<3$, this is an ordinary integral. If not it is
$$=\frac{1}{(t-2)(t-3)}E\int_0^\infty\frac{f''(x)\,dx}{x^{t-3}},$$
and so on; finally we obtain
$$\frac{1}{(t-2)(t-3)\dots(t-m-1)}\int_0^\infty\frac{f^{(m)}(x)\,dx}{x^{t-m-1}}.$$

We have supposed t real, but it is easy to see that this restriction is in no way necessary, and that the transformation may be applied to the integrals on the right of (33). We thus obtain the formula (in which m is any positive integer)

$$\left.\begin{aligned}&-2\pi i\zeta_2(s,a,\omega_1,\omega_2)=s(s+1)\dots(s+r+m)\\&\left[E\int_0^{\infty\omega_1}\phi_{r+m+3}(u-a)\,u^{-s-r-m-1}du-E\int_0^{\infty\omega_2}\psi_{r+m+3}(u-a)\,u^{-s-r-m-1}du\right]\end{aligned}\right\}\text{.....(39),}$$

* Cauchy, *Exercices de Mathématique*, t. I., p. 57. Cauchy's notation is rather inconvenient.

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and from this, by differentiation with respect to a,

$$\left.\begin{aligned}&-2\pi i\frac{\Gamma(s+\kappa)}{\Gamma(s+r+m+1)}\zeta_2(s+\kappa,\ a,\ \omega_1,\ \omega_2)\\&=E\int_0^{\infty\omega_1}\phi_{r+m+s-\kappa}(u-a)\,u^{-s-r-m-1}du-E\int_0^{\infty\omega_2}\psi_{r+m+s-\kappa}(u-a)\,u^{-s-r-m-1}du\end{aligned}\right\}\ \ldots\ldots(40).$$

Some particular cases of the general formulae are interesting. If, for instance, we make $r=0$ in (34), so that $-1<\Re(s)<0$, and write t for $s+\kappa$, and then $\kappa+2$ for κ, we find, since

$$\phi_{1-\kappa}(u-a)=\wp^{(\kappa)}(u-a),$$

$$-2\pi i\frac{\Gamma(t)}{\Gamma(t-\kappa-1)}\zeta_2(t,\ a,\ \omega_1,\ \omega_2)=\int_0^{\infty\omega_1}\wp^{(\kappa)}(u-a)\frac{du}{u^{t-\kappa-1}}-\int_0^{\infty\omega_2}\wp^{(\kappa)}(u-a)\frac{du}{u^{t-\kappa-1}}\quad\ldots(41).$$

$(\kappa\geqq 1,\ 1<t-\kappa<2)$.

And from (40) we deduce

$$-2\pi i(t-1)\,\zeta_2(t,\ a,\ \omega_1,\ \omega_2)=E\int_0^{\infty\omega_1}\wp(u-a)\frac{du}{u^{t-1}}-E\int_0^{\infty\omega_2}\wp(u-a)\frac{du}{u^{t-1}}\qquad(2<\Re(t))\ldots(42).$$

§ 21. Before I proceed to deduce expressions for the logarithm of the Double Gamma-function and its derivatives, I shall show how to calculate the first few constants C_ν and the corresponding constants C_ν' which occur in the functions ψ_ν.

Since
$$\int_0^{\omega_1}\phi_1(u-a)\,du=0,$$

$$\begin{aligned}C_1&=-\frac{1}{\omega_1}\int_0^{\omega_1}\wp(u-a)\,du\\&=\frac{1}{\omega_1}\left[\zeta(u-a)-\zeta(-a)\right]_0^{\omega_1}\\&=\frac{\eta_1}{\omega_1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43).\end{aligned}$$

Again
$$\begin{aligned}C_2&=\frac{1}{\omega_1}\int_0^{\omega_1}\left\{\zeta(u-a)-\frac{\eta_1}{\omega_1}(u-a)\right\}du\\&=\frac{1}{\omega_1}\left[\log\sigma(u-a)-\frac{\eta_1}{2\omega_1}(u-a)^2\right]_0^{\omega_1}.\end{aligned}$$

Now
$$\frac{\sigma(\omega_1-a)}{\sigma(-a)}=-e^{\eta(\frac{1}{2}\omega_1-a)};$$

hence
$$C_2\omega_1=(2k+1)\pi i,$$

where k is an integer*.

§ 22. Now it is known that

$$\zeta(u)-\frac{\eta_1 u}{\omega_1}=\frac{\pi}{\omega_1}\cot\frac{\pi u}{\omega_1}+\frac{4\pi}{\omega_1}\sum_1^\infty\frac{q^{2n}}{1-q^{2n}}\sin\frac{2n\pi u}{\omega_1}$$

* For a discussion of the value of $\int_0^{\omega_1}\zeta(u-a)\,du$ in the case of a real discriminant, see Halphen, *Fonctions Elliptiques*, t. I., p. 199.

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if the series is convergent; *i.e.* if u lies within the infinite strip formed by the two lines through $\pm\omega_2$ parallel to ω_1. As this condition is satisfied by all values of u on the line joining $-a$ and $\omega_1 - a$, we may write $u-a$ for u and integrate from $u=0$ to $u=\omega_1$. Thus

$$\int_0^{\omega_1}\left\{\zeta(u-a)-\frac{\eta_1}{\omega_1}(u-a)\right\}du = \frac{\pi}{\omega_1}\int_0^{\omega_1}\cot\frac{\pi(u-a)}{\omega_1}\,du$$

$$= \left[\log\sin\frac{\pi(u-a)}{\omega_1}\right]_0^{\omega_1}$$

$$= \pi i,$$

by § 9. Thus

$$C_2 = \frac{\pi i}{\omega_1} \quad\text{.....(44).}$$

Again

$$C_3 = \frac{1}{\omega_1}\int_0^{\omega_1}\left\{\log\sigma(u-a) - \frac{\eta_1}{2\omega_1}(u-a)^2 - \frac{\pi i}{\omega_1}(u-a)\right\}du.$$

Now*

$$\log\sigma(u-a) - \frac{\eta_1}{2\omega_1}(u-a)^2 = \log\left\{\frac{\omega_1}{\pi}\sin\frac{\pi(u-a)}{\omega_1}\right\} + 2\sum_1^\infty \frac{q^{2n}}{1-q^{2n}}\,\frac{1-\cos\dfrac{2n\pi(u-a)}{\omega_1}}{n},$$

if suitable determinations of the logarithms are chosen. The infinite series gives on integration

$$2\omega_1\sum_1^\infty\frac{q^{2n}}{n(1-q^{2n})}.$$

Also

$$-\int_0^{\omega_1}\frac{\pi i}{\omega_1}(u-a)\,du = -\pi i\left(\tfrac{1}{2}\omega_1 - a\right),$$

and by § 9

$$\int_0^{\omega_1}\log\left\{\frac{\omega_1}{\pi}\sin\frac{\pi(u-a)}{\omega_1}\right\}du = \omega_1\log\frac{\omega_1}{2\pi} - a\pi i,$$

so that finally

$$C_3 = -\log\frac{2\pi}{\omega_1} - \tfrac{1}{2}\pi i + 2\sum_1^\infty\frac{q^{2n}}{n(1-q^{2n})} \quad\text{.....(45).}$$

Thus $\phi_3(u-a)$ is determined. The branch of $\log(2\pi/\omega_1)$ may remain arbitrary, as it appears in $\log\sigma(u-a)$ and also in C_3 and does not affect $\phi_3(u-a)$.

The infinite series which occurs in C_3 may be evaluated in finite form. For† if Δ is the discriminant

$$16(e_2-e_3)^2(e_3-e_1)^2(e_1-e_2)^2,$$

$$\frac{1}{12}\log\Delta = \log\frac{2\pi}{\omega_1} + \frac{1}{6}\log q + 2\sum_1^\infty\log(1-q^{2n}),$$

and

$$\Sigma\log(1-q^{2n}) = -\Sigma\Sigma\frac{q^{2np}}{p} = -\Sigma\frac{q^{2p}}{p(1-q^{2p})},$$

so that

$$C_3 = -\tfrac{1}{2}\pi i + \tfrac{1}{6}\log q - \tfrac{1}{12}\log\Delta \quad\text{.....(46),}$$

if suitable branches of the logarithms are chosen.

Thus

$$\phi_1(u-a) = \wp(u-a) + \frac{\eta_1}{\omega_1} \quad\text{.....(47),}$$

$$\phi_2(u-a) = -\zeta(u-a) + \frac{\eta_1}{\omega_1}(u-a) + \frac{\pi i}{\omega_1} \quad\text{.....(48),}$$

$$\phi_3(u-a) = -\log\sigma(u-a) + \frac{\eta_1}{2\omega_1}(u-a)^2 + \frac{\pi i}{\omega_1}(u-a) - \tfrac{1}{2}\pi i + \tfrac{1}{6}\log q - \tfrac{1}{12}\log\Delta \quad\text{.....(49).}$$

* Halphen, *loc. cit.*, t. I., p. 428.

† Jordan, *Cours d'Analyse*, t. II., pp. 426—7.

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§ 23. It is easy to see that the constant C_1' in ψ_1 is

$$\frac{\eta_2}{\omega_2} \quad \text{.....(50)},$$

and that

$$C_2' = -\frac{\pi i}{\omega_2} \quad \text{.....(51)}.$$

The exact specification of C_3' is more difficult. It follows from § 22 that

$$\phi_3(u-a) = -\log\sigma(u-a) + \tfrac{1}{2}C_1(u-a)^2 + C_2(u-a) + C_3$$

$$= -\log\sin\frac{\pi(u-a)}{\omega_1} + \frac{\eta_1}{2\omega_1}(u-a)^2 + \frac{\pi i}{\omega_1}(u-a) - \log 2 - \frac{1}{2}\pi i - 2\Sigma\frac{q^{2n}}{n(1-q^{2n})}\cos\frac{2n\pi(u-a)}{\omega_1},$$

where that branch of $\log\sin\frac{\pi(u-a)}{\omega_1}$ is chosen which is real on $(a, a+\omega_1)$. Also

$$\psi_3(u-a) = -\log\sigma(u-a) + \tfrac{1}{2}C_1'(u-a)^2 + C_2'(u-a) + C_3',$$

where the value of C_3' depends on which branch of $\log\sigma(u-a)$ is chosen. We may, for instance, agree that that branch is chosen which is obtained by going from near 0 with the branch chosen in ϕ_3 and returning near 0 by a loop round a nearly coinciding with the line $(0, a)$. The value of C_3' is then fixed. To find its explicit expression would, I think, be troublesome, as we should need to use expansions for elliptic functions in terms of

$$q' = e^{-\frac{\pi i\omega_1}{\omega_2}},$$

which are not given in the books, and cannot be written down at once from those in terms of

$$q = e^{\frac{\pi i\omega_2}{\omega_1}},$$

because

$$\mathfrak{R}\left(\frac{\omega_1}{i\omega_2}\right) < 0.$$

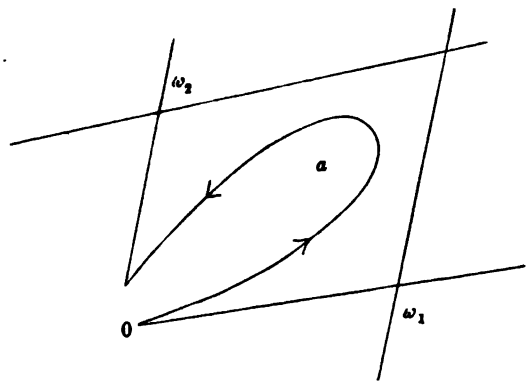

FIG. 1.

However, it is not in any way necessary for us to know an explicit expression for C_3'.

It is evident that if we make $u=0$ in ϕ_3 and ψ_3 the corresponding values of $\log\sigma(-a)$ differ by $2\pi i$.

The Double Gamma-function.

§ 24. I shall now deduce an expression for the logarithm of the double Gamma-function by means of the equation*

$$\log\frac{\Gamma_2(a, \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} = \left[\frac{\partial}{\partial s}\zeta_2(s, a, \omega_1, \omega_2)\right]_{s=0} \quad \text{.....(52)}.$$

By (37) (§ 19)

$$-2\pi i\zeta_2(s, a, \omega_1, \omega_2) \qquad (0 < \mathfrak{R}(s) < 1)$$

$$= s\int_0^{\infty\omega_1}\{\phi_3(u-a) - \phi_3(-a)\}\frac{du}{u^{s+1}} - s\int_0^{\infty\omega_2}\{\psi_3(u-a) - \psi_3(-a)\}\frac{du}{u^{s+1}}.$$

* Barnes, *loc. cit.*, pp. 330—334.

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$$\left.\begin{aligned}\text{Now}\quad & \frac{\partial}{\partial s}\left[s\int_0^{\infty\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{du}{u^{s+1}}\right]\\ &=\int_0^{\infty\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{du}{u^{s+1}}-s\int_0^{\infty\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{\log u\,du}{u^{s+1}}\end{aligned}\right\}\quad\ldots(53).$$

Each of the last two integrals becomes divergent for $s=0$.

Let h be a small fixed positive quantity. Then

$$\int_0^{\infty\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{du}{u^{s+1}}=\left(\int_0^{h\omega_1}+\int_{h\omega_1}^{\infty\omega_1}\right)\{\phi_3(u-a)-\phi_3(-a)\}\frac{du}{u^{s+1}}$$

$$=\int_0^{h\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{du}{u^{s+1}}+\int_{h\omega_1}^{\infty\omega_1}\phi_3(u-a)\frac{du}{u^{s+1}}-\frac{\phi_3(-a)}{s(h\omega_1)^s}.$$

Each of the integrals on the right is continuous for $s=0$*.

The last term may be expanded in the form

$$-\phi_3(-a)\left\{\frac{1}{s}-\log(h\omega_1)+\ldots\right\}.$$

Now
$$-\int_{h\omega_1}^{\infty\omega_1}\frac{\phi_3(-a)}{u(1+u)}du=-\phi_3(-a)\log\left(\frac{1+h\omega_1}{h\omega_1}\right),$$

and
$$-\int_0^{h\omega_1}\phi_3(-a)\left(\frac{1}{1+u}-1\right)\frac{du}{u}=\phi_3(-a)\log(1+h\omega_1).$$

Hence $\displaystyle\int_0^{\infty\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{du}{u^{s+1}}=-\frac{\phi_3(-a)}{s}+\int_0^{\infty\omega_1}\left\{\phi_3(u-a)-\frac{\phi_3(-a)}{1+u}\right\}\frac{du}{u}$

plus terms which vanish for $s=0$.

Again $\displaystyle\int_0^{\infty\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{\log u\,du}{u^{s+1}}=\int_0^{h\omega_1}\{\phi_3(u-a)-\phi_3(-a)\}\frac{\log u\,du}{u^{s+1}}$

$$+\int_{h\omega_1}^{\infty\omega_1}\phi_3(u-a)\frac{\log u\,du}{u^{s+1}}+\phi_3(-a)\frac{d}{ds}\left[\frac{1}{s(h\omega_1)^s}\right]=-\frac{\phi_3(-a)}{s^2}$$

plus terms which remain finite for $s=0$. Hence by (37) and (53)

$$-2\pi i\log\frac{\Gamma_2(a,\omega_1,\omega_2)}{\rho_2(\omega_1,\omega_2)}=\int_0^{\infty\omega_1}\left\{\phi_3(u-a)-\frac{\phi_3(-a)}{1+u}\right\}\frac{du}{u}-\int_0^{\infty\omega_2}\left\{\psi_3(u-a)-\frac{\psi_3(-a)}{1+u}\right\}\frac{du}{u}\ldots\ldots(54).$$

We can at once deduce formulae for the various logarithmic derivates of the Double Gamma-function. For if, in Mr Barnes' notation,

$$\psi_2^{(s)}(a,\omega_1,\omega_2)=\frac{d^s}{da^s}\log\Gamma_2(a,\omega_1,\omega_2)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55),$$

we obtain at once

$$\left.\begin{aligned}(-)^{m+1}2\pi i\psi_2^{(m)}(a,\omega_1,\omega_2)=&\int_0^{\infty\omega_1}\left\{\phi_{3-m}(u-a)-\frac{\phi_{3-m}(-a)}{1+u}\right\}\frac{du}{u}\\ &-\int_0^{\infty\omega_2}\left\{\psi_{3-m}(u-a)-\frac{\psi_{3-m}(-a)}{1+u}\right\}\frac{du}{u}\end{aligned}\right\}\ldots\ldots\ldots\ldots(56).$$

* Here, as in § 12, I omit the proof. This remark also applies to the differentiations under the integral sign in this section and elsewhere. I have proved elsewhere a number of theorems concerning the uniform convergence of certain classes of integrals which enable us to justify the assertions of this kind made in this paper.

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Special cases of the Double Zeta-function.

§ 25. We have now found formulae for the double Zeta-function for all values of s for which $\mathfrak{R}(s)$ is not integral, and also when s is a positive integer >2, in which case

$$\zeta_2(s, a, \omega_1, \omega_2) = \frac{(-)^s}{(s-1)!}\psi_2^{(s)}(a, \omega_1, \omega_2) \quad\text{.....(57).}$$

For $s=1$ and $s=2$ the Zeta-function becomes infinite, the finite terms in its expansions in powers of $s-1$ and $s-2$ being respectively $-\psi_2^{(1)}(a)$ and $\psi_2^{(2)}(a)$.

There remain two cases to consider, that in which s is complex, and $\mathfrak{R}(s)$ integral, and that in which s is zero or a negative integer.

By (33) of § 17, if $0 \leqq r < \mathfrak{R}(-s) < r+1$,

$$-2\pi i\zeta_2(s, a, \omega_1, \omega_2) = s(s+1)\dots(s+r)\left[\int_0^{\infty\omega_1}\phi_{r+3}(u-a)\,u^{-s-r-1}\,du - \int_0^{\infty\omega_2}\psi_{r+3}(u-a)\,u^{-s-r-1}\,du\right],$$

or
$$= s(s+1)\dots(s+r+1)\left[\int_0^{\infty\omega_1}\{\phi_{r+4}(u-a)-\phi_{r+4}(-a)\}\,u^{-s-r-2}\,du\right.$$
$$\left.-\int_0^{\infty\omega_2}\{\psi_{r+4}(u-a)-\psi_{r+4}(-a)\}\,u^{-s-r-2}\,du\right],$$

on integrating once by parts.

It can be shown that if $s=-r-1+i\sigma$ this last formula must be replaced by

$$\left.\begin{aligned}-2\pi i\zeta_2(s, a, \omega_1, \omega_2) = s(s+1)\dots(s+r+1)\Bigg[&H\{\phi_{r+4}(-a)-\psi_{r+4}(-a)\}\\ &+\int_0^{\infty\omega_1}\left\{\phi_{r+4}(u-a)-\frac{\phi_{r+4}(-a)}{1+u}\right\}u^{-s-r-2}\,du\\ &-\int_0^{\infty\omega_2}\left\{\psi_{r+4}(u-a)-\frac{\psi_{r+4}(-a)}{1+u}\right\}u^{-s-r-2}\,du\Bigg]\end{aligned}\right\}\quad\text{.....(57),}$$

where
$$H = -\frac{1}{i\sigma}\int_0^\infty\frac{u^{-i\sigma}}{(1+u)^2}\,du = \frac{\pi i}{\sinh\sigma\pi}\quad\text{.....(58).}$$

Suppose $s=-r-1+\epsilon+i\sigma$, where ϵ is small, and let h, as before, be a small fixed positive quantity. Then, if we neglect terms which vanish with ϵ,

$$\int_0^{\infty\omega_1}\{\phi_{r+4}(u-a)-\phi_{r+4}(-a)\}\,u^{-1-\epsilon-i\sigma}\,du$$
$$= \int_0^{h\omega_1}\{\phi_{r+4}(u-a)-\phi_{r+4}(-a)\}\,u^{-1-i\sigma}\,du + \int_{h\omega_1}^{\infty\omega_1}\phi_{r+4}(u-a)\,u^{-1-i\sigma}\,du - \frac{\phi_{r+4}(-a)}{i\sigma(h\omega_1)^{i\sigma}}.$$

This expression differs from the first integral in (21) by

$$\left[-\frac{1}{i\sigma(h\omega_1)^{i\sigma}} - \int_0^{h\omega_1}\frac{u^{-i\sigma}}{1+u}\,du + \int_{h\omega_1}^{\infty\omega_1}\frac{u^{-1-i\sigma}}{1+u}\,du\right]\times\phi_{r+4}(-a).$$

The quantity in square brackets is independent of h, and therefore

$$= \lim_{h=0}\left\{\int_{h\omega_1}^{\infty\omega_1}\frac{u^{-1-i\sigma}}{1+u}\,du - \frac{1}{i\sigma(h\omega_1)^{i\sigma}}\right\} = \lim_{h=0}\left\{\frac{1}{i\sigma(h\omega_1)^{i\sigma}(1+h\omega_1)} - \frac{1}{i\sigma(h\omega_1)^{i\sigma}}\right.$$
$$\left.-\frac{1}{i\sigma}\int_{h\omega_1}^{\infty\omega_1}\frac{u^{-i\sigma}}{(1+u)^2}\,du\right\} = -\frac{1}{i\sigma}\int_0^{\infty\omega_1}\frac{u^{-i\sigma}}{(1+u)^2}\,du.$$

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The integral along the axis of ω_2 may be treated in the same way. Since $\frac{u^{-i\sigma}}{(1+u)^2}$ has no poles whose real part is positive,

$$\int_0^{\infty\omega_1} \frac{u^{-i\sigma}}{(1+u)^2}\,du = \int_0^{\infty\omega_2} \frac{u^{-i\sigma}}{(1+u)^2}\,du = \int_0^{\infty} \frac{u^{-i\sigma}}{(1+u)^2}\,du = \frac{\sigma\pi}{\sinh\sigma\pi},$$

by an easy calculation.

Differentiating (57) κ times gives

$$\left.\begin{aligned} -2\pi i\frac{\Gamma(s+\kappa)}{\Gamma(s+r+2)}\zeta_2(s+\kappa, a, \omega_1, \omega_2) = H\{\phi_{r-\kappa+4}(-a) - \psi_{r-\kappa+4}(-a)\} \\ + \int_0^{\infty\omega_1}\left\{\phi_{r-\kappa+4}(u-a) - \frac{\phi_{r-\kappa+4}(-a)}{1+u}\right\} u^{-s-r-2}\,du \\ - \int_0^{\infty\omega_2}\left\{\psi_{r-\kappa+4}(u-a) - \frac{\psi_{r-\kappa+4}(-a)}{1+u}\right\} u^{-s-r-2}\,du \end{aligned}\right\} \quad \text{.....(59).}$$

If $r-\kappa+4 \leqq 0$ the first term vanishes. If $\sigma = 0$, so that $-s = r+1$, and $r-\kappa+4 = 3-m$, $\kappa = r+m+1$, $s+\kappa = m$, while $m \geqq 3$, we find that

$$-2\pi i\,\Gamma(m)\,\zeta_2(m, a, \omega_1, \omega_2)$$

is equal to the integral expression already obtained for

$$(-)^{m+1}\,2\pi i\psi_2^{(m)}(a, \omega_1, \omega_2),$$

which is of course correct.

There still remains the case in which s is actually a negative integer. Suppose that

$$s = -r-1+\epsilon,$$

where ϵ is small. Then

$$\int_0^{\infty\omega_1} \phi_{r+3}(u-a)\,u^{-s-r-1}\,du = (s+r+1)\int_0^{\infty\omega_1}\{\phi_{r+4}(u-a) - \phi_{r+4}(-a)\}\,u^{-s-r-2}\,du = \epsilon\int_0^{h\omega_1} + \epsilon\int_{h\omega_1}^{\infty\omega_1}.$$

The first term vanishes for $\epsilon = 0$, and the second becomes

$$-\phi_{r+4}(-a)\lim_{\epsilon=0}\epsilon\int_{h\omega_1}^{\infty\omega_1} u^{-1-\epsilon}\,du = -\phi_{r+4}(-a).$$

Thus $$-2\pi i\zeta_2(-r-1, a, \omega_1, \omega_2) = (-)^r(r+1)!\,\{\phi_{r+4}(-a) - \psi_{r+4}(-a)\} \quad \text{.....(60),}$$

$(0 \leqq r)$, which is a polynomial of degree $r+3$. This agrees with the known fact that for negative integral values of s the double Zeta-function is substantially the double Bernoullian function of order $-s$*.

To verify the work differentiate with respect to a and put $r = 0$. We find

$$2\pi i\zeta_2(0, a, \omega_1, \omega_2) = \phi_3(-a) - \psi_3(-a) \quad \text{.....(61).}$$

Now $$\phi_3(-a) = -\log\sigma(-a) + \tfrac{1}{2}C_1a^2 - C_2a + C_3,$$

$$\psi_3(-a) = -\log\sigma(-a) + \tfrac{1}{2}C_1'a^2 - C_2'a + C_3',$$

where $$C_1 = \frac{\eta_1}{\omega_1}, \quad C_1' = \frac{\eta_2}{\omega_2},$$

$$C_2 = \frac{\pi i}{\omega_1}, \quad C_2' = -\frac{\pi i}{\omega_2},$$

* Barnes, *loc. cit.*, p. 340.

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so that (remembering the remark made at the end of § 23)

$$\left.\begin{aligned}2\pi i\zeta_2(0, a, \omega_1, \omega_2) &= \frac{1}{2}a^2\left(\frac{\eta_1}{\omega_1}-\frac{\eta_2}{\omega_2}\right)-a\pi i\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)+C_3-C_3'-2\pi i\\ &= \frac{a^2\pi i}{\omega_1\omega_2}-\frac{a\pi i(\omega_1+\omega_2)}{\omega_1\omega_2}+C_3-C_3'-2\pi i\end{aligned}\right\}\quad\ldots\ldots(62).$$

Thus (apart from a constant) $\zeta_2(0, a, \omega_1, \omega_2)$ is

$$\frac{a^2}{2\omega_1\omega_2}-\frac{a(\omega_1+\omega_2)}{2\omega_1\omega_2}.$$

But it is known* that

$$\zeta_2(0, a, \omega_1, \omega_2) = {}_2S_1'(a) = \frac{a^2}{2\omega_1\omega_2}-\frac{a(\omega_1+\omega_2)}{2\omega_1\omega_2}+\frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2}\quad\ldots\ldots\ldots\ldots(63);$$

which agrees with our result. To attempt to verify that

$$C_3-C_3'-2\pi i = 2\pi i\,\frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2},$$

would lead us too far from our present subject.

We have now found formulae for the double Zeta-function for all values of s save $s=1$ and $s=2$, for which it becomes infinite, and $\psi_2^{(1)}$ and $\psi_2^{(2)}$ take its place.

A Formula for $\log\rho_2(\omega_1, \omega_2)$.

§ 26. We can obtain a formula for $\log\rho_2(\omega_1, \omega_2)$ by means of the equations

$$\lim_{a=0}\{\log\Gamma_2(a, \omega_1, \omega_2)+\log a\} = 0,$$

and (54). We require to consider the behaviour of

$$\int_0^{\infty\omega_1}\left\{\phi_3(u-a)-\frac{\phi_3(-a)}{1+u}\right\}\frac{du}{u} = \int_0^{h\omega_1}+\int_{h\omega_1}^{\infty\omega_1}$$

when a tends to zero by a path lying entirely within the principal period parallelogram.

Under these circumstances (see § 11) $\phi_3(u-a)$ tends to a definite value for all points on the axis of ω_1 except the points $0, \omega_1, 2\omega_1 \ldots$. These limiting values define a function $\phi_3(u)$ which becomes logarithmically infinite at $\omega_1, 2\omega_1, \ldots$, but which is easily seen to possess the characteristic properties of $\phi_3(u-a)$, in that it has ω_1 as a period, and

$$\int_0^{\omega_1}\phi_3(u)\,du = 0.$$

It is not difficult to prove that

$$\int_{h\omega_1}^{\infty\omega_1}\frac{\phi_3(u-a)}{u}\,du$$

is uniformly convergent for a range of values of a including 0, and so continuous for $a=0$.

Thus
$$\int_{h\omega_1}^{\infty\omega_1} = \int_{h\omega_1}^{\infty\omega_1}\phi_3(u)\frac{du}{u}-\phi_3(-a)\log\frac{1+h\omega_1}{h\omega_1},$$

if we neglect terms which vanish with a, as I shall do without remark in what follows.

* Barnes, *loc. cit.*, p. 273.

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Again $$\int_0^{h\omega_1} = \phi_3(-a)\log(1+h\omega_1) - \int_0^{h\omega_1}\{\log(u-a)-\log(-a)\}\frac{du}{u}$$
$$+\int_0^{h\omega_1}\{\phi_3(u-a)+\log(u-a)-\phi_3(-a)-\log(-a)\}\frac{du}{u},$$

$\log(u-a)$, $\log(-a)$ being chosen so that

$$\lim_{u=0}\log(u-a) = \log(-a),$$

and $$\lim_{a=0}\{\phi_3(-a)+\log(-a)\} = C_3.$$

The last integral tends for $a=0$ to

$$\int_0^{h\omega_1}\{\phi_3(u)+\log u - C_3\}\frac{du}{u}.$$

We may therefore replace the original integral by

$$\int_{h\omega_1}^{\infty\omega_1}\phi_3(u)\frac{du}{u} + \phi_3(-a)\log(h\omega_1)$$
$$-\int_0^{h\omega_1}\{\log(u-a)-\log(-a)\}\frac{du}{u} + \int_0^{h\omega_1}\{\phi_3(u)+\log u - C_3\}\frac{du}{u},$$

which is easily seen to be equal to

$$\int_0^{\infty\omega_1}\left\{\phi_3(u)+\frac{\log u - C_3}{1+u}\right\}\frac{du}{u} + \phi_3(-a)\log(h\omega_1) - \int_0^{h\omega_1}\{\log(u-a)-\log(-a)\}\frac{du}{u}$$
$$+\int_0^{h\omega_1}\frac{\log u - C_3}{1+u}du - \int_{h\omega_1}^{\infty\omega_1}\frac{\log u - C_3}{u(1+u)}du.$$

We may replace $\phi_3(-a)$ by $-\log(-a)+C_3$, and then C_3 may be omitted except in the first integral, since

$$\log(h\omega_1) - \int_0^{h\omega_1}\frac{du}{1+u} + \int_{h\omega_1}^{\infty\omega_1}\frac{du}{u(1+u)} = 0,$$

as appears on working out the integrations.

The integral along the axis of ω_2 may be treated in the same way. Hence the right-hand side of (54) becomes

$$\left.\begin{aligned}
&\int_0^{\infty\omega_1}\left\{\phi_3(u)+\frac{\log u - C_3}{1+u}\right\}\frac{du}{u}\\
&-\int_0^{\infty\omega_2}\left\{\psi_3(u)+\frac{\operatorname{Log} u - C_3'}{1+u}\right\}\frac{du}{u}\\
&-\log(-a)\log(h\omega_1)+\operatorname{Log}(-a)\log(h\omega_2)\\
&-\int_0^{h\omega_1}\{\log(u-a)-\log(-a)\}\frac{du}{u}\\
&+\int_0^{h\omega_2}\{\operatorname{Log}(u-a)-\operatorname{Log}(-a)\}\frac{du}{u}\\
&+\int_0^{h\omega_1}\frac{\log u}{1+u}du - \int_0^{h\omega_2}\frac{\operatorname{Log} u}{1+u}du\\
&-\int_{h\omega_1}^{\infty\omega_1}\frac{\log u}{u(1+u)}du + \int_{h\omega_2}^{\infty\omega_2}\frac{\operatorname{Log} u}{u(1+u)}du
\end{aligned}\right\}\quad\text{.....(64).}$$

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In this formula log and Log denote different branches of the logarithm.

The last four integrals are equal to

$$-\int_{h\omega_1}^{h\omega_2}\left(1+\frac{1}{u}\right)\frac{\log u}{1+u}\,du-2m\pi i\log(h\omega_2),$$

(where the integral is taken along the straight line from $h\omega_1$ to $h\omega_2$, and m is an integer)

$$=-\tfrac{1}{2}\{\log(h\omega_2)\}^2+\tfrac{1}{2}\{\log(h\omega_1)\}^2-2m\pi i\log(h\omega_2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(65).$$

Now let us consider the integral

$$\int\log\left(\frac{u-a}{-a}\right)\frac{du}{u},$$

taken round the contour C shown in the figure, in which 0_1, 0_2 are points nearly

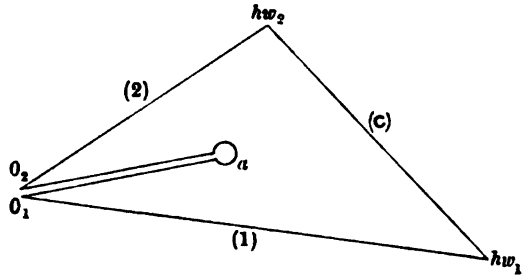

FIG. 2.

coinciding with 0. We start from 0_1 with $\log\{(u-a)/-a\}$ nearly zero. When we get to 0_2 its value is nearly $2\pi i$. Hence the contribution of (1) is nearly equal to the integral in the fourth line of (64), and the contribution of (2) exceeds the fifth line of (64) by a quantity nearly equal to

$$-2\pi i\int_{0_2}^{h\omega_2}\frac{du}{u}.$$

The contribution of the loop is practically

$$2\pi i\int_{0_2}^{a}\frac{du}{u}.$$

Hence, by Cauchy's theorem, we may replace the fourth and fifth lines of (64) by

$$\int_{h\omega_1}^{h\omega_2}\log\left(\frac{u-a}{-a}\right)\frac{du}{u}-2\pi i\int_{a}^{h\omega_2}\frac{du}{u}$$

$$=\tfrac{1}{2}\{\log(h\omega_2)\}^2-\tfrac{1}{2}\{\log(h\omega_1)\}^2$$

$$-\log(-a)\{\log(h\omega_2)-\log(h\omega_1)\}$$

$$-2\pi i\log(h\omega_2)+2\pi i\log a,$$

neglecting terms which vanish with a. The sum of this, (65), and the third line of (64) is $2\pi i\log a$. There is apparently an additional term, viz. an undetermined integral multiple of $2\pi i\log(h\omega_2)$; but this must vanish, as the result cannot contain h.

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Hence, finally,

$$\left.\begin{aligned}2\pi i \log \rho_2(\omega_1, \omega_2) &= \lim_{a=0}\left[-2\pi i \log \frac{\Gamma_2(a, \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} - 2\pi i \log a + 2\pi i \log a\Gamma_2(a, \omega_1, \omega_2)\right] \\ &= \int_0^{\infty\omega_1}\left\{\phi_3(u) + \frac{\log u - C_3}{1+u}\right\}\frac{du}{u} - \int_0^{\infty\omega_2}\left\{\psi_3(u) + \frac{\log u - C_3'}{1+u}\right\}\frac{du}{u}\end{aligned}\right\} \quad \ldots(66),$$

it being understood that the right branch of the left-hand side is selected, and that the logarithms under the sign of integration are so chosen that the limits of $\phi_3(u) + \log u - C_3$ and $\psi_3(u) + \log u - C_3'$ for $u = 0$ are each zero.

§ 27. I shall next find a similar formula for the modular constant $\gamma_{22}(\omega_1, \omega_2)$ defined by the equation

$$\gamma_{22}(\omega_1, \omega_2) = -\left[\psi_2^{(1)}(a, \omega_1, \omega_2) + \frac{1}{a}\right]_{a=0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(67).$$

We found that

$$2\pi i\psi_2^{(1)}(a, \omega_1, \omega_2) = \int_0^{\infty\omega_1}\left\{\phi_2(u-a) - \frac{\phi_2(-a)}{1+u}\right\}\frac{du}{u} - \int_0^{\infty\omega_2}\left\{\psi_2(u-a) - \frac{\psi_2(-a)}{1+u}\right\}\frac{du}{u}.$$

Now it is easy to prove that

$$\int_0^{\infty\omega_1}\left\{\phi_2(u-a) - \frac{\phi_2(-a)}{1+u}\right\}\frac{du}{u} = -\phi_2(-a) + \int_0^{\infty\omega_1}\left\{\phi_3(u-a) - \phi_3(-a) - \frac{u\phi_2(-a)}{1+u}\right\}\frac{du}{u^2} \quad \ldots(68).$$

For

$$\int_0^{h\omega_1}\left\{\phi_2(u-a) - \frac{\phi_2(-a)}{1+u}\right\}\frac{du}{u} = \int_0^{h\omega_1}\{\phi_3(u-a) - \phi_3(-a) - u\phi_2(-a)\}\frac{du}{u^2}$$

$$+ \{\phi_3(h\omega_1 - a) - \phi_3(-a) - h\omega_1\phi_2(-a)\}/h\omega_1 + \int_0^{h\omega_1}\frac{\phi_2(-a)}{1+u}du,$$

and $$\int_{h\omega_1}^{\infty\omega_1} = \int_{h\omega_1}^{\infty\omega_1}\{\phi_3(u-a) - \phi_3(-a)\}\frac{du}{u^2} - \{\phi_3(h\omega_1 - a) - \phi_3(-a)\}/h\omega_1 - \int_{h\omega_1}^{\infty\omega_1}\frac{\phi_2(-a)}{u(1+u)}du,$$

and the sum of these two expressions at once reduces to the form required. Hence

$$\left.\begin{aligned}2\pi i\psi_2^{(1)}(a, \omega_1, \omega_2) &= \int_0^{\infty\omega_1}\left\{\phi_3(u-a) - \phi_3(-a) - \frac{u\phi_2(-a)}{1+u}\right\}\frac{du}{u^2} \\ &- \int_0^{\infty\omega_2}\left\{\psi_3(u-a) - \psi_3(-a) - \frac{u\psi_2(-a)}{1+u}\right\}\frac{du}{u^2}\end{aligned}\right\} \ldots\ldots\ldots\ldots(69).$$

§ 28. Now

$$\int_0^{h\omega_1}\{\phi_3(u-a) - \phi_3(-a) - u\phi_2(-a)\}\frac{du}{u^2}$$

$$= \int_0^{h\omega_1}\left\{\phi_3(u-a) + \log(u-a) - C_3 - C_2(u-a) - \phi_3(-a) - \log(-a) + C_3 - C_2 a\right.$$

$$\left. - u\phi_2(-a) + \frac{u}{a} + C_2 u\right\}\frac{du}{u^2} - \int_0^{h\omega_1}\left\{\log(u-a) - \log(-a) + \frac{u}{a}\right\}\frac{du}{u^2}$$

$$= \int_0^{h\omega_1}\{\phi_3(u) + \log u - C_3 - C_2 u\}\frac{du}{u^2} - \int_0^{h\omega_1}\left\{\log(u-a) - \log(-a) + \frac{u}{a}\right\}\frac{du}{u^2},$$

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if we neglect terms which vanish with a. Hence

$$\int_0^{h\omega_1}\left\{\phi_3(u-a)-\phi_3(-a)-\frac{u\phi_2(-a)}{1+u}\right\}\frac{du}{u^2}$$

$$=\int_0^{h\omega_1}\{\phi_3(u)+\log u-C_3-C_2u\}\frac{du}{u^2}+\phi_3(-a)\log(1+h\omega_1)-\int_0^{h\omega_1}\left\{\log(u-a)-\log(-a)+\frac{u}{a}\right\}\frac{du}{u^2}.$$

Again
$$\int_{h\omega_1}^{\infty\omega_1}\left\{\phi_3(u-a)-\phi_3(-a)-\frac{u\phi_2(-a)}{1+u}\right\}\frac{du}{u^2}$$

$$=\int_{h\omega_1}^{\infty\omega_1}\phi_3(u)\frac{du}{u^2}-\frac{\phi_2(-a)}{h\omega_1}-\phi_2(-a)\log\left(\frac{1+h\omega_1}{h\omega_1}\right).$$

Thus
$$\int_0^{\infty\omega_1}\left\{\phi_3(u-a)-\phi_3(-a)-\frac{u\phi_2(-a)}{1+u}\right\}\frac{du}{u^2}$$

$$=\int_0^{\infty\omega_1}\left\{\phi_3(u)+\log u-C_3-\frac{C_2u}{1+u}\right\}\frac{du}{u^2}$$

$$+\phi_3(-a)\log(h\omega_1)-\frac{\phi_2(-a)}{h\omega_1}$$

$$-\int_{h\omega_1}^{\infty\omega_1}\left\{\log u-C_3-\frac{C_2u}{1+u}\right\}\frac{du}{u^2}$$

$$-\int_0^{h\omega_1}\frac{C_2\,du}{1+u}$$

$$-\int_0^{h\omega_1}\left\{\log(u-a)-\log(-a)+\frac{u}{a}\right\}\frac{du}{u^2}.$$

We may replace $\phi_2(-a)$ by $\frac{1}{a}+C_2$, and $\phi_3(-a)$ by $-\log(-a)+C_3$. Then it is easy to see that C_2 and C_3 may be omitted except in the first line.

Hence
$$\left.\begin{aligned}
&2\pi i\psi_2^{(1)}(a,\omega_1,\omega_2)\\
&=\int_0^{\infty\omega_1}\left\{\phi_3(u)+\log u-C_3-\frac{C_2u}{1+u}\right\}\frac{du}{u^2}\\
&\quad-\int_0^{\infty\omega_2}\left\{\psi_3(u)+\operatorname{Log}u-C_3'-\frac{C_2'u}{1+u}\right\}\frac{du}{u^2}\\
&\quad+\frac{1}{a}\{\log(h\omega_1)-\log(h\omega_2)\}+\frac{\log(-a)}{h\omega_1}-\frac{\operatorname{Log}(-a)}{h\omega_2}\\
&\quad-\int_{h\omega_1}^{\infty\omega_1}\frac{\log u}{u^2}\,du+\int_{h\omega_2}^{\infty\omega_2}\frac{\operatorname{Log}u}{u^2}\,du\\
&\quad-\int_0^{h\omega_1}\left\{\log(u-a)-\log(-a)+\frac{u}{a}\right\}\frac{du}{u^2}\\
&\quad+\int_0^{h\omega_2}\left\{\operatorname{Log}(u-a)-\operatorname{Log}(-a)+\frac{u}{a}\right\}\frac{du}{u^2}
\end{aligned}\right\}\quad\text{.....(70).}$$

We now take the integral

$$\int\left\{\log(u-a)-\log(-a)+\frac{u}{a}\right\}\frac{du}{u^2}$$

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round the contour C considered before. We find that the last two integrals in (70) may be replaced by

$$\int_{h\omega_1}^{h\omega_2}\left\{\log(u-a)-\log(-a)+\frac{u}{a}\right\}\frac{du}{u^2}-2\pi i\int_{0_2}^{h\omega_2}\frac{du}{u^2}+2\pi i\int_{0_2}^{a}\frac{du}{u^2},$$

$$=\int_{h\omega_1}^{h\omega_2}\frac{\log u}{u^2}du-\frac{\log(-a)}{h\omega_1}+\frac{\log(-a)}{h\omega_2}+\frac{1}{a}\log\left(\frac{h\omega_2}{h\omega_1}\right)-2\pi i\left(\frac{1}{a}-\frac{1}{h\omega_2}\right).$$

And since
$$\left(\int_{h\omega}^{h\omega_2}-\int_{h\omega_1}^{\infty\omega_1}+\int_{h\omega_2}^{\infty\omega_2}\right)\frac{\log u}{u^2}du=0,$$

(by Cauchy's theorem) we obtain finally by reasoning similar to that at the end of § 26

$$\left.\begin{aligned}-2\pi i\,\gamma_{22}(\omega_1,\omega_2)&=2\pi i\left[\psi_2^{(1)}(a,\omega_1,\omega_2)+\frac{1}{a}\right]_{a=0}\\&=\int_0^{\infty\omega_1}\left\{\phi_3(u)+\log u-C_3-\frac{C_2u}{1+u}\right\}\frac{du}{u^2}\\&-\int_0^{\infty\omega_2}\left\{\psi_3(u)+\log u-C_3'-\frac{C_2'u}{1+u}\right\}\frac{du}{u^2}\end{aligned}\right\}\quad\text{.....(71).}$$

§ 29. If we differentiate (69) we obtain

$$\left.\begin{aligned}&-2\pi i\psi_2^{(2)}(a,\omega_1,\omega_2)\\&=\int_0^{\infty\omega_1}\left\{\phi_2(u-a)-\phi_2(-a)-\frac{u\phi_1(-a)}{1+u}\right\}\frac{du}{u^2}\\&-\int_0^{\infty\omega_2}\left\{\psi_2(u-a)-\psi_2(-a)-\frac{u\psi_1(-a)}{1+u}\right\}\frac{du}{u^2}\end{aligned}\right\}\quad\text{.....(72).}$$

By a transformation similar to that of § 27,

$$\left.\begin{aligned}&-2\pi i\psi_2^{(2)}(a,\omega_1,\omega_2)\\&=2\int_0^{\infty\omega_1}\left\{\phi_3(u-a)-\phi_3(-a)-u\phi_2(-a)-\frac{\frac{1}{2}u^2\phi_1(-a)}{1+u}\right\}\frac{du}{u^3}\\&-2\int_0^{\infty\omega_2}\left\{\psi_3(u-a)-\psi_3(-a)-u\psi_2(-a)-\frac{\frac{1}{2}u^2\psi_1(-a)}{1+u}\right\}\frac{du}{u^3}\end{aligned}\right\}\quad\text{.....(73).}$$

And by analysis similar to that of §§ 26, 28 we can deduce

$$\left.\begin{aligned}2\pi i\gamma_{21}(\omega_1,\omega_2)&=-2\pi i\left[\psi_2^{(2)}(a,\omega_1,\omega_2)-\frac{1}{a^2}\right]_{a=0}\\&=2\int_0^{\infty\omega_1}\left\{\phi_3(u)+\log u-C_3-C_2u-\frac{\frac{1}{2}C_1u^2}{1+u}\right\}\frac{du}{u^3}\\&-2\int_0^{\infty\omega_1}\left\{\psi_3(u)+\log u-C_3'-C_2'u-\frac{\frac{1}{2}C_1'u^2}{1+u}\right\}\frac{du}{u^3}\end{aligned}\right\}\quad\text{.....(74).}$$

Thus Mr Barnes' three modular functions are expressed as definite integrals containing elliptic functions under the sign of integration.

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General values of a.

§ 30. We have so far restricted ourselves to the case in which a lies within the principal period parallelogram.

Now suppose that a lies in the parallelogram

$$k_1\omega_1 + k_2\omega_2,\ (k_1+1)\,\omega_1 + k_2\omega_2,\ (k_1+1)\,\omega_1 + (k_2+1)\,\omega_2,\ k_1\omega_1 + (k_2+1)\,\omega_2,$$

$(k_1,\ k_2 > 0)$. Under these circumstances the right side of (31) must be replaced by

$$-2\pi i\,(s-1)\sum_{-k_1}^{p_1 n-k_1}\sum_{-k_2}^{p_2 n-k_2}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s}.$$

Now
$$\sum_{-k_1}^{p_1 n-k_1}\sum_{-k_2}^{p_2 n-k_2} = \sum_{0}^{p_1 n-k_1}\sum_{0}^{p_2 n-k_2} + \sum_{-k_1}^{-1}\sum_{0}^{p_2 n-k_2} + \sum_{0}^{p_1 n-k_1}\sum_{-k_2}^{-1} + \sum_{-k_1}^{-1}\sum_{-k_2}^{-1}.$$

The constant terms in the expansions of the first three sums are respectively

$$\zeta_2(s,\ a,\ \omega_1,\ \omega_2),\qquad \sum_{-k_1}^{-1}\zeta_1(s,\ a+m_1\omega_1,\ \omega_2),$$

and
$$\sum_{-k_2}^{-1}\zeta_1(s,\ a+m_2\omega_2,\ \omega_1).$$

Thus
$$\left.\begin{aligned}
&-2\pi i\Bigg[\zeta_2(s,\ a,\ \omega_1,\ \omega_2) + \sum_{-k_1}^{-1}\zeta_1(s,\ a+m_1\omega_1,\ \omega_2)\\
&\qquad + \sum_{-k_2}^{-1}\zeta_1(s,\ a+m_2\omega_2,\ \omega_1) + \sum_{-k_1}^{-1}\sum_{-k_2}^{-1}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s}\Bigg]\\
&= s(s+1)\ldots(s+r)\Bigg[\int_0^{\infty\omega_1}\phi_{r+s}(u-a)\,u^{-s-r-1}\,du\\
&\qquad -\int_0^{\infty\omega_2}\psi_{r+s}(u-a)\,u^{-s-r-1}\,du\Bigg]\\
&(0 \leq r < \Re(-s) < r+1).
\end{aligned}\right\}\ldots\ldots(75),$$

Now in § 11 we found integral expressions for each of the terms

$$\zeta_1(s,\ a+m_2\omega_2,\ \omega_1).$$

For by (28)

$$\left.\begin{aligned}
&-2\pi i\Bigg[\zeta_1(s,\ a+m_2\omega_2,\ \omega_1) + \sum_{m_1=-k_1}^{-1}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s}\Bigg\}\\
&= s(s+1)\ldots(s+r)\Bigg[\int_0^{\infty\omega_1}\phi_{1,r+s}(u-a-m_2\omega_2)\,u^{-s-r-1}\,du\\
&\qquad -\int_0^{\infty\omega_2}\lambda_{r+s}(u-a-m_2\omega_2)\,u^{-s-r-1}\,du\Bigg]
\end{aligned}\right\}\ldots\ldots\ldots(76),$$

and there is a similar expression for

$$-2\pi i\left\{\zeta_1(s,\ a+m_1\omega_1,\ \omega_2) + \sum_{m_2=-k_2}^{-1}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s}\right\}.$$

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Summing these expressions from $m_2 = -k_2$ to -1, and from $m_1 = -k_1$ to -1, and subtracting from (75), we obtain an expression for

$$-2\pi i \left\{ \zeta_2(s, a, \omega_1, \omega_2) - \sum_{-k_1}^{-1} \sum_{-k_2}^{-1} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s} \right\} \quad \text{.....(77).}$$

We may also, if we like, express the algebraic terms of (77) as integrals along the axes of ω_1 and ω_2, and so obtain a formula for $-2\pi i\, \zeta_2(s, a, \omega_1, \omega_2)$ simply: but the formula is complicated. Of course formulae for

$$\log \frac{\Gamma_2(a, \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)},$$

for general values of a, may be deduced. There is also the case in which a lies on the other side of one or both axes to consider; but my present object is only to indicate the way in which the formulae of this paper must be modified for values of a other than those especially considered.

Special Cases.

§ 31. Special cases occur when a is on a side or at a corner of a period parallelogram. In such cases we may proceed as in § 12, if the special value of a is not one for which the function which we want an expression for becomes infinite.

The most interesting case is however that in which $a = 0$. Suppose $0 < \mathfrak{R}(s) < 1$. Then by (37)

$$-2\pi i\, \zeta_2(s, a, \omega_1, \omega_2) = s \int_0^{\infty\omega_1} \{\phi_3(u-a) - \phi_3(-a)\} \frac{du}{u^{s+1}} - s \int_0^{\infty\omega_2} \{\psi_3(u-a) - \psi_3(-a)\} \frac{du}{u^{s+1}}.$$

Suppose now that a tends to zero by a path lying inside the period parallelogram. Then it may be shown by an argument similar to that of §§ 26—29 that

$$\left.\begin{aligned} -2\pi i \left[\zeta_2(s, a, \omega_1, \omega_2) - \frac{1}{a^s} \right]_{a=0} = s \Bigg[& \int_0^{\infty\omega_1} \{\phi_3(u) + \log u - C_3\} \frac{du}{u^{s+1}} \\ & - \int_0^{\infty\omega_2} \{\psi_3(u) + \log u - C_3'\} \frac{du}{u^{s+1}} \Bigg] \end{aligned}\right\} \text{.....(79);}$$

and generally it appears that if $\mathfrak{R}(s) > 0$ and is not integral,

$$\left.\begin{aligned} -2\pi i \left[\zeta_2(s, a, \omega_1, \omega_2) - \frac{1}{a^s} \right] = s \Bigg[& E \int_0^{\infty\omega_1} \{\phi_3(u) + \log u\} \frac{du}{u^{s+1}} \\ & - E \int_0^{\infty\omega_2} \{\psi_3(u) + \log u\} \frac{du}{u^{s+1}} \Bigg] \end{aligned}\right\} \text{.....(80).}$$

If $\mathfrak{R}(s)$ is integral a new set of formulae is required, of which we have already worked out those which correspond to the cases $s = 0, 1, 2$. The form of the general formula for positive integral values of s may be readily inferred from these examples.

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PART III.

THE CONNECTION OF THE FORMULAE OF THE PAPER WITH MR BARNES' CONTOUR INTEGRALS FOR THE DOUBLE ZETA- AND GAMMA-FUNCTIONS.

§ 32. When the real parts of ω_1, ω_2 and a are positive, the double Zeta-function may be defined* for all values of s for which it exists by the contour integral

$$\frac{i\Gamma(1-s)}{2\pi}\int\frac{e^{-au}(-u)^{s-1}}{(1-e^{-\omega_1 u})(1-e^{-\omega_2 u})}du \quad\text{.....(81)},$$

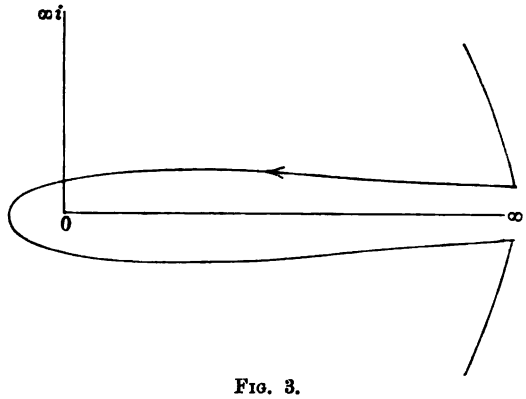

FIG. 3.

where $(-u)^{s-1}=e^{(s-1)\log(-u)}$, $\log(-u)$ being real when u is real and negative, and rendered uniform by a cut along the positive half of the real axis. The contour of integration is a loop from $u=+\infty$ to $u=+\infty$ enclosing 0 but no other pole of the subject of integration.

Suppose that the contour is separated at ∞ and completed by drawing a very large circle whose centre is 0. Then under certain conditions which I shall formulate shortly the value of the integral when the radius of the circle is made infinite will still be the same. If this is so it is clear that

$$\frac{2\pi}{i\Gamma(1-s)}\zeta_2(s, a, \omega_1, \omega_2)$$

will be equal to $-2\pi i$ times the sum of the residues for poles within the contour.

Now the roots of $1-e^{-\omega_1 u}$ are

$$u=\frac{2n\pi i}{\omega_1},\ (n=-\infty, \ldots \infty),$$

and the residue of

$$\frac{e^{-au}(-u)^{s-1}}{(1-e^{-\omega_1 u})(1-e^{-\omega_2 u})},$$

at $u=\dfrac{2n\pi i}{\omega_1}$ is

$$\frac{1}{\omega_1}\frac{e^{-\frac{2na\pi i}{\omega_1}}}{1-e^{-\frac{2n\omega_2\pi i}{\omega_1}}}\left(-\frac{2n\pi i}{\omega_1}\right)^{s-1}.$$

* Barnes, *loc. cit.*, p. 314.

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The sum of all these residues from $n=-\infty$ to ∞ (exclusive of 0) will be convergent if

$$\lim_{n=\pm\infty} \frac{1}{e^{\frac{2na\pi i}{\omega_1}} - e^{\frac{2n(a-\omega_2)\pi i}{\omega_1}}} = 0 \quad \text{.....(82)},$$

which will be the case if $\mathfrak{R}\left(\frac{ai}{\omega_1}\right)$ and $\mathfrak{R}\left\{\frac{(a-\omega_2)i}{\omega_1}\right\}$ have opposite signs.

But if u_1 and u_2 are two complex quantities $=x_1+iy_1$ and x_2+iy_2 respectively, the meaning of $\mathfrak{R}\left(\frac{u_2 i}{u_1}\right)>0$ is that $\frac{y_1}{x_1}>\frac{y_2}{x_2}$.

Hence it is easy to see that (82) is satisfied if a lies within the strip bounded by $(0, \infty\,\omega_1)$ and $(\omega_2, \omega_2+\infty\,\omega_1)$. This is in fact obvious on a reference to Fig. 4.

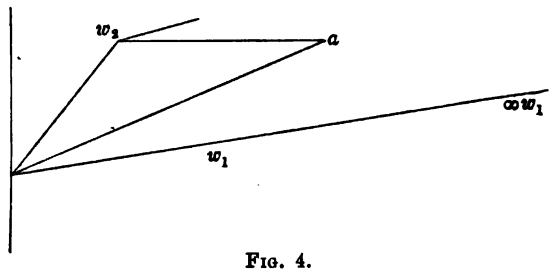

FIG. 4.

The roots of $1-e^{-\omega_2 u}$ give another series of residues which is certainly convergent if a lies in the strip bounded by $(0, \infty\,\omega_2)$ and $(\omega_1, \omega_1+\infty\,\omega_2)$. Hence if a lies within the principal period parallelogram both series will be convergent. Moreover it is easy to see that if this is the case, and if the circular part of the contour, in its progress to infinity, always has its radius so chosen that it never passes at less than some assigned distance from any pole, the value of its contribution to the contour integral ultimately vanishes.

For (neglecting the power of u) the subject of integration is

$$\frac{1}{e^{au} - e^{(a-\omega_1)u} - e^{(a-\omega_2)u} + e^{(a-\omega_1-\omega_2)u}} \quad \text{.....(83)}.$$

Now if $\alpha=\alpha_1+i\alpha_2$, $u=x+iy$, and $|u|$ is large, $|e^{\alpha u}|$ will be large or small according as $\alpha_1 x \gtrless \alpha_2 y$.

Draw the line $(0, \infty\,\alpha)$ and its image with respect to $x=y$. Then the part *below* or *above* this line is the part of the plane in which $|e^{\alpha u}|$ is infinite at infinity, according as $\alpha_2 \gtrless 0$. A reference to Fig. 5, in which the different angles within which the four terms in the denominator of (83) become infinite are indicated by semicircular arcs, now shows that

$$e^{au} - e^{(a-\omega_1)u} - e^{(a-\omega_2)u} + e^{(a-\omega_1-\omega_2)u}$$

becomes infinite, and so (83) vanishes, all along the circular part of the contour. Hence the contribution of this part ultimately vanishes. Hence

$$\frac{\zeta_2(s, a, \omega_1, \omega_2)}{\Gamma(1-s)} = \frac{1}{\omega_1}\sum_{-\infty}^{\infty}{}' \frac{e^{-\frac{2na\pi i}{\omega_1}}}{1-e^{-\frac{2n\omega_2\pi i}{\omega_1}}}\left(-\frac{2n\pi i}{\omega_1}\right)^{s-1} + \frac{1}{\omega_2}\sum_{-\infty}^{\infty}{}' \frac{e^{-\frac{2na\pi i}{\omega_2}}}{1-e^{-\frac{2n\omega_1\pi i}{\omega_2}}}\left(-\frac{2n\pi i}{\omega_2}\right)^{s-1} \quad \text{...(84)}.$$

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Now let us consider the first series. Take first the general term for which $n = \nu$ is positive. In it

$$\left(-\frac{2\nu\pi i}{\omega_1}\right)^{s-1} = e^{(s-1)\log\left(-\frac{2\nu\pi i}{\omega_1}\right)},$$

where the imaginary part of the logarithm lies between $\pm\pi i$.

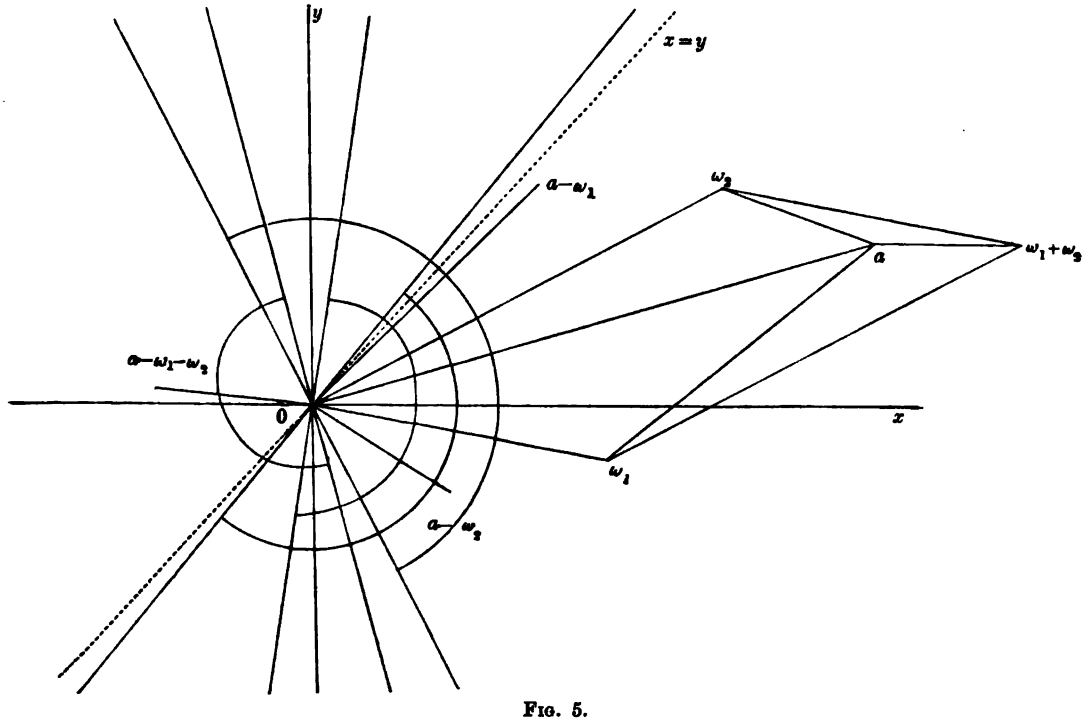

FIG. 5.

Similarly in the corresponding term for which $n = -\nu$ is negative

$$\left(\frac{2\nu\pi i}{\omega_1}\right)^{s-1} = e^{(s-1)\log\left(\frac{2\nu\pi i}{\omega_1}\right)},$$

where again the imaginary part of the logarithm lies between $\pm\pi i$.

Now if ω_1 makes an angle θ_1 with $0x$ the imaginary part of $\log\left(\frac{2\nu\pi i}{\omega_1}\right)$ is

$$(\tfrac{1}{2}\pi - \theta_1 + 2k\pi)\, i,$$

and as $\frac{1}{2}\pi - \theta_1$ lies between 0 and π, $k = 0$. Similarly that of $\log\left(-\frac{2\nu\pi i}{\omega_1}\right)$ is

$$\{\tfrac{1}{2}\pi - \theta_1 + (2k+1)\,\pi\}\, i$$

and here $k = -1$.

Thus the terms for which $n = \pm\nu$ give together

$$\left(\frac{2\nu\pi}{|\omega_1|}\right)^{s-1}\left\{\frac{e^{-\frac{2\nu a\pi i}{\omega_1} - (s-1)(\frac{1}{2}\pi+\theta_1)i}}{1 - e^{-\frac{2\nu\omega_2\pi i}{\omega_1}}} + \frac{e^{\frac{2\nu a\pi i}{\omega_1} - (s-1)(-\frac{1}{2}\pi+\theta_1)i}}{1 - e^{\frac{2\nu\omega_2\pi i}{\omega_1}}}\right\}$$

$$= \left(\frac{2\nu\pi}{\omega_1}\right)^{s-1} \frac{e^{\frac{\nu\omega_2\pi i}{\omega_1}}}{1 - e^{\frac{2\nu\omega_2\pi i}{\omega_1}}}\left\{e^{\frac{2\nu(a-\frac{1}{2}\omega_2)\pi i}{\omega_1} + \frac{1}{2}(s-1)\pi i} - e^{-\frac{2\nu(a-\frac{1}{2}\omega_2)\pi i}{\omega_1} - \frac{1}{2}(s-1)\pi i}\right\}$$

$$= 2i\left(\frac{2\nu\pi}{\omega_1}\right)^{s-1} \frac{q^\nu}{1 - q^{2\nu}} \sin\left\{\frac{2\nu\pi\,(a - \frac{1}{2}\omega_2)\,\pi}{\omega_1} + \frac{1}{2}(s-1)\,\pi\right\}.$$

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Thus the first term in the expression of $\zeta_2(s, a, \omega_1, \omega_2)$ is

$$2i\,\frac{(2\pi)^{s-1}\,\Gamma(1-s)}{\omega_1^{\,s}}\sum_1^\infty \frac{q^\nu \nu^{s-1}}{1-q^{2\nu}}\sin\left\{\frac{2\nu\pi\,(a-\frac{1}{2}\omega_2)\,\pi}{\omega_1}+\tfrac{1}{2}(s-1)\,\pi\right\} \quad \text{.....(85)},$$

where ω_1^{-s} has its principal value.

§ 33. Now*

$$\wp(u-a-\tfrac{1}{2}\omega_2)+\frac{\eta_1}{\omega_1}=-2\left(\frac{2\pi}{\omega_1}\right)^2\sum_1^\infty\frac{nq^n}{1-q^{2n}}\cos\frac{2n\,(u-a)\,\pi}{\omega_1} \quad \text{.....(86)},$$

for all values of u and a such that $u-a$ lies in the strip bounded by

$$(-\infty\,\omega_1+\tfrac{1}{2}\omega_2,\quad \infty\,\omega_1+\tfrac{1}{2}\omega_2),$$

and

$$(-\infty\,\omega_1-\tfrac{1}{2}\omega_2,\quad \infty\,\omega_1-\tfrac{1}{2}\omega_2),$$

a condition which is evidently satisfied if a lies in the strip and u on the axis of ω_1.

Hence

$$\int_0^{\infty\omega_1}\left\{\wp(u-a-\tfrac{1}{2}\omega_2)+\frac{\eta_1}{\omega_1}\right\}\frac{du}{u^{s-1}}=-2\left(\frac{2\pi}{\omega_1}\right)^2\sum_1^\infty\frac{nq^n}{1-q^{2n}}\int_0^{\infty\omega_1}\cos\frac{2n\,(u-a)\,\pi}{\omega_1}\,\frac{du}{u^{s-1}} \quad \text{...(87)}.$$

Here it is supposed that $1<\mathfrak{R}(s)<2$, and that

$$u^{-s}=e^{-s\log u},$$

$\log u$ having its principal value. Now

$$\int_0^{\infty\omega_1}\exp\left\{\frac{2n\,(u-a)\,\pi i}{\omega_1}\right\}\frac{du}{u^{s-1}}=e^{-\frac{2na\pi i}{\omega_1}}\int_0^{\infty\omega_1}e^{\frac{2nu\pi i}{\omega_1}}\,\frac{du}{u^{s-1}}.$$

Along the path of integration $\dfrac{u}{\omega_1}=x$, where x is real, and $u^{1-s}=\omega_1 x\,.\,e^{-s\log(\omega_1 x)}=\omega_1^{\,1-s}x^{1-s}$ where ω_1^{-s} has its principal value. Hence the integral is

$$\omega_1^{\,2-s}e^{-\frac{2na\pi i}{\omega_1}}\int_0^{\infty\omega_1}e^{2nx\pi i}x^{1-s}dx=\omega_1^{\,2-s}\,(2n\pi)^{s-2}\,\Gamma(2-s)\exp\left\{-\left[\frac{2na}{\omega_1}+\tfrac{1}{2}(2-s)\right]\pi i\right\},$$

and so

$$\left.\begin{aligned}\int_0^{\infty\omega_1}\left\{\wp(u-a+\tfrac{1}{2}\omega_2)+\frac{\eta_1}{\omega_1}\right\}\frac{du}{u^{s-1}}&=-\left(\frac{2\pi}{\omega_1}\right)^s\Gamma(2-s)\sum_1^\infty\frac{n^{s-1}q^n}{1-q^{2n}}\\ &\left\{\exp\left[-\frac{2na}{\omega_1}+\tfrac{1}{2}(2-s)\right]\pi i+\exp\left[\frac{2na}{\omega_1}-\tfrac{1}{2}(2-s)\right]\pi i\right\}\\ &=-2\left(\frac{2\pi}{\omega_1}\right)^s\Gamma(2-s)\sum_1^\infty\frac{n^{s-1}q^n}{1-q^{2n}}\cos\left\{\frac{2na}{\omega_2}-\tfrac{1}{2}(2-s)\right\}\pi\end{aligned}\right\} \quad \text{.....(88)}.$$

Here ω_1^{-s} has its principal value, and a lies within the limits previously defined. If we write $a-\frac{1}{2}\omega_2$ for a, the new a will lie in the strip bounded by the axis of ω_1 and the parallel to it through ω_2, and for such an a

$$\left.\begin{aligned}\int_0^{\infty\omega_1}\left\{\wp(u-a)+\frac{\eta_1}{\omega_1}\right\}\frac{du}{u^{s-1}}&=-2\left(\frac{2\pi}{\omega_1}\right)^s\Gamma(2-s)\sum_1^\infty\frac{n^{s-1}q^n}{1-q^{2n}}\cos\left\{\frac{2n\pi\,(a-\frac{1}{2}\omega_2)}{\omega_1}-\tfrac{1}{2}(2-s)\,\pi\right\}\\ &=-2\left(\frac{2\pi}{\omega_1}\right)^s\Gamma(2-s)\sum_1^\infty\frac{n^{s-1}q^n}{1-q^{2n}}\sin\left\{\frac{2n\pi\,(a-\frac{1}{2}\omega_2)}{\omega_1}+\tfrac{1}{2}(s-1)\,\pi\right\}\end{aligned}\right\}\text{...(89)}.$$

Now since $\zeta_2(s, a, \omega_1, \omega_2)$ is the sum of two series, of which

$$2i\,\frac{(2\pi)^{s-1}\,\Gamma(1-s)}{\omega_1^{\,s}}\sum_1^\infty\frac{n^{s-1}q^n}{1-q^{2n}}\sin\left\{\frac{2n\pi\,(a-\frac{1}{2}\omega_2)}{\omega_1}+\tfrac{1}{2}(s-1)\,\pi\right\}$$

* Halphen, *Fonctions Elliptiques*, I. 426. The notation is different.

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is one, it follows that $-2\pi i(s-1)\zeta_2(s, a, \omega_1, \omega_2)$ is the sum of two series, of which one is the series which occurs in (89).

The second series may be treated in the same way; and so we arrive at the formula

$$-2\pi i(s-1)\zeta_2(s, a, \omega_1, \omega_2) = \int_0^{\infty\omega_1}\left\{\wp(u-a)+\frac{\eta_1}{\omega_1}\right\}\frac{du}{u^{s-1}} - \int_0^{\infty\omega_2}\left\{\wp(u-a)+\frac{\eta_2}{\omega_2}\right\}\frac{du}{u^{s-1}} \quad\text{.....(90).}$$

Now if in (34) we suppose $r+3-\kappa=1$, $\kappa=r+2$, we obtain

$$\left.\begin{aligned}-2\pi i(s+r+1)\zeta_2(s+r+2, a, \omega_1, \omega_2) = &\int_0^{\infty\omega_1}\phi_1(u-a)\,u^{-s-r-1}\,du\\ &-\int_0^{\infty\omega_2}\psi_1(u-a)\,u^{-s-r-1}\,du\end{aligned}\right\}\quad\text{.....(91),}$$

which agrees with the above formula when we put $s+2=t$ and suppose $r=0$, so that

$$-1<\Re(s)<0, \quad 1<\Re(t)<2.$$

Conclusion.

§ 34. The investigations of §§ 32, 33 connect the formulae of this paper, which were derived from the definition of the double Zeta-function as the constant term in a certain asymptotic expansion, with the contour integral by means of which the double Zeta-function is first defined by Mr Barnes. The series which occur in §§ 32, 33 are very interesting; we can of course deduce expressions for

$$\log\frac{\Gamma_2(a, \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)}, \text{ etc.,}$$

as the sum of two infinite series. These expressions are analogous to Kummer's series for $\log\Gamma(a)$. But the length of this paper forbids that I should pursue the subject further here.

We cannot (at any rate without undertaking a difficult and laborious investigation) deduce the corresponding formulae for the G-function directly from those for the double Gamma-function. The G-function, which is the degenerate form assumed by the double Gamma-function in the limiting case in which the parameters ω_1, ω_2 become equal, is a very interesting function; more so, perhaps, than the general double Gamma-function itself. But when the periods become equal the elliptic functions cease to exist, and to determine the limiting forms assumed by formulae containing them is in general a matter of considerable difficulty.

Mr Barnes has however remarked that the properties of the G-function may be deduced from those of the function

$$-\frac{1}{s}\frac{\partial}{\partial\omega}\zeta_1(s, a, \omega),$$

in the same way as that in which the properties of the Gamma-function are deduced from those of the Zeta-function. There is therefore no difficulty in deducing expressions for the G-function and its logarithmic derivates from the formulae of Part I.

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II. *The Law of Error.* By Professor F. Y. Edgeworth.

[*Received* 22 August 1904.]

INTRODUCTORY SUMMARY.

The law which is the subject of this paper is the approximate expression for the frequency with which different values are assumed by a magnitude that depends on a number of independently varying magnitudes, here called elements.

The first Part deals with a case called "typical," in which the elements and the compound vary only in one dimension, and the relation of the compound to the elements is particularly simple. Approximate expressions for the law of error are found by a new method which proceeds on the assumption that a form may be regarded as approximate to a given frequency-locus when the mean powers of deviation measured from the centre of gravity—second, third, fourth powers and so on up to some high power—are approximately the same for the representative as for the given locus. An approximating curve of this character is found in the typical case by means of certain properties of the multinomial law. The general expression thus found for the law of error may be written

$$e^{-k_1\frac{1}{3!}\left(\frac{d}{dx}\right)^3+k_2\frac{1}{4!}\left(\frac{d}{dx}\right)^4-\text{etc.}+(-1)^t\frac{1}{t+2!}\left(\frac{d}{dx}\right)^{t+2}}\frac{1}{\sqrt{\pi 2k}}\,e^{-\frac{x^2}{2k}};$$

where k_0 is the mean square of deviation for the compound; $k_1, k_2, \ldots k_t$, form a descending series corresponding to the second, third, $\ldots (t+1)$th approximation to the actual locus; k_1 is the difference between the mean third power for the actual locus and what it would be if the first approximation were accurate, k_2 is the difference between the mean fourth power for the actual locus and what it would be if the second approximation were accurate, and so on up to the tth, where t is not large with respect to m the number of the elements. The formula comprises as particular cases the familiar normal law which forms the first approximation, and the second approximation investigated by Poisson.

The properties of the multinomial law on which the new proof rests are also required in order to establish and extend the proof by way of partial differential equations which was initiated by Professor Morgan Crofton.

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The system of partial differential equations for the typical case is:—

$$\frac{dy}{dk_0} = \frac{1}{2}\frac{d^2y}{dx^2},$$

$$\frac{dy}{dk_1} = -\frac{1}{3!}\frac{d^3y}{dx^3},$$

$$\frac{dy}{dk_2} = \frac{1}{4!}\frac{d^4y}{dx^4},$$

$$\vdots \qquad \vdots$$

$$\frac{dy}{dk_t} = \frac{(-1)^t}{t+2!}\frac{d^{t+2}y}{dx^{t+2}}.$$

From this system, combined with certain subsidiary conditions, and regard being had to the orders of magnitude to which the respective ks belong, there is obtained a solution which is identical with the expression given in the preceding paragraph.

The properties of the multinomial law which have been referred to are next employed to extend Laplace's method of investigation beyond the second approximation found by Poisson, to the third and generally the tth approximation. The "difficult point" which Czuber* has pointed out as obstructing the general application of Laplace's method even to the first approximation is here surmounted.

A variant proof is formed by combining some of the premisses on which Mr Morgan Crofton's method is based with a new condition which is handled by Laplace's analysis. The new condition is that the law of error must be *reproductive* in the sense that, if there be two (or more) magnitudes each fluctuating according to the law of error (as being each the sum of a number of elements fluctuating according to any law of frequency), then the sum of these two (or more) magnitudes must fluctuate according to the law of error.

The results found for the typical case are verified by showing them to be true in some simple varieties of the case for which the law of approximation can be investigated by ordinary analysis. Thus in the case of the binomial the second, third and fourth approximations as found by ordinary analysis (employed by Todhunter for the second approximation, by Czuber and Karl Pearson for the third approximation) agree with the results of the more general theory.

Further verification is obtained by a new use of ordinary analysis to prove the law of error in the simplest case of multinomial laws of frequency with which Laplace begins his investigation (the case in which the frequency-locus for each element is a horizontal line) and the fairly complicated case at which Laplace stops, the case in which each of the elements have any the same locus.

The results thus established for the typical case are extended in Part II. to the general case in which there are several degrees of freedom, the compound may be a function other than linear of the elements; and other simplifications are withdrawn, only essential conditions being retained.

* *Theorie der Beobachtungsfehler*, referring to Laplace's Method of Least Squares.

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PART I.

The law of error may be described as the approximate expression of the frequency with which in the long run different values are assumed by a quantity which is dependent on a number of variable items or elements; given certain conditions which seem to be adequately fulfilled in common experience.

The first approximation to the law of error thus understood was given by Laplace* in connexion with his method of least squares. The second approximation was investigated by Poisson on the same lines. Another proof of the first approximation has been given by Professor Morgan Crofton†; and has been extended by the present writer‡ to the second approximation. It is here attempted to confirm the results which have been found, and to find new results—namely the third, fourth, and generally the tth approximation—partly by a new method, and partly by a modification and extension of the methods which were originated by Laplace and by Professor Crofton.

It will be convenient first, in Part I., to discuss a case which is neither the simplest nor yet the most general, but may be regarded as *typical*, presenting the most characteristic difficulties in their least complicated form. The theories proved for the typical case will then, in Part II., be extended to the general case of which only the conditions necessary to the fulfilment of the law are predicated.

The typical case which is the subject of the first part is defined by the following attributes: *first*, essential conditions which must be satisfied in order that the law of error should be perfectly fulfilled, and *second*, limitations which are introduced only for convenience of exposition, and are to be afterwards withdrawn.

1. The elements assume different values in random fashion.

2. The values assumed by each element in the long run recur with a proportionate frequency capable of being represented by a single definite frequency-curve or -locus.

3. The set of values each of which is contributed by the same element to a different value of the compound quantity is formed by variations which are independent of each other.

4. The set of values each of which is contributed by a different element to the same value of the compound quantity, is formed by variations which are independent of each other.

5. The method of aggregation by which the elements are compounded is the simplest possible, namely summation.

6. Each element varies only in one and the same dimension.

* *Théorie Analytique des Probabilités*, B. II. ch. 4: cp. Glaisher (*Memoirs of the Astronomical Society*, Vol. XXXIX. p. 104) on the wide applicability of this theory.

† *Encyclopædia Britannica*, Art. "Probability," p. 781.

‡ *Philosophical Magazine*, 1896, Vol. XLI. p. 93 *et sqq.* The third approximation is enounced (but not proved) by the present writer in the article on the "Law of Error" in the Supplement to the *Encyclopædia Britannica*.

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7. The distance between the centre of gravity and either extremity for the frequency-locus of each element is not infinite; "centre of gravity" denoting a point on the axis on which are measured the values of the elements, the point which would be the centre of gravity, if at every point corresponding to a value assumed by the element, there were placed a particle of mass proportionate to the frequency with which that value occurs in the long run, and the "extremities" denoting the greatest and smallest values which an element in the long run assumes.

8. The distance between the centre of gravity and either extremity for the frequency-locus of each element is not infinitesimal.

9. The number of the elements is large, in relation to certain quantities which will be defined in the sequel.

10. The frequency-loci for the different elements, though not in general identical, are approximately equivalent in that each mean power of deviation from the centre of gravity (the first, second, third, and so on up to some considerable power, say the tth), is approximately the same for any one of the elements as the corresponding power for any other of the elements.

11. The values which any element assumes in the long run are all multiples of one and the same finite difference.

The properties which have been enumerated may thus be expressed in symbols. Let any the pth value of the compound magnitude—the pth observation as it may be called—be $A + {}_px$; where ${}_px$ is measured from the centre of gravity pertaining to the set of values assumed by the compound quantity in the long run; A is the distance of that centre from an assigned origin. Then

$$A + {}_px = (\mathrm{a}_1 + {}_p\xi_1) + (\mathrm{a}_2 + {}_p\xi_2) + \ldots + (\mathrm{a}_m + {}_p\xi_m);$$

where $\mathrm{a}_q + {}_p\xi_q$ is the contribution of the qth element to the pth observation, ${}_p\xi_q$ being measured from the centre of gravity pertaining to the set of values assumed by the qth element in the long run. It is proper to regard $\mathrm{a}_1, \mathrm{a}_2 \ldots \mathrm{a}_m$ as each measured from the same origin, the same as the origin of x, in such wise that $A = \sum_{q=1}^{q=m} \mathrm{a}_q$, and ${}_px = \sum_{q=1}^{q=m} {}_p\xi_q$. It is postulated that the values contributed by the same element to different observations, e.g. ${}_1\xi_q, {}_2\xi_q \ldots {}_p\xi_q \ldots$, are assumed at random independently of each other. The values contributed by different elements to the same observation, e.g. ${}_p\xi_1, {}_p\xi_2 \ldots {}_p\xi_q \ldots$, are also assumed at random independently of each other. The number of the elements, m, is large, in such wise that a descending series is formed by $\lambda_0, \lambda_1/m^{\frac{1}{2}}, \lambda_2/m, \lambda_3/m^{\frac{3}{2}}$, where $\lambda_0, \lambda_1 \ldots$ are functions of mean powers of an element specified in the sequel. The values which any the qth element assumes occur at points lying on the axis x at intervals equal to the finite difference Δx. Thus if the (left) extremity of any the qth element is taken as origin the values assumed in the long run, with more or less frequency, by the element are

$$0, \quad \Delta x, \quad 2\Delta x \ldots \nu\Delta x,$$

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where ν is an integer, $\nu\Delta x$ is the distance of the one extremity from the other, the range, as it may be called, of that element. But if the centre of gravity is taken as the origin, the series of values which the element assumes is

$$\ldots -(2-\alpha)\Delta x, \quad -(1-\alpha)\Delta x, \quad \alpha\Delta x, \quad (1+\alpha)\Delta x, \quad (2+\alpha)\Delta x \ldots,$$

where $\alpha\Delta x$, $(1-\alpha)\Delta x$ are the distances* of the centre of gravity from the nearest values assumed by the element. The frequency with which the different values are assumed by the qth element is represented by a frequency-locus: $\eta_q = \phi_q(\xi)$ signifying that the proportionate frequency with which any value of the element, e.g. ξ_q', is assumed in the long run is equal to $\Delta x \phi_q(\xi')$. Accordingly $[S\phi_q(\xi)\Delta\xi]$†, if the summation extend from one extremity of the element to the other, $=1$. The mean tth power of deviation from the centre of gravity may be expressed as $[S\xi^t\phi_q(\xi)\Delta\xi]$, if the centre of gravity is taken as the origin and the square brackets denote that the summation extends from one extremity of the element to another. It is postulated that the last written expression is approximately equal to $[S\xi^t\phi_p(\xi)\Delta\xi]$, the mean tth power of deviation for any other the pth element.

Such being the data, the *quaesitum* is an expression in terms of x, say y, which may be put into the form $y_0 + (y_1 - y_0) + (y_2 - y_1) +$ etc., such that $y_0\Delta x$ forms a first approximation to the proportionate frequency with which any particular value of x is assumed by the compound magnitude, $y_1\Delta x$ forms a second approximation, and so on; the first correction $y_1 - y_0$, the second $y_2 - y_1$ and so on, forming a series descending in respect of magnitude.

This problem will now be attacked *first* by a new method, *secondly* by an extension of the method which Professor Morgan Crofton originated, *thirdly* by an extension of the method which Laplace originated, *fourthly* by a method which combines some of the data utilised by Professor Crofton with a fresh datum to be handled by Laplace's analysis. *Fifthly* the conclusions thus obtained will be tested by being applied to certain simple cases for which the solution is known or discoverable by simple algebra, independently of the higher methods.

SECTION I.

The proposed new Method.

The following investigation starts from a special definition of the *quaesitum*, an approximating frequency-locus. It is here assumed that the sought representative locus will be approximate to the given locus when every mean power of deviation from the centre of gravity for the one locus is approximately equal to the corresponding mean power for the other locus, up to some power, say the tth, where t, though a considerable number, is yet small with respect to m, the number of the elements.

This criterion of approximation may be thus defended: whereas correspondence between the actual and representative locus is to be expected rather on the whole than in each detail, the mean powers of deviation form a proper feature of comparison, being an outcome of all

* α is in general a proper fraction; zero is a particular case.

† S is used here rather than Σ; the latter symbol being appropriated to the summation of contributions made by the different elements.

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the observations*. Accordingly it has been found, both by Professor Karl Pearson†, and by Mr Bowley‡, that when a representative curve has several—e.g. four or five—mean powers identical with the actual one, it also satisfies the more familiar test of approximation, namely that the "misfit"—the sum of the differences between the corresponding ordinates of the actual and the representative curve—should be small.

A curve, or series of curves, satisfying this criterion in the typical case will be found if any power (up to the tth) of the compound quantity can be broken up into a series of groups descending in order of magnitude, say G_0, G_{-1}, $G_{-2}\ldots$, each of which e.g. G_{-p} is equateable to a definite integral of the form $\int_{-\infty}^{\infty} x^t (y_p - y_{p-1})\, dx$; where y_0, $y, \ldots y_{p-1}$ are functions of x, constituting first, second, ... pth approximations to the actual frequency-locus of the compound quantity.

The required separation of the mean powers may be effected by the method of Generating Functions. The generating function for the powers of a particular observation, each divided by the corresponding factorial, is $e^{\theta_p x}$,

$$\equiv 1 + \theta_p x + \tfrac{1}{2}\theta^2{}_p x^2 + \ldots \frac{1}{r!}\theta^r{}_p x^r + \ldots.$$

But
$$e^{\theta_p x} = e^{\theta\,(p\xi_1 + p\xi_2 + \ldots + p\xi_m)} = e^{\theta_p \xi_1} \times e^{\theta_p \xi_2} \ldots e^{\theta_p \xi_m}$$
$$= (1 + \theta_p \xi_1 + \tfrac{1}{2}\theta^2{}_p \xi_1^2 + \ldots)(1 + \theta_p \xi_2 + \ldots) \ldots (1 + \theta_p \xi_m + \ldots).$$

Identifying the two expressions with which $e^{px\theta}$ is identical, and obtaining similar identities for another and another up to an indefinitely large number of observations, we find that the *mean* value of the first-written development of $e^{\theta x}$ is identical with the mean value of the second. Let the mean value of x^q be denoted by $x^{(q)}$, with similar notation for the mean values of the powers of the elements. We have then

$$1 + \theta x^{(1)} + \frac{1}{2!}\theta^2 x^{(2)} + \ldots \equiv \text{Mean of } \left(1 + \theta\xi_1 + \frac{1}{2!}\theta^2 \xi_1^2 + \ldots\right)\left(1 + \theta\xi_2 + \frac{1}{2!}\theta^2 \xi_2^2 + \ldots\right) \ldots.$$

The right-hand side of this identity may be transformed by the fundamental principle that the mean of the product of two independent statistical quantities—the pairs being supposed to be repeated with indefinitely great frequency under unaltered conditions—is equal to the product of their respective means§. Therefore

$$1 + \theta x^{(1)} + \frac{1}{2!}\theta^2 x^{(2)} + \frac{1}{3!}\theta^3 x^{(3)} \ldots$$
$$\equiv \text{Mean of } \left(1 + \theta\xi_1^1 + \frac{1}{2!}\theta^2 \xi_1^2 + \ldots\right) \times \text{Mean of} \left(1 + \theta\xi_2^1 + \frac{1}{2!}\theta^2 \xi_2^2 + \ldots\right)$$
$$\equiv \left(1 + \theta\xi_1^{(1)} + \frac{1}{2!}\theta^2 \xi_1^{(2)} + \ldots\right)\left(1 + \theta\xi_2^{(1)} + \frac{1}{2!}\theta^2 \xi_2^{(2)} \ldots\right) \ldots$$
$$\equiv e^{\log(1 + \theta\xi_1^{(1)} + \frac{1}{2!}\theta^2\xi_1^{(2)} + \ldots) + \log(1 + \theta\xi_2^{(1)} + \frac{1}{2!}\theta^2\xi_2^{(2)} + \ldots) + \ldots}.$$

* Compare Prof. Karl Pearson: "Our equation then depends on *all* the observations" [when obtained from a moment, i.e. mean power of deviation], *Phil. Trans.* 1894, A. p. 92. See also *ibid.* p. 74.

† *Loc. cit.* and *Phil. Trans.* 1895, ly to k_1—of the

‡ *Journal of the Statistical Societ*

§ "Wie sich leicht erweise *Theorie der Beobachtungsfehler*

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The last written expression is therefore the generating function of the mean powers of the compound divided by the corresponding factorial, viz. $\frac{x^{(1)}}{1}, \frac{x^{(2)}}{2!} \dots \frac{x^{(t)}}{t!}$. If each element is measured from its centre of gravity as origin, $\xi_1^{(1)}, \xi_2^{(1)}, \dots$ all vanish. Any term in the index of the form $\log\left(1+\frac{1}{2!}\theta^2\xi_q^{(2)}+\frac{1}{3!}\theta^3\xi_q^{(3)}+\&c.\right)$, being expanded in ascending powers of θ, becomes

$$\theta^2\frac{1}{2!}\xi_q^{(2)}+\theta^3\frac{1}{3!}\xi_q^{(3)}+\theta^4\left(\frac{1}{4!}\xi_q^{(4)}-\frac{1}{2}\frac{1}{2^2}(\xi_q^{(2)})^2\right)+\dots.$$

Call the coefficients of $\theta^2, \theta^3, \theta^4 \dots \frac{1}{2!}\kappa_0, \frac{1}{3!}\kappa_1, \frac{1}{4!}\kappa_2 \dots$ respectively. These coefficients are not in general the same for all the elements; but by the postulate numbered 10, they are not very different. Any κ subscribed r belonging to any element may be regarded as of the order $1/(\sqrt{m})^{r+2}$ relatively to a magnitude which it is convenient to take as unity*, viz.

$$x^{(2)}\,(=\xi_1^{(2)}+\xi_2^{(2)}+\dots+\xi_m^{(2)}).$$

For as by postulate no. 7 the range of each element is finite the successive mean powers of any the qth element may be equated to 0, $b_q\gamma_q^2$, $b_q'\gamma_q^3$, $b_q''\gamma_q^4 \dots$, where γ_q is a finite parameter, and b_q, $b_q' \dots$ are finite numerical coefficients. The coefficient b_q for each element may without loss of generality be put equal to unity. Then by postulate no. 10 the γs for all the elements are nearly equal; also the coefficients b pertaining to the same power for different elements, e.g. b_q'', b_r'', &c. are nearly equal to each other. Therefore, if γ, the average value of the γs, be of the order $1/\sqrt{m}$ corresponding to the convention that $m\gamma^2$ is of the order unity, the third, fourth, and mean powers for any element, e.g. $\xi_q^{(3)}, \xi_q^{(4)}, \xi_q^{(5)}$, are respectively of the orders

$$\frac{1}{m^{\frac{1}{2}}}, \frac{1}{m^2}, \frac{1}{m^{\frac{5}{2}}} \dots.$$

Therefore, as any κ coefficient, e.g. κ_r for any e.g. the qth element, is a homogeneous function of the mean powers $\xi_q^{(2)}, \xi_q^{(3)}, \dots$ the order of κ_r for any element is $1/(\sqrt{m})^{r+2}$. Put $\Sigma\kappa_r = k_r$, the summation extending over all the elements. Then k_r is of the order $1/(\sqrt{m})^r$. The generating function for the mean powers of x may now be written

$$G = e^{+\frac{1}{2!}k_0\theta^2+\frac{1}{3!}k_1\theta^3+\frac{1}{4!}k_2\theta^4+\dots+\frac{1}{t+2!}k_t\theta^{t+2}},$$

where the coefficient of any power of θ, e.g. θ^t, in the expansion of G is identical with $\frac{1}{t!}x^{(t)}$.

As k_r and every product of ks that with the sum of its subscripts $= r$ is of the order $1/(\sqrt{m})^r$, [illegible]ry mean power is hereby broken up into groups forming a descending series; except indeed up t[illegible]st and second powers each consisting of a single group, k_0 and k_1 respectively. For example to m, t[illegible]

This [illegible] $x^{(4)} = \frac{4!}{2!\,2^2}k_0^2, \quad +k_2;$

the actual a[illegible]

the mean pow[illegible] $x^{(6)} = \frac{6!}{3!\,2^3}k_0^3, \quad +\frac{6!}{4!\,2!}k_0k_2+\frac{6!}{3!\,3!}\frac{k_1^2}{2!}, +k_4;$

$\vdots \qquad \vdots \qquad \vdots$

* α is in general a [illegible] case.

† S is used here rather [illegible] the orders of magnitude with which we have to deal see the Appendix.

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$$x^{(2p)} = \frac{2p\,!}{p\,!\,2^p} k_0^p, \; + \frac{2p\,!}{4\,!\,p-2\,!\,2^{p-2}} k_0^{p-2} k_2$$

$$+ \frac{2p\,!}{3\,!\,3\,!\,p-3\,!\,2^{p-3}} k_0^{p-3} \frac{k_1^2}{2\,!} + \frac{2p\,!}{6\,!\,p-2\,!\,2^{p-2}} k_0^{p-2} k_4 + \ldots, + \ldots.$$

Also $$x^{(5)} = \frac{5\,!}{3\,!\,2} k_0 k_1, \; + k_3;$$

$$x^{(7)} = \frac{7\,!}{3\,!\,2\,!\,2^2} k_0^2 k_1, \; + \frac{7\,!}{5\,!\,2!} k_0 k_3 + \frac{7\,!}{3\,!\,4!} k_1 k_2 + \ldots, + \ldots$$

$$\vdots \qquad\qquad \vdots$$

$$x^{(2p+1)} = \frac{2p+1\,!}{3\,!\,p-1\,!\,2^{p-1}} k_0^{p-1} k_1, \; + \frac{2p+1\,!}{5\,!\,p-2\,!\,2^{p-2}} k_0^{p-2} k_3 + \ldots.$$

From the point which has been reached it is an easy step to the sought representation of the groups by means of definite integrals. Observing that $\frac{2p\,!}{p\,!\,2^p} k^p$ is the integral between limits $\pm\infty$ of the expression $\frac{1}{\sqrt{\pi 2k}} e^{-\frac{x^2}{2k}}$ we obtain this function for y_0 as the first approximation to the law of error, since the difference between $\int_{-\infty}^{+\infty} x^t y_0 dx$ and the mean tth power of the actual locus is small. The difference is small, in the case of an even power, relatively to the value of the power whether the actual value or that given by the approximation is taken, being of the order $\frac{1}{m}$ with respect to either. In the case of an odd power there is some difficulty in applying this test, as the odd mean powers are zero for the representative locus. The difficulty would not exist if the test of approximation were, that the ratio of any mean power, say the tth, to a certain even mean power, say the $2r$th (preferably the *second*), raised to the power $t/2r$, should be approximately the same for the actual and the representative locus.

The difficulty does not affect the further approximations. The second approximation y_1 ought to be such that $\int_{-\infty}^{\infty} x^{2t+1}(y_1)\,dx$ generally should be equal to $x^{(2t+1)}$ to within a small fraction—$\frac{1}{m}$th as it proves to be—of the true value of that power. The requisite correction is afforded by putting for y_1, $y_0 - k_1 \frac{1}{3\,!} \frac{d^3 y_0}{dx^3}$. For when p is 1, $x^{(2p+1)}$, that is $x^{(3)}$, is exactly equal to $\int_{-\infty}^{\infty} x^3 (y_1)\,dx$. And when p is greater than 1,

$$\int_{-\infty}^{\infty} x^{2p+1} \left(y_0 - k_1 \frac{1}{3\,!} \frac{d^3 y_0}{dx^3}\right) dx = (2p+1)(2p)(2p-1) \frac{k_1}{3\,!}$$

$$\times \int_{-\infty}^{\infty} x^{2p-2} y_0 dx = \frac{2p+1\,!}{2p-2\,!\,3\,!} k_1 \frac{2p-2\,!}{p-1\,!\,2^{p-1}} k_0^{p-1},$$

which is exactly the value for $x^{(2p+1)}$ when quantities of the order of $\frac{1}{m}$ relatively to k_1—of the order $\frac{1}{m^{\frac{3}{2}}}$ relatively to k_0—are neglected.

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By parity of reasoning the second correction, forming the third approximation, is found to be

$$k_2 \frac{1}{4!} \frac{d^4 y_0}{dx^4} + \frac{k_1^2}{2!} \frac{1}{3!\,3!} \frac{d^6 y_0}{dx^6}.$$

A general expression for the series of corrections may be found by a slight change of the expression for the generating function for mean powers. In that expression omit from the index the first term $\frac{1}{2}k_0\theta^2$; substitute for every other θ the symbol $-\frac{d}{dx}$; and place after the operator thus constituted, as operand, the function $\frac{1}{\sqrt{\pi 2k}} e^{-\frac{x^2}{2k}}$. We have then for y_t, the tth approximation,

$$e^{-k\frac{1}{2!}\left(\frac{d}{dx}\right)^2 + k_2 \frac{1}{4!}\left(\frac{d}{dx}\right)^4 - \ldots + (-1)^t \frac{1}{t+2!}\left(\frac{d}{dx}\right)^{t+2}} \frac{1}{\sqrt{\pi 2k}} e^{-\frac{x^2}{2k}}.$$

For the mean tth power as determined from this curve, viz. $\int_{-\infty}^{+\infty} x^t y_t dx$, is equal, to within quantities of the tth order, to the mean tth power as determined by the generating function. For consider any term in the expansion of $x^{(t)}$ as determined in the latter way. It is, if t is even, of the form

$$t!\left(\frac{1}{\alpha+2!}\right)^{\mathrm{a}}\left(\frac{1}{\beta+2!}\right)^{\mathrm{b}} \cdots \frac{k_0^r}{2^r r!} \frac{k_\alpha^{\mathrm{a}}}{\mathrm{a}!} \frac{k_\beta^{\mathrm{b}}}{\mathrm{b}!} \ldots, \text{ (where } 2r + \mathrm{a}(\alpha+2) + \mathrm{b}(\beta+2) + \ldots = t\text{); say } t!\, Q \frac{k_0^r}{2^r r!}.$$

An identical term is given by the expansion of the operator, namely,

$$\int_{-\infty}^{\infty} x^t Q \frac{d^{t-2r}}{dx^{t-2r}} y_0 dx, \text{ (where } \mathrm{a}(\alpha+2) + \mathrm{b}(\beta+2) + \ldots = t - 2r)$$

$$= \frac{t!}{2r!} Q \int_{-\infty}^{\infty} x^{2r} y_0 dx = t!\, Q \frac{k_0^r}{2^r r!} \ldots.$$

In like manner if t is odd the terms of the two expressions for y_t may be identified.

It will be observed that any coefficient of the type k_t may be regarded as the difference between the mean tth power as it actually is, $x^{(t)}$, and what it would have been if the actual frequency-locus of the compound coincided with the tth approximation y_{t-1}. The coefficient k_t may also be regarded as the sum of coefficients of the type κ_t, pertaining to the different elements; where κ_t for any the qth element is the difference between the mean tth powers for that element as it actually is and what it would have been if the actual frequency-locus for the element coincided with the law of error η_{t-1}, involving the coefficients $\kappa_0, \kappa_1 \ldots \kappa_{t-1}$ in the same way as y_{t-1}, the tth approximation to the frequency-locus for the compound quantity, involves $k_0, k_1, \ldots k_{t-1}$.

In considering the order of the successive groups formed by the expansion of the generating function, or the equivalent operator, account should be taken of the number of terms which go to each group and of the numerical coefficient $1/t!$ with which the coefficient is affected. Let ρ_t be the proportionate increase of the tth group compared with the $(t-2)$th group due to these causes—any homogeneous product of the tth degree of the ks being treated as equal to any others of that degree, and equal to $\frac{1}{m}$th of any homogeneous product of the

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$(t-2)$th degree (except k_0 and k_1*). Then m must be sufficiently large that ρ_t/m should be small. For any given value of m *ceteris paribus* the approximation becomes slower as with the increase of t ρ_t approaches m†.

With these explanations we may write the general law of error

$$y_0, -k_1 \frac{1}{3!}\frac{d^3y_0}{dx^3}, \; +k_2 \frac{1}{4!}\frac{d^4y_0}{dx^4} + \frac{k_1^2}{2!}\frac{1}{3!\,3!}\frac{d^6y_0}{dx^6}, \; -k_3 \frac{d^5y_0}{dx^5} \ldots;$$

where y_0 is put for $\frac{1}{\sqrt{\pi 2k_0}} e^{-\frac{x^2}{2k_0}}$, the successive groups being given by the coefficients of successive powers of ϕ in the expansion of

$$e^{-\phi k_1 \frac{1}{3!}\left(\frac{d}{dx}\right)^3 + \phi^2 k_2 \frac{1}{4!}\left(\frac{d}{dx}\right)^4 + \ldots + (-1)^t \phi^t k_t \frac{1}{t+2!}\left(\frac{d}{dx}\right)^{t+2}} y_0.$$

SECTION II.

The Method originated by Professor Morgan Crofton.

The fundamental propositions which are required in order to find a law of frequency which is approximative in the sense that the mean powers of the representative and the actual locus are nearly equal, are also required in order to find a law of frequency which is approximative in the more familiar sense that the ordinate of the representative locus at any point and that of the actual locus at the corresponding point should be nearly equal. Of the methods which pursue the latter *quaesitum* Professor Crofton especially requires the support of the propositions here regarded as fundamental. His proof of the law of error is based on a datum obtained by observing the effect which the introduction of a new element produced on the frequency-locus for the aggregate of elements. It seems to be assumed, very properly, that the sought function involves as constants some at least of the mean powers of the aggregate‡.

We may pass rapidly over Professor Crofton's first step which is to obtain a partial differential equation of the first order by which he deduces that, when a new element is taken in, the mean first power of the aggregate is increased by the mean first power of that element. This inference may be obtained immediately from the fundamental theorem

$$x^{(1)} = \xi_1^{(1)} + \xi_2^{(1)} + \ldots + \xi_m^{(1)}\S.$$

This being observed we may without loss of generality refer each of the elements (and accordingly the aggregate) to its respective centre of gravity. Then if $y, = f(x)$, is the ordinate of the frequency-locus for the aggregate before taking in a new element, and $y + \partial y$ the ordinate after that operation, by a well-known principle‖, $y + \partial y = [S\phi_m(\xi) f(x-\xi)\,\Delta\xi]$

* k_1 is of the order $k_0 m^{-\frac{1}{2}}$.

† On the value of ρ see the Appendix.

‡ It is *a priori* evident that the actual locus can be exactly expressed by a function involving as constants as many mean powers as there are values—multiples of Δx—assumed by the compound quantity in the long run.

§ A simple case of the multinomial law for a mean power, as to which see the Appendix.

‖ Morgan Crofton, *loc. cit.* p. 781, col. *a*. The principle had been used by the present writer in his first publication on these subjects in the *Philosophical Magazine*, 1883, Vol. XVI. p. 301.

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where $\eta = \phi_m(\xi)$ is the frequency-locus for the new element, and the square brackets indicate that the summation is to extend over the whole range of values assumed by that element. Expanding in ascending powers of (each value of) ξ, and neglecting powers above the second, as is found to be legitimate under the conditions specified*, we have (since the first mean power of the element vanishes)

$$\partial y = \tfrac{1}{2}\,[S\xi^2\phi_m(\xi)\,\Delta\xi]\,\frac{d^2f}{dx^2}.$$

From the fundamental proposition that

$$x^{(2)} = \xi_1^{(2)} + \xi_2^{(2)} + \ldots + \xi_m^{(2)}\dagger,$$

it follows that $[S\xi^2\phi_m(\xi)\,\Delta\xi]$ the mean second power of deviation for the mth element is equal to ∂k‡ the addition to k the mean second power of deviation for the aggregate. There is thus obtained a partial differential equation of the second order

$$\frac{dy}{dk} = \tfrac{1}{2}\,\frac{d^2y}{dx^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In order to obtain a subsidiary condition wherewith to supplement this leading equation a further recourse to the theory here proposed as fundamental may be made. A subsidiary equation is (in effect) obtained by Professor Crofton from the property that if the unit according to which the axis of x is graduated is altered in any assigned ratio, there must be a corresponding alteration both of the ordinate expressing the frequency of the aggregate and of the mean square of deviation for the aggregation. By supposing the alteration indefinitely small he obtains a second partial differential equation, viz. (in our notation)

$$y + x\,\frac{dy}{dx} + 2k\,\frac{dy}{dk} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now the form of y which is given by the general solution of this equation, viz.

$$y = \frac{1}{\sqrt{k}}\,\psi\left(\frac{x}{\sqrt{k}}\right),$$

where ψ is an arbitrary function, may at once be presumed from the relations of the ordinate and the mean-square of deviation to the abscissa which have just been adverted to; provided that we assume—as Professor Crofton seems to have assumed—that the sought function involves only one of the mean powers as a constant, namely the second. Otherwise why should we not have in the last written general expression, instead of $\sqrt{k}$, as the denominator of x (and Δx) the *fourth* root of the mean *fourth* power; with a corresponding modification of Professor Crofton's subsidiary partial differential equation? The justification for omitting $x^{(4)}$ (and higher mean powers) is to be sought in the fundamental principle that the mean fourth (and every higher even mean power) differs from a certain function of the mean second power by a quantity which may be neglected in a first approximation§; while the mean odd powers differ from zero by a quantity which may be likewise neglected‖.

* See the appended note on the orders of magnitude involved.

† A simple case of the multinomial law for a mean power, as to which see the Appendix.

‡ Identical with the k_0 of Section I. Where no mistake can arise k has been sometimes used instead of k_0.

§ See above, Section I.

‖ Above, p. 42.

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The subsidiary condition having been established in either of the forms above written, it may be combined with the leading equation either by treating the partial differential equations as simultaneous after the procedure adopted by Professor Crofton*, or by restricting the general solution of equation (1) in the manner proposed by the present writer†. It may be remarked that Professor Crofton's procedure seems to require yet another recourse to the fundamental theory. By combining equation (1) with (2) there is obtained a first integral $yx + \frac{dy}{dx} = C$, from which the normal law of error is obtained, if C is zero. C is inferred to be zero because, when x vanishes, $\frac{dy}{dx}$ also vanishes. But can that premiss be obtained otherwise than by the fundamental theory that the odd powers (above the first, which is zero by the construction here employed) are negligible in a first approximation, and accordingly that y may be regarded as involving only even powers?

The first and main part of the structure having been placed on a firm foundation, let us proceed to repair and add to that superstructure which is formed by the second and subsequent approximations. By continuing the expansion for the ordinate of the sought locus when varied by taking in a new element, as above represented, we obtain the additional term

$$-\frac{1}{3!}[S\xi^3\phi_m(\xi)\,\Delta\xi]\frac{d^3y}{dx^3}.$$

By the fundamental theory the quantity represented by the bracketed sum is ∂k_1 the addition made to k_1 the mean third power of deviation for the aggregate. As the increment ∂k_1 $(=\kappa_1)$ is evidently independent of the increment ∂k $(=\kappa_0)$ we obtain for the first approximation y_1 the partial differential equation

$$\frac{dy_1}{dk_1} = -\frac{1}{3!}\frac{d^3y_1}{dx^3} \qquad \text{(3)}.$$

The employment of this equation is restricted by the condition that k_1 is small relatively to y_0 and k_0. Accordingly y may presumably be expanded in ascending powers of k_1 as thus:

$$y = y_0 + k_1\left[\frac{dy_1}{dk_1}\right]_{k_1=0} + \ldots$$

Employing equation (3), and neglecting quantities below the order $1/\sqrt{m}$, we have for that function of x of which k_1 is the coefficient in the expansion of y, $-\frac{1}{3!}\frac{d^3y_0}{dx^3}$. The first correction $y_1 - y_0$, is thus $-k_1\frac{1}{3!}\frac{d^3y_0}{dx^3}$‡.

When we go on to the third approximation, there occurs the difficulty that the fourth mean power of the superadded element cannot, like the third, be treated as an independent variable. In assigning $\partial k = \kappa_0 = \xi^{(2)}$, the increment of k, we have also assigned a part of $\xi_m^{(4)}$. What may be regarded as independent is the difference between the fourth power as it actually is and what it would be if the element were perfectly "normal," viz. $\xi^{(4)} - 3\kappa_0^2$. By the fundamental theory

* *loc. cit.*

† *Phil. Mag.* 1896, Vol. XLI. p. 96, top.

‡ Cp. *Phil. Mag.* 1896, Vol. XLI. p. 93 *et seq.* (by the present writer).

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this quantity, κ_2, is the increment contributed by the new element to k_2 $(= x^{(4)} - 3k^2)$. To connect this increment with the increment of y that depends on it, there must be utilised a fresh datum, namely that the aggregate formed by taking on a new element is itself obedient to the law of error; and accordingly satisfies equation (1). Whence

$$\frac{d}{dk}(y + \delta y) = \frac{1}{2}\frac{d^2}{dx^2}(y + \delta y),$$

where δy is the addition made to the ordinate by taking on a new element. Therefore

$$\frac{d}{dk}\partial k \frac{dy}{dk} = \frac{1}{2}\frac{d^2}{dx^2}\partial k \frac{dy}{dk} = \frac{1}{2}\frac{d^2}{dx^2}\partial k \frac{1}{2}\frac{d^2y}{dx^2}; \quad \left(\frac{d}{dk}\right)^2 \equiv \left(\frac{1}{2}\frac{d^2}{dx^2}\right)^2.$$

Now continuing to expand the variation of y due to the introduction of a new element, we have

$$y, + \delta k \frac{dy}{dk}, + \delta k_1 \frac{dy}{dk_1}, + \frac{1}{2}\delta k^2 \frac{d^2y}{dk^2} + \delta y_2, + \ldots$$

$$= y, + \frac{1}{2}[S\xi^2\phi_m(\xi)\,\Delta\xi]\frac{d^2y}{dx^2}, - \frac{1}{3!}[S\xi^3\phi_m(\xi)\,\Delta\xi]\frac{d^3y}{dx^3}, + \frac{1}{4!}[S\xi^4\phi_m(\xi)\,\Delta\xi]\frac{d^4y}{dx^4};$$

where δy_2 is the variation of y dependent on the increment δk_2, and the other symbols have the meaning already assigned. By the condition just established

$$\frac{1}{2}\delta k^2 \frac{d^2y}{dk^2} = \frac{1}{2}\delta k^2 \frac{1}{4}\frac{d^4y}{dx^4}.$$

Bringing over the term in the latter form from the left to the right-hand side of the equation, omitting terms that have been proved to be equal on both sides, and substituting equivalent symbols in the expression for the remainder, we have

$$\delta y_2 = (\xi^{(4)} - 3(\delta k)^2)\frac{1}{4!}\frac{d^4y_2}{dx^4} = \delta k_2 \frac{1}{4!}\frac{d^4y_0}{dx^4} \text{ (approximately).}$$

Whence

$$\frac{dy_2}{dk_2} = \frac{1}{4!}\frac{d^4y_0}{dx^4};$$

subject to the condition that k_2 is small, of the order $\frac{1}{m}$. By that condition y may presumably be expanded in ascending powers of k_2; and thus, by parity of reasoning with that employed for the first correction, we obtain for $(y_2 - y_1)$

$$k_2 \frac{1}{4!}\frac{d^4y_0}{dx^4} + R;$$

where R is of the nature of a constant with respect to k_2—involving only k_1. What R is may be seen by continuing the expansion of y in ascending powers of k_1:

$$R = \frac{1}{2}k_1^2 \frac{d^2y}{dk_1^2} = \frac{k^2}{2}\frac{1}{3!}\frac{1}{3!}\frac{d^6y_0}{dx^6},$$

since by a parity of reasoning to that just now employed in the case of the condition

$$\frac{dy}{dk} = \frac{1}{2}\frac{d^2y}{dx^2},$$

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not only is
$$\left(\frac{dy}{dk_1}\right) = -\frac{1}{3\,!}\frac{d^3y_0}{dx^3},$$
but also
$$\left(\frac{d}{dk_1}\right)^3 y = \left(-\frac{1}{3\,!}\frac{d}{dx}\right)^3 y_0.$$

With the aid of the fundamental theory the method of partial differential equations can be extended to any number of approximations. The general, the tth, correction may be expressed as follows. Defining k as the mean square of deviation from the centre of gravity of the compound quantity whose law of frequency is to be represented, k_1 as the difference between the mean third power of deviation for that quantity as it actually is and as it would be if the first approximation y_0 were perfectly accurate, k_2 the difference between the mean fourth power as it actually is and as it would be if the second approximation y_1 were perfectly accurate, and so on, expand y_t by Taylor's theorem as if it were a function of the variables $k_1\theta$, $k_2\theta^2$, ..., $k_t\theta^t$, and write out the coefficient of θ^t in the expansion thus:
$$\left[\frac{dy_t}{dk_t}\right]_0 k_t + \left[\frac{d^2y_t}{dk_{t-1}dk_1}\right]_0 k_{t-1}k_1 + \ldots + \frac{1}{t\,!}\left[\frac{d^t}{dk_1{}^t}\right]_0 k_1{}^t;$$
where after the differentiations k_1, k_2, ... k_t are replaced by zero. Now substitute for every operator $\frac{d}{dk_p}$ operating on y_t, the operator $\frac{(-1)^p}{p+2\,!}\frac{d^{p+2}}{dx^{p+2}}$ operating on y_0; and for every power of the former operator the corresponding power of the latter. The resulting expression is the tth correction, the difference between the $(t+1)$th approximation, y_t, and the tth approximation, y_{t-1}. For assuming that the required expression y_t involves mean powers of deviation* as constants, we know by the fundamental theory that they must enter in the forms k, k_1, k_2, &c., with orders of magnitude proportioned to 1, $\frac{1}{\sqrt{m}}$, $\frac{1}{m}$, ..., $\frac{1}{m^{\frac{1}{2}t}}$. Therefore the above-written rule for the tth approximation will hold good, provided that

(I)
$$\frac{dy_q}{dk_q} = \frac{(-1)^q}{q+2\,!}\frac{d^{q+2}}{dx^{q+2}}y_0,$$

and (II)
$$\left(\frac{d}{dk_p}\right)^s\left(\frac{d}{dk_q}\right)^r \ldots y_q = \left(\frac{(-1)^p}{p+2\,!}\frac{d}{dx}\right)^s\left(\frac{(-1)^q}{q+2\,!}\frac{d}{dx}\right)^r \ldots y_0,$$

where p, q, ... are integers up to t. Now proposition II. follows from proposition I. by parity of reasoning with that above employed† with reference to equations (1) and (3). And it may be proved that if proposition I. holds good for k_{t-2} and all coefficients with lower subscripts, it holds good also for k_t.

For continuing the expansion of the ordinate‡ when a new element is taken in up to the tth order of magnitude we have for that order on the right-hand side of the equation
$$(-1)^{t+2}\frac{1}{t+2\,!}\left[S\phi_m(\xi)\xi^{t+2}\Delta\xi\right]\frac{d^{t+2}y}{dx^{t+2}} = (-1)^{t+2}\frac{1}{t+2\,!}\,\xi^{(t+2)}\frac{d^{t+2}}{dx^{t+2}}y.$$

* Cf. note to p. 45 above. † Above, pp. 48, 49. ‡ Above, p. 46.

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On the left side of the equation we have

$$\frac{dy}{dk}\delta k, + \frac{dy}{dk_1}\delta k_1, + \tfrac{1}{2}\frac{d^2y}{dk^2}\delta k^2, \ldots,$$

up to terms of the tth order of magnitude. The terms preceding the $(t+2)$nd group* are cancelled by equal terms on the right hand of the equation. The terms following the $(t+2)$nd group may be neglected (with reference to the tth correction). There remain on the left hand to be set against the term on the right which has just been written a group of terms of the type $\frac{\kappa_0^r}{r!}\frac{\kappa_\alpha^{\mathrm{a}}}{\mathrm{a}!}\frac{\kappa_\beta^{\mathrm{b}}}{\mathrm{b}!}\frac{\ldots d^{t+2}}{d\kappa_0^r d\kappa_\alpha^{\mathrm{a}} d\kappa_\beta^{\mathrm{b}}}y$, where $2r + \mathrm{a}(\alpha+2) + \mathrm{b}(\beta+2)\ldots = t+2$†. Beside these terms, of which the contribution to the variation of the ordinate y by the $(t+2)$nd mean power of the new element is involved in the assumptions which have been already made with regard to earlier mean powers of the element, there is to be placed a contribution to y_t which is not thus dependent on preceding assumptions, ∂y_t as it may be called, or $\partial(y_t - y_{t-1})$, dependent on the independent portion of $\xi_m^{(t+2)}$.

Now proposition I, and therefore proposition II, are supposed to hold good up to and inclusive of the equation

$$\left(\frac{d}{dk_{t-2}}\right)^r y = \left(\frac{(-1)^t}{t!}\frac{d^t}{dx^t}\right)^r y_0.$$

Observing that no differential of the type $\frac{dy}{dk_p}$ occurs in the group on the left with subscript higher than $t-2$, we replace each symbol $\left(\frac{d}{dk_q}\right)^r$ by its equivalent $\left(\frac{(-1)^q}{q+2!}\frac{d^{q+2}}{dx^{q+2}}\right)^r$; and thus obtain on the left hand an expression of the form $\xi_m'^{(t+2)}\frac{1}{t+2!}\frac{d^{t+2}}{dx^{t+2}}y_0 + \partial y_t$, where $\xi_m'^{(t+2)}$ is the $(t+2)$nd power of the mth, the newly added element, on the supposition that its frequency-locus exactly coincides with the law of error η_{t-1}, a frequency-function of the same form as y_{t-1} but involving as constants the coefficients of the element $\kappa_0, \kappa_1, \ldots, \kappa_{t-1}$ which have been already utilised—the portion therefore of the $(t+2)$nd power of the mth element which can no longer be regarded as independently variable. What may be regarded as variable in the mean $(t+2)$nd power is $\xi_m^{(t+2)} - \xi_m'^{(t+2)}$, the difference between the power as it actually is and what it would be if the approximations hitherto made were exact; in other symbols κ_t, which by the fundamental theory may be regarded as δk_t. Thus bringing over from the left to the right the portion affected with $\xi_m'^{(t+2)}$ we obtain

$$\partial y_t = \frac{(-1)^t}{t+2!}\delta k_t \frac{d^{t+2}y_t}{dx^{t+2}},$$

$$\frac{dy_t}{dk_t} = \frac{(-1)^t}{t+2!}\frac{d^{t+2}y_0}{dx^{t+2}}, \text{ (approximately)};$$

* Or the tth group in the notation of p. 41 above.

† It will be remembered that κ_0 is of the order k_0/m, i.e. $\frac{1}{m}$; κ_p is of the order γ^{p+2} (above, p. 42) that is of the order $\left(\frac{1}{\sqrt{m}}\right)^{p+2}$, or one mth of the order of k_p.

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which was to be proved. Therefore if proposition I. holds good up to the $(t-2)$nd approximation it holds good also for the tth. And it has been shown to hold good for the first and second approximations. Therefore the proposition holds good in general. Therefore in general

$$(y_t - y_{t-1}) = \frac{1}{t+2\,!} k_t \frac{d^{t+2} y_0}{dx^{t+2}} + R;$$

where R consists of terms of the tth order of magnitude not involving k_t. What those terms are is found by expanding y in powers of $k_1\theta$, $k_2\theta^2$, ..., $k_{t-1}\theta^{t-1}$ and selecting the coefficient of θ^t, and substituting for every symbol of the type $\left(\frac{d}{dk_p}\right)_0^r$ the symbol $\left(\frac{(-1)^p}{p+2\,!}\frac{d^{p+2}}{dx^{p+2}}\right)$ operating on y_0. The series of terms thus formed are identical with that obtained in Section I.

A variant solution of the system of partial differential equations, in the form of the real or significant portion of an expression involving the symbol $\sqrt{2\frac{d}{dk}}$, will be given in the Appendix.

SECTION III.

The Method originated by Laplace.

The method originated by Laplace and developed by Poisson seems also to require for its extension, if not for its support, the propositions here regarded as fundamental. In a free version of this classical investigation let us begin with the case to which Laplace confined himself, the comparatively simple case in which all the elements have one and the same locus. Let that locus be $\eta = \phi(\xi)$, signifying that the probability of the element having a value ξ is $\phi(\xi)\Delta\xi$, where $[S\phi(\xi)\Delta\xi] = 1$, the square brackets denoting summation between extreme limits. Put $\chi(\alpha)$ for $[S\phi(\xi)e^{\sqrt{-1}\alpha\xi}\Delta\xi]$, where, as all along, ξ is an integer multiple of $\Delta\xi$, (or $\Delta x) = \rho\Delta x$, say. Form the mth power of $\chi(\alpha)$. The coefficient of $e^{\sqrt{-1}\alpha r\Delta x}$ in $(\chi(\alpha))^m$ is the probability that the sum of the values of the elements should be equal to $r\Delta x$; a probability which is equal to $y\Delta x$ where y is the ordinate of the locus representing the frequency of the compound quantity. Now multiply $\chi^m(\alpha)$ by $e^{-\sqrt{-1}t\Delta x\alpha}$. The product will consist of a sum of terms all of the form $e^{(r-t)\Delta x\alpha\sqrt{-1}}y_{r\Delta\xi}$ (where r assumes every possible value); except one which is free from the transcendental expression, viz. $y_{t\Delta x}\Delta x$. Each of the other terms may be put in the form

$$\Delta x\,(\cos(r-t)\Delta x\alpha + \sqrt{-1}\sin(r-t)\Delta x\alpha)\,y_{r\Delta x}.$$

Put β for $\Delta x\alpha$ and integrate the expression thus transformed, with respect to β between limits $\beta = 0$ and $\beta = \pi$. All the *real* terms of the integral vanish except $\pi y_{t\Delta x}\Delta x$. An imaginary part in general subsists. Thus we obtain as equal to $y_{t\Delta x}\Delta x$ the expression

$$\Re^* \Delta x \frac{1}{\pi}\int_0^\pi \chi^m(\beta/\Delta x)\,e^{-\sqrt{-1}t\beta}d\beta.$$

* A symbol used to denote the useful conception of the *real part* of an expression (a conception used with effect by Czuber in his *Beobachtungsfehler*).

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Substituting x for $t\Delta x$, omitting Δx from both sides of the equation, and changing the independent variable from β to α, we have

$$y_x = \Re \frac{1}{\pi}\int_0^{\pi/\Delta x} \chi^m(\alpha)\, e^{-\sqrt{-1}\alpha x} d\alpha.$$

The path from this point is easiest when* we take the frequency-function of the element to be symmetrical. The expression which has been found for y may then be evaluated by expanding $\chi(\alpha)$ in ascending mean powers of the element

$$\chi(\alpha) = \left[S\left(1 - \frac{1}{2}\alpha^2\xi^2 + \frac{1}{4!}\alpha^4\xi^4 + \ldots + (\sqrt{-1})^{2p}\frac{1}{2p!}\alpha^{2p}\xi^{2p}\right)\Delta\xi\right];$$

where ξ assumes all possible values between its extremities. Whence

$$\chi(\alpha) = 1 - \frac{1}{2}\alpha^2\xi^{(2)} + \frac{1}{4!}\alpha^4\xi^{(4)} - \ldots.$$

To determine $\{\chi(\alpha)\}^m$ take the logarithm of $\chi(\alpha)$.

$$\operatorname{Log}\chi(\alpha) = 0 - \frac{1}{2!}\alpha^2\kappa_0 + \frac{1}{4!}\alpha^4\kappa_2 - \ldots + (\sqrt{-1})^{2p}\alpha^{2p}\frac{1}{2p!}\kappa_{2p-2};$$

where $\kappa_0, \kappa_2, \ldots$ have the signification attached to them in our first section. Accordingly

$$\log\{\chi(\alpha)\}^m = m\log\chi(\alpha) = -\frac{1}{2}\alpha^2 k_0 + \frac{1}{4!}k_2\alpha^4 - \ldots + (\sqrt{-1})^{2p}\alpha^{2p}\frac{1}{2p!}k_{2p-2};$$

where $k_0, k_2, \ldots$, having the signification before assigned to them, form a series descending in magnitude. Thus

$$y = \Re\frac{1}{\pi}\int_0^{\pi/\Delta x} e^{-\frac{1}{2}\alpha^2 k_0 + \frac{1}{4!}k_2\alpha^4 - \ldots}\, e^{-\sqrt{-1}\alpha x} d\alpha;$$

$$\Re\frac{1}{\pi}\int_0^{\pi/\Delta x} e^{-\frac{1}{2}\alpha^2 k_0 + \frac{1}{4!}k_2\alpha^4 - \ldots}(\cos\alpha x - \sqrt{-1}\sin\alpha x)\, d\alpha;$$

$$= \frac{1}{\pi}\int_0^{\pi/\Delta x} e^{-\frac{1}{2}\alpha^2 k_0^2 + \frac{1}{4!}k_2\alpha^4 - \ldots}\cos\alpha x\, d\alpha;$$

which may be expanded in powers of small quantities

$$\frac{1}{\pi}\int_0^{\pi/\Delta x} e^{-\frac{1}{2}\alpha^2 k_0}\left\{1 + \frac{1}{4!}k_2\alpha^4 - \frac{1}{6!}k_4\alpha^6 + \frac{1}{2}\left(\frac{k_2}{4!}\right)^2\alpha^8 + \ldots\right\}\cos\alpha x\, d\alpha;$$

the rather as each of the definite integrals thus presented is not only finite, but also is made up of finite elements. The first approximation is therefore $\frac{1}{\pi}\int_0^{\pi/\Delta x} e^{-\frac{1}{2}\alpha^2 k_0}\cos\alpha x\, d\alpha$; where Δx is at most of the order $1/\sqrt{m}$† and may be indefinitely small. The expression therefore differs by a very small quantity from

$$\int_0^\infty e^{-\frac{1}{2}\alpha^2 k_0}\cos\alpha x\, d\alpha = \frac{1}{\sqrt{\pi 2k}}e^{-\frac{x^2}{2k}},$$

the normal law of error.

* The facilitation proper to the simpler case is introduced by Laplace at an earlier stage.

† E.g. in the case of the Binomial the range of the element is Δx which with reference to magnitudes that are equated to unity is of the order $1/\sqrt{m}$. See below, Section V. and Appendix.

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A nearer approximation is made by taking in the term $\frac{1}{4!}k_2\int_0^{\pi/\Delta x}\alpha^4 e^{-\frac{1}{2}\alpha^2 k_0}\cos\alpha x\,dx$. Now this may be identified with

$$\frac{1}{4!}k_2\int_0^{\pi/\Delta x}\frac{d^4}{dx^4}e^{-\frac{1}{2}\alpha^2 k_0}\cos\alpha x\,dx=\frac{1}{4!}k_2\frac{d^4}{dx^4}y_0\,;$$

where y_0 is the above written expression for the function which forms the first approximation. In like manner any other term of the expansion of y in the series of definite integrals may be identified with a term in the series obtained by the method of Section I. The general term for the $2p$th correction by the present method is

$$(\sqrt{-1})^P\frac{k_\alpha{}^{\mathrm{a}}}{\mathrm{a}\,!}\frac{k_\beta{}^{\mathrm{b}}}{\mathrm{b}\,!}\cdots\frac{1}{(\alpha+2\,!)^{\mathrm{a}}}\frac{1}{(\beta+2\,!)^{\mathrm{b}}}\cdots\int_0^\infty\frac{d^P}{dx^P}e^{-\frac{1}{2}\alpha^2 k_0}\cos\alpha x\,d\alpha^*,$$

where $\qquad 2p=\mathrm{a}\alpha+\mathrm{b}\beta+\ldots,\qquad P=\mathrm{a}(\alpha+2)+\mathrm{b}(\beta+2)+\ldots;$

which is identical with a term obtained by the earlier method since

$$\frac{d^{2r}}{dx^{2r}}\cos\alpha x=(\sqrt{-1})^{2r}\alpha^{2r}\cos\alpha x.$$

When the odd powers of the element do not vanish $\chi(\alpha)$ is to be written

$$\left(1+\sqrt{-1}\,\alpha\xi^{(1)}-\frac{1}{2}\alpha^2\xi^{(2)}-\frac{1}{3!}\alpha^3\sqrt{-1}\,\xi^{(3)}+\ldots\right);$$

where $\xi^{(1)}$ vanishes, the elements being referred each to its centre of gravity as origin. Thus

$$\log\chi(\alpha)=-\frac{1}{2}\alpha^2\kappa_0-\frac{1}{3!}\alpha^3\sqrt{-1}\,\xi^{(3)}+\alpha^4\left(\frac{1}{4!}\xi^{(4)}-\frac{1}{2}\frac{1}{4}(\xi^{(2)})^2\right)+\sqrt{-1}\alpha^5\left(-\frac{1}{5!}\xi^{(5)}+\frac{1}{2}\xi^{(2)}\frac{1}{3!}\xi^{(3)}\right)$$

$$=-\frac{1}{2}\alpha^2\kappa_0-\frac{1}{3!}\alpha^3\sqrt{-1}\,\kappa_1+(\sqrt{-1})^4\alpha^4\frac{1}{4!}\kappa_2+(\sqrt{-1})^5\alpha^5\frac{1}{5!}\kappa_5+\ldots.$$

Which differs from the index in the generating function (or the corresponding operator) pertaining to an element, given in Section I. only in this, that for the $\theta\left(\text{or}-\frac{d}{dx}\right)$ there employed there is now substituted $\sqrt{-1}\,\alpha$. The expression for $m\log\chi(\alpha)=\log(\chi(\alpha))^m$ is similarly related to the generating function and operator pertaining to the compound. Thus

$$y=\Re\int e^{-\frac{1}{2}\alpha^2 k_0-\frac{1}{3!}\alpha^3\sqrt{-1}\,k_1+\ldots(\sqrt{-1})^t\alpha^t\frac{1}{t!}k_{t-2}}(\cos\alpha x-\sqrt{-1}\sin\alpha x)\,dx.$$

Expanding in powers of the small quantities k_1, k_2, etc., and selecting the real terms we obtain two sets of terms of the respective types

$$(\sqrt{-1})^{2P}\frac{1}{\pi}l\int_0^\infty\alpha^{2P}e^{-\frac{1}{2}k\alpha^2}\cos\alpha x\,d\alpha$$

and

$$-(\sqrt{-1})^{2P+2}\frac{1}{\pi}l'\int_0^\infty\alpha^{2P+1}e^{-\frac{1}{2}k\alpha^2}\sin\alpha x\,d\alpha,$$

where l and l' are products of ks (divided by factorials).

* ∞ being substituted for $\pi/\Delta\xi$ as before.

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Every term in the expression obtained before, by expansion of the form

$$e^{S\frac{k_t}{t+2!}\left(-\frac{d}{dx}\right)^{t+2}} y_0,$$

corresponds to a term in one or other of these two series, as will be found by observing that when t is even

$$\left(-\frac{d}{dx}\right)^t \cos ax = (\sqrt{-1})^t \alpha^t \cos ax;$$

and when t is odd

$$\left(-\frac{d}{dx}\right)^t \cos ax = -(\sqrt{-1})^{t+1} \alpha^t \sin ax^*.$$

So far it has been assumed with Laplace that each element has the same frequency-function. But the reasoning is not affected when we relax this condition to the extent permissible in the typical case; substituting for $(\chi(\alpha))^m$ the product $\chi_1(\alpha)\chi_2(\alpha)\ldots\chi_m(\alpha)$, where each of the χs pertains to a different function; and making other slight changes.

The reason which legitimates the substitution of $e^{-\frac{1}{2}k\alpha^2}$ for $\chi_1(\alpha)$, $\chi_2(\alpha)\ldots\chi_m(\alpha)$ as part of the definite integral (with respect to α) which is equivalent to the required function of x has not in general been clearly stated by the mathematicians who have treated this subject. Laplace indeed (*Théorie Analytique des Probabilités*, Book IV. Art. 22, p. 336, ed. 1847), with reference to the case in which all the frequency-curves have the same locus, has expressed the true reason, namely that the *logarithm* of the above-written expression in α may be expanded in ascending powers of α of which the coefficients form a descending series of magnitudes. So too Professor Czuber, with reference to the case of symmetrical elements (*Theorie der Beobachtungsfehler*, p. 91). Yet later, referring to Ellis' hesitation about the Method of Least Squares, Professor Czuber argues (p. 271, Art. 117) that there is a "difficult point" not yet cleared up†.

SECTION IV.

A variant Method.

The preceding results may be confirmed by a method of proof based partly on conditions which have been utilised by Professor Crofton, and partly on a fresh condition which will be utilised by means of Laplace's analysis. The fresh condition is that the sought frequency-function must be *reproductive*, in the sense that if two or more independently fluctuating quantities $A, B, \ldots$ assume different values with a frequency designated by a member of the family represented by the sought function, then Q a quantity formed by adding together each pair (triplet, etc.) of concurrent values presented by $A, B, \ldots$ will also assume different values with a frequency designated by a member of the sought family. For as A is—or may be

* Cp. Czuber, *Theorie der Beobachtungsfehler*, p. 90, on the successive differentiations of the integral.

† See appended note on the 'Ατόπια of Ellis and Czuber.

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replaced by—an aggregate of a great number, say m_1, elements of the typical sort, and likewise B is equivalent to the aggregate of m_2 other elements, and so on; Q being the aggregate of $m_1 + m_2 +$ etc. elements of the typical sort must fulfil the law of error.

Let us at first limit the enquiry by supposing that $A, B, \ldots$ each range under the same member of the family of frequency-curves—or more generally loci—which have the property in question; and that this common member is symmetrical. Let the common frequency-locus be $y = f(x)$. It is required to find the form of f such that, if $A, B, \ldots$ etc. each range under a locus of that form, then Q the aggregate of $A, B, \ldots$ shall range under a locus of the form $y = \frac{1}{c} f\left(\frac{x}{c}\right)$.

By Laplace's analysis if

$$\chi(\alpha) = [Sf(x)\,\Delta x \cos \alpha x],$$

then as the ordinate of the frequency-locus for the aggregate we have

$$y = \frac{1}{\pi}\int_0^\infty (\chi(\alpha))^m \cos \alpha x\, d\alpha;$$

where m is the number of the components, not in general (in the present enquiry) a large number. Put $e^{\psi(\alpha)}$ for $\chi(\alpha)$. Then the condition of reproduction will be satisfied if

$$\frac{1}{\pi}\int_0^\infty e^{m\psi(\alpha)} \cos \alpha x\, d\alpha$$

can be put into the form

$$\frac{1}{c}\frac{1}{\pi}\int_0^\infty e^{\psi(\alpha)} \cos \alpha \frac{x}{c}\, d\alpha.$$

Put $\alpha/c = \beta$; and the last written integral becomes

$$\frac{1}{\pi}\int_0^\infty e^{\psi(\beta c)} \cos \beta x\, d\beta.$$

Whence it appears that the condition will be satisfied if $\psi(c\beta) \equiv m\psi(\beta)$; where c is a constant depending on m and $\psi(\beta)$ is a function not involving m.

Now the general solution* of the functional equation $\psi(c\alpha) \equiv m\psi(\alpha)$ is of the form

$$\alpha^t (\mathrm{a}_0 + \mathrm{a}_1 \cos 2\pi t \log \alpha/\log m + \mathrm{a}_2 \cos 4\pi t \log \alpha/\log m + \ldots$$
$$+ \mathrm{b}_1 \sin 2\pi t \log \alpha/\log m + \mathrm{b}_2 \sin 4\pi t \log \alpha/\log m + \ldots).$$

But in the case with which we have to do the coefficients $\mathrm{a}_1 \ldots \mathrm{b}_1 \ldots$ must all vanish as m does not enter into $\psi(\beta)$. The required function is therefore of the form

$$\frac{1}{\pi}\int_0^\infty e^{\mathrm{a}\alpha^t} \cos \alpha x\, d\alpha.$$

It is necessary that the function should be of this form in order that it should be *reproductive* when the components $A, B, \ldots$ have the same frequency curve; and it is evidently sufficient that the function should be of this form in order that it should be reproductive when the components being of different magnitude belong to different members of the reproductive family.

* The solution given by Boole, *Finite Differences*, Ch. XI. Art. 5; the symbols being altered.

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The general form which has been obtained may be restricted by a condition which has been utilised in Professor Crofton's method, namely that any member of the sought family must be approximately at least of the form $\frac{1}{q\sqrt{k}} f\left(\frac{x}{q\sqrt{k}}\right)$; where k is the mean square of deviation from the centre of gravity, and q is a constant such that $\int_{-\infty}^{\infty} \frac{x^2}{q\sqrt{k}} f\left(\frac{x}{q\sqrt{k}}\right) dx = k$. This form may be combined with the recently found form $\frac{1}{\pi}\int_0^\infty e^{a\alpha^t} \cos \alpha x d\alpha$ by putting $\alpha = \beta/q\sqrt{k}$, so that the latter expression becomes

$$\frac{1}{q\sqrt{k}} \frac{1}{\pi} \int_0^\infty e^{a\beta^t/(q\sqrt{k})^t} \cos \frac{\beta x}{q\sqrt{k}} d\beta.$$

When a fluctuating quantity having this frequency-locus is compounded with $m-1$ others of the same species, the frequency-locus for the compound is

$$\frac{1}{\sqrt{q^2k}} \frac{1}{\pi} \int_0^\infty e^{am\beta^t/(q\sqrt{k})^t} \cos \beta x/q\sqrt{k} d\beta;$$

or, putting $\beta = \gamma/m^{\frac{1}{t}}$,

$$\frac{1}{m^{1/t}\sqrt{q^2k}} \frac{1}{\pi} \int_0^\infty e^{a\gamma^t/(q\sqrt{k})^t} \cos \frac{\gamma}{m^{1/t}q\sqrt{k}} d\gamma.$$

But by the proposition just now adduced the composite function is of the form $\frac{1}{Q\sqrt{K}} \Psi\left(\frac{x}{Q\sqrt{K}}\right)$, where K is the mean square of deviation for the compound, and Q is a constant such that $\left[\Sigma \frac{x^2}{Q\sqrt{K}} \Psi \Delta x\right] = K$. As Ψ differs from ψ only in respect of constants, Q is evidently equal to q. Also by a fundamental proposition $K = mk$. Accordingly comparing the identical forms of the composite locus, we have $m^{1/t}\sqrt{k}$ identical with $\sqrt{mk}$, whence $t = 2$. The sought function is of the form

$$\frac{1}{\pi} \int_0^\infty e^{a\beta^2/q^2k} \cos \frac{\beta x}{\sqrt{q^2k}} \delta\beta.$$

In order to secure a finite form a must evidently be negative. And thus by familiar transformations we reach the normal law of error.

When we start without limiting the enquiry by the assumption that the sought form is symmetrical it is proper to put

$$\chi(\alpha) = \left[S \frac{1}{c} f\left(\frac{x}{c}\right) \Delta x e^{+\sqrt{-1}\alpha x}\right],$$

and to multiply $[\chi(\alpha)]^m$ by $e^{-\sqrt{-1}\alpha x}$. By parity of reasoning it may thus be found that the required function is of the form

$$\frac{1}{\pi} \int_0^\infty e^{(G+\sqrt{-1}H)\alpha^2} e^{-\sqrt{-1}\alpha x/q\sqrt{k}} d\alpha.$$

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This is no doubt a reproductive form, but it is one which being imaginary does not concern us, or only concerns us in respect of its real part. But that real part, which may be written $\frac{1}{\pi}\int_0^\infty e^{-g^2\alpha^2}\cos\left(H\alpha^2-\frac{\alpha x}{q\sqrt{k}}\right)d\alpha$ (as G is evidently negative), is not reproductive except indeed when H vanishes*; in which case we obtain a function of the same form as before (by putting $g^2=2k$). Without putting symmetry in the premisses we find it in the conclusion.

Some varieties of proof by which the datum peculiar to this section is combined with the principal partial differential equation of Section II. may be relegated to an appendix. It need only be added here that just as that leading differential equation is satisfied by other forms beside the normal, provided that the coefficients which enter into those forms do not exceed certain orders of magnitude, so the condition of reproductivity is fulfilled by those other forms under the same restrictions. In the expression found for a reproductive law of error, viz. $\frac{1}{\pi}\int_0^\infty e^{-\frac{1}{2}\alpha^2 k}\cos\alpha x d\alpha$, if we add to the index *any* term affected with α^4 or generally α^p, where p is other than 2, the character of reproductivity will in general be destroyed. But if the coefficient of α^4 is of the order -2 relatively to k, and generally the coefficient of α^p (p integral) is of the order $-(p-2)$, then we may expand in such wise that the resulting series of approximations are of nearly the same form for a single component (A or B ...) and for the aggregate of several such quantities (Q).

SECTION V.

Verifications.

Before extending to more complicated cases the theory which has been proved for the typical case it may be well to test the validity of the methods employed by showing that they lead to correct results in some simpler cases, which admit of being treated by more elementary methods. Such a test case is afforded by the Binomial locus, considered as a particular case of the typical circumstances, the case in which all the elements have the same frequency-locus, viz. two points separated by an interval Δx representing two values which are enjoyed by any element with respective frequency p and q. If the centre of gravity is taken as the origin, the abscissæ of these points are $-q\Delta x$ and $+p\Delta x$; the corresponding ordinates being $p/\Delta x$ and $q/\Delta x$. It is proposed to compare the approximate frequency with which the sum of m such elements will enjoy an assigned value of x, as determined on the one hand by the simpler analysis of which the case admits, and on the other hand by the more general methods investigated in the preceding sections.

Let us begin with the case in which mp is an integer, and accordingly x the sum of m elements measured from the centre of gravity is an integer multiple of Δx.

* Or at least is very small, in which case as explained in the next paragraph the curve will be in a sense reproductive.

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According to the simpler methods the approximate law of frequency is deduced from the general expression for a term in the expansion of $(p+q)^m$, viz.

$$\frac{m!}{mp-r!\ mq+r!}p^{mp-r}q^{mq+r} \quad (p+q=1),$$

where r is the excess of the number of elements that enjoy the upper of the two possible values, viz. $+p\Delta x$, of which the probability is q, the number over and above mq the most probable number (out of m trials). By a well-known approximation employed by Laplace and others this expression may be represented as the product of three factors $P \times S \times T$, of which the product of the first two $P \times S$ is formed by putting

$$m! = \sqrt{2\pi m}\, m^m e^{-m}\left[1+\frac{1}{12m}+\dots\right],$$

with like expressions for the factorials $mp-r!$, $mq+r!$, and multiplying the portions of these expressions which are outside the brackets on each other and on $p^{mp-r}q^{mq+r}$; while T is formed from the tail, so to speak, of Stirling's formula, by multiplying together the parts of the expressions for the three factorials, which are within the brackets. The expression PST represents the probability that at any particular trial the sum of elements will be exactly $r\Delta x$, the proportionate frequency with which in the long run that value will be enjoyed: that is in our notation $y_{r\Delta x}\Delta x$, or $y_x\Delta x$. P is a first approximation to this *quaesitum*. It is found to be

$$\frac{1}{\sqrt{\pi 2mpq}}e^{-\frac{r^2}{2mpq}},$$

where r, or more exactly rpq, is of (i.e. $\not>$) the order $\sqrt{m}$. Put x for $r\Delta x$ and c for $\sqrt{2mpq}\,\Delta x$, a quantity of the same order as those values of x up to which the approximation is available. We have then for $y_x\Delta x$ the expression $\frac{\Delta x}{\sqrt{\pi}c}e^{-\frac{x^2}{c^2}}$, the normal law of error; in which the constant c corresponds to $\sqrt{2} \times$ the constant k (or k_0), employed in former sections.

When the same substitutions are made in S and T, it is found that they respectively assume the forms $e^{A_1+A_2+\dots}$ and $(1+B_1+B_2\dots)$, where A_1, $A_2\dots$, and likewise B_1, $B_2\dots$, form a series descending in order of magnitudes, for values of x not exceeding the order of c. Accordingly the groups forming the successive corrections of the first approximation are

$$P(A_1+B_1),\quad P(\tfrac{1}{2}A_1^2+A_2+A_1B_1+B_2),$$
$$P\left(\frac{1}{3!}A_1^3+A_1A_2+A_3+A_2B_1+A_1B_2+B_3+B_1B_2+\frac{1}{3}B_1^3\right)\dots,$$

where the As and Bs have values found by Laplace and his successors, or to be found by continuing their work. Thus it has been found that

$$A_1=\frac{q-p}{2mpq}\left(r-\frac{2}{3}\frac{r^3}{2mpq}\right),$$

which becomes, when x is substituted for $r\Delta x$ and c for $\sqrt{2mpq}\,\Delta x$,

$$\frac{q-p}{\sqrt{2mpq}}\left\{\frac{x}{c}-\frac{2}{3}\left(\frac{x}{c}\right)^3\right\};$$

an expression which may be simplified by putting x for x/c.

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Again, A_2 has been found to be (in this notation)

$$\frac{1-2pq}{2mpq}\mathrm{x}^3 - \frac{1}{3}\frac{1-3pq}{mpq}\mathrm{x}^4.$$

Also
$$B_1 = 0,$$
$$B_2 = -(1-pq)/12mpq.$$

We have thus for the first correction

$$P\frac{q-p}{\sqrt{2mpq}}(\mathrm{x} - \tfrac{2}{3}\mathrm{x}^3).$$

The second correction is found as follows:

$$\frac{1}{2}A_1^2 = \frac{1-4pq}{4mpq}\mathrm{x}^2 - \frac{1-4pq}{mpq}\frac{1}{3}\mathrm{x}^4 + \frac{1-4pq}{9mpq}\mathrm{x}^6,$$
$$A_2 = \frac{1-2pq}{2mpq}\mathrm{x}^3 - \frac{1-3pq}{mpq}\frac{1}{3}\mathrm{x}^4,$$
$$B_2 = -\frac{1-pq}{12mpq},$$
$$A_1B_1 = 0,$$

$$\text{Second correction } = P\left(-\frac{1-pq}{12mpq} + \frac{3-8pq}{4mpq}\mathrm{x}^2 - \frac{2-7pq}{3mpq}\mathrm{x}^4 + \frac{1-4pq}{9mpq}\mathrm{x}^6\right).$$

These expressions for the first and second corrections are now to be compared with those which are obtained by the methods set forth in the preceding sections. According to those methods the first correction is

$$-\frac{1}{3!}k_1\frac{d^3y_0}{dx^3}, \text{ where } y_0 = \frac{1}{\sqrt{\pi}c}e^{-\frac{x^2}{c^2}},$$

or if we put x for $\frac{x}{c}\left(\frac{d}{dx} = \frac{1}{c}\frac{d}{d\mathrm{x}}\right)$,

$$-\frac{1}{3!}\frac{k_1}{c^3}\frac{d^3\mathrm{y}_0}{d\mathrm{x}^3} = -\frac{1}{3!}\mathrm{k}_1\frac{d^3\mathrm{y}_0}{d\mathrm{x}^3}, \text{ if } \mathrm{k}_1 = k_1/c^3.$$

For the second correction, using similar notation, we have

$$\frac{1}{4!}\mathrm{k}_2\frac{d^4\mathrm{y}_0}{d\mathrm{x}^4} + \frac{\mathrm{k}_1^2}{2}\frac{1}{3!\,3!}\frac{d^6\mathrm{y}_0}{d\mathrm{x}^6}.$$

To effect these corrections, we have

$$\mathrm{k}_1 = k_1/c^3; \qquad k_1 = \sum_{p=1}^{p=m}\xi_p^{(3)}, \text{ in our former notation, } = m(\Delta x)^3 pq(p-q);$$
$$c = \sqrt{2mpq}\,\Delta x; \qquad c^3 = (2mpq)^{\frac{3}{2}}\Delta x^3; \qquad \mathrm{k}_1 = \frac{(p-q)}{2\sqrt{2mpq}}.$$

Likewise

$$\mathrm{k}_2 = k_2/c^4 = (\Sigma\xi^{(4)} - 3(\xi^{(2)})^2)/c^4 * = \frac{1-6pq}{4mpq}.$$

* The values of k_1 and k_2 might also be constructed from the compound curve, being respectively $\mu_3\Delta\xi^3$, $(\mu_4 - 3\mu_2^2)\Delta\xi^4$ where μ_2, μ_3, μ_4 are put for the second, third and fourth *moments* in Professor Karl Pearson's notation (*Trans. Roy. Soc.* 1895, A. p. 347).

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The derivees of y_0 which will be required in the sequel are as follows:

$$\frac{d^3}{dx^3} y_0 = (12x - 8x^3) y_0,$$

$$\frac{d^4}{dx^4} y_0 = (12 - 48x^2 + 16x^4) y_0,$$

$$\frac{d^5}{dx^5} y_0 = (-120x + 160x^3 - 32x^5) y_0,$$

$$\frac{d^6}{dx^6} y_0 = (-120 + 720x^2 - 480x^4 + 64x^6) y_0.$$

Utilising these data we obtain for the first correction, $y_1 - y_0$,

$$-k_1 \frac{1}{3!} \frac{d^3 y_0}{dx^3} = \frac{q-p}{\sqrt{2mpq}} \left(x - \frac{2}{3} x^3\right) y_0.$$

For the second correction

$$k_2 \frac{1}{4!} \frac{d^4 y_0}{dx^4} = \frac{3-18pq}{8mpq} - \frac{2(1-6pq)}{4mpq} x^2 + \frac{2(1-6pq)}{12mpq} x^4,$$

$$\frac{k_1^2}{2!} \frac{1}{(3!)^2} \frac{d^6 y_0}{dx^6} = \frac{-(5-20pq)}{24mpq} + \frac{5(1-4pq)}{4mpq} x^2 - \frac{10(1-4pq)}{12mpq} x^4 + \frac{1}{9} \frac{1-4pq}{mpq} x^6,$$

$$y_2 - y_1 = \left(-\frac{1-pq}{12mpq} + \frac{3-8pq}{4mpq} x^2 - \frac{2-7pq}{3mpq} x^4 + \frac{1-4pq}{9mpq} x^6\right) y_0.$$

Both results are in perfect agreement with those obtained by the more familiar method.

It may be interesting to proceed one step further and to determine at least the first term—the term involving the first power of x—for the *third* correction both by the simpler and by the more general method, for the purpose of further testing the latter.

On the one hand, continuing the expansion of the general term of the Binomial beyond the stage to which it had been carried by Professor Karl Pearson*, Professor Czuber† and others, we obtain $-\frac{1}{3}\frac{q-p}{C^3} x$ for the first term of B_3, if we put C for the *number* $2mpq$, and as before x for x/c where $x = r\Delta x$, $c = \sqrt{2mpq}\Delta x$, C being thus related to c as r is to x. It is unnecessary to write down A_3 as it is seen to involve no lower term of x than x^3. Employing the values of A_1, A_2, B_1, B_2, which have been already given, we find for the first term of the third correction

$\frac{1}{3!} A_1^3$	does not involve first power of x,	
$A_1 A_2$	" " " " " "	
A_3	" " " " " "	
$A_2 B_1$	$= 0$,	
$B_1 B_2$	"	
$\frac{1}{3!} B_1^3$	"	
$A_1 B_2$	$= \frac{(p-q)(1-pq)}{6C^3}$,	
B_3	$= \frac{p-q}{6C^3}$,	

* *Trans. Roy. Soc.* 1895, p. 347. † *Theorie der Beobachtungsfehler*, p. 85.

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$$\text{Coefficient of } x \text{ in } y_3 - y_2 = y_0 \frac{(p-q)(3-pq)}{6C^3}.$$

On the other hand we have by the general method for the third correction

$$-k_3 \frac{1}{5!}\frac{d^5y_0}{dx^5} - k_1k_2 \frac{1}{3!\,4!}\frac{d^7y_0}{dx^7} - \frac{k_1^3}{3!}\frac{1}{(3!)^2}\frac{d^9y_0}{dx^9},$$

where k_1 and k_2 have the significations already assigned to them; and

$$k_3 = k^3/c^5 = \Sigma(\xi^{(3)} - 10\xi^{(2)}\xi^{(2)})/(2pqm)^{\frac{5}{2}}(\Delta x^5) = mpq(p-q)(1-12pq)(\Delta x)^5/2pqm \times (2pqm)^{\frac{5}{2}}(\Delta x)^5$$
$$= (p-q)(1-12pq)/2C^{\frac{5}{2}}*.$$

The additional derivees of y_0 which are required for this calculation are as follows:

$$\frac{d^7y_0}{dx^7} = 1680x - 3360x^3,$$

$$\frac{d^9y_0}{dx^9} = -30240x + \ldots.$$

We have thus for the first term of $y_3 - y_2$:

$$-k_3 \frac{1}{5!}\frac{d^5y_0}{dx^5} = \frac{(p-q)(1-12pq)}{2C^3}x + \ldots$$

$$-k_1k_2 \frac{1}{3!\,4!}\frac{d^7y_0}{dx^7} = -\frac{(p-q)(1-6pq)}{4C^3}\frac{35}{3}x + \ldots$$

$$-k^3 \frac{1}{(3!)^2}\frac{d^9y_0}{dx^9} = \frac{(p-q)(1-4pq)}{4C^3}\frac{70}{3}x + \ldots$$

$$y_3 - y_2 = y_0 \frac{(p-q)(\frac{1}{2} - \frac{1}{6}pq)}{C^3}x + \ldots$$

which agrees with the result obtained by the more direct method.

So far we have supposed mp to be an integer. The modification required when this is not the case is thus expressed by Todhunter. Put for the number of times that the "event" corresponding to the left or lower value of one of our elements is most likely to occur, out of a total number of trials m, not now mp but $mp + z$, where z is a proper fraction. The most probable number for the non-occurrence of the event is accordingly $mq - z$. These corrected values are to be substituted for mp and mq respectively in the expression for the general term of the Binomial; in which $mq + r$ becomes now $mq + (r - z)$, $mp - r$ becomes $mp - (r - z)$. The expression thus transformed gives the probability that the number of times the event occurs will be less by the integer r than the most probable number, now $mp + z$. Now this modification is in accordance with that which is required by the general formula in order to answer the same question. The general formula, when the centre of gravity is taken as the origin for each of the elements, and so for the compound, gives the probability that any assigned number out of the m elements should have the upper—

* The coefficient k_3 may also be calculated from the compound locus; for which the mean fifth power may be obtained by a continuation of a method employed by the present writer to find the mean second and third power of the binomial (*Phil. Mag.* 1886, Vol. XXI. p. 30), or by the more general method pointed out by Prof. Karl Pearson (*Trans. Roy. Soc.* 1895, A. p. 346).

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and likewise the lower—value in terms of x, the distance of the point corresponding to the assigned number, from G the centre of gravity for the aggregate. When mp is not an integer, that distance is not an integer multiple of Δx. If it is required to express the said probability in terms of that integer multiple of Δx by which the point designated x is distant from the point corresponding to the most probable number of events, say the point G', distant by less than Δx from G, then we must transform the expression for the probability in question from the origin G to the origin G'—on the left, at a distance $z\Delta x$ from G, corresponding to Todhunter's supposition that the most probable number of events is now $mp+z$. Let the abscissa measured from G' be x'. Put $x=x'-z\Delta x$; corresponding to the change of r in Todhunter's expression. The modification of the second approximation obtained by him is thus equally obtained by the general formula.

The simpler analysis proper to the Binomial may be applied to verify the general theory in a less degraded case, in fact the most general case to which Laplace extended his proof, the case in which all the elements have the same law of frequency.

Consider first the particular law of frequency with which Laplace introduces his general proof, the case in which the law of frequency common to all the elements is the next simplest after the binomial, viz. such that over a finite range of the axis x say from 0 to $2a$, one value of each element occurs as often as another in the long run. Let these values occur at intervals equal to Δx; and let there be ν such intervals in the range of the element; so that $\nu=2a/\Delta x$. To begin with, let ν be a power of 2, say 2^τ. Then the frequency with which an aggregate of m such elements taken at random enjoys any assigned value, may be determined by the following construction. Suppose a body of which the dimensions may be neglected, a grain or particle* moving along a line from zero in a positive direction, to take τ *steps* of the following description. The first step is either zero or a, the alternatives being equally probable. Likewise with equal probability, the second step is either zero or $\frac{1}{2}a$; the third step is either zero or $\frac{1}{4}a$; and so on up to the τth step which is either zero 0 or $a/2^{\tau-1}=2a/2^\tau=2a/\nu=\Delta x$. Let m grains be started at the origin O on this course, and distribute themselves over the range OA, $=2a$, by taking τ steps of the kind described. As each step is independent of the others, it comes to the same whether we aggregate m parcels, each parcel comprising the results of τ steps, of respectively a, $\frac{1}{2}a \ldots \frac{1}{2^\tau}2a$, or τ parcels each comprising m steps, the first parcel comprising only steps of range a, the second parcel only steps of range $\frac{1}{2}a$, and so on. But each of the latter parcels corresponds to an even binomial, with ranges respectively a, $\frac{1}{2}a$, $\frac{1}{4}a, \ldots \Delta x$. Therefore by the preceding paragraphs the aggregate for each of these τ parcels fulfils the law of error; the coefficient k for the successive aggregates being $\frac{1}{4}a^2$, $\frac{1}{16}a^2 \ldots \frac{1}{4}(\Delta x)^2$. Now when two or more quantities each fulfil the normal law of error, their sum also fulfils that law; the coefficient k for the sum being the sum of the corresponding coefficients pertaining to each of the components†. Thus the quantity is formed by the aggregation of all the steps pertaining to any one particle

* The construction might be illustrated by a modification of the *error machine* devised by Mr Francis Galton.

† In the present connexion this proposition had better be proved by actual superposition of one normal curve upon another according to the formula given on p. 45 above: in general the proposition follows from the postulate that the law of error is *reproductive*.

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fluctuating according to a regular* law of error of which the standard deviation squared is

$$\frac{1}{4}a^2\left(1+\frac{1}{4}+\frac{1}{16}+\dots\frac{1}{2^{2\nu}}\right)=\frac{1}{4}a^2\frac{1}{1-\frac{1}{4}}, \quad =\frac{1}{3}a^2,$$

when ν is indefinitely great, Δx being indefinitely small; a result which agrees with the result which Laplace obtains for the mean square of error on that supposition, viz.,

$$2\int_0^a \frac{1}{2a}x^2dx=\frac{1}{3}a^2.$$

When Δx is supposed to decrease, ν to increase indefinitely, ν may be treated as an integer-power of 2 without loss of generality†.

The method of changing the order in the accumulation of parcels may be employed to obtain a more general verification. Let the range of each element be as before $\nu\Delta x$. Let there be taken steps now each of the same range 0 or Δx, these alternatives occurring with different probability for each step, for the first step the probability of 0 being 0_1, that of Δx being q_1 $(p_1+q_1=1)$, for the second step the probability of those alternatives being respectively p_2 and q_2, and so on. The distribution of a large number of grains each of which has taken ν steps of the kind described, is set forth in the following scheme:

Stages on the course	$0,$	$\Delta x,$	$2\Delta x\dots$	$(\nu-1)\Delta x,$	$2a.$
Probability of a particle coming to a stop at a stage	$p_1p_2\dots p_n,$	$Sq_1p_2\dots p_n,$	$Sq_1q_2p_3\dots p_n\dots$	$Sp_1q_2q_3\dots q_n,$	$q_1q_2\dots q_n.$
Given distribution of frequency	$P_0,$	$P_1,$	$P_2\dots$	$P_{\nu-1},$	$P_\nu.$

The last line is intended to represent the frequency with which an element enjoys each particular value 0, Δx, $2\Delta x\dots 2a$, respectively $P_0, P_1\dots$ where $\Sigma P=1$. It appears therefore that the supposed given frequency distribution of an element can be replaced by a system of ν steps such as have been described, provided that each of the probabilities in the second row is equal to the corresponding proportionate frequency in the third row. As the sum of the fractions in the second as well as in the third row is equal to unity, the $(\nu+1)$ conditions involve only ν equations for the ν quantities $p_1, p_2\dots p_n$. The equivalence between the constructed system of steps and the given multinomial distribution of an element will be attained if the following ν equations are satisfied.

$$(1)\quad \frac{Sq_1p_2p_3\dots p_\nu}{p_1p_2\dots p_n}=\frac{P_1}{P_0}, \qquad (2)\quad \frac{Sq_1q_2p_3\dots p_\nu}{p_1p_2\dots p_n}=\frac{P_2}{P_0}\dots.$$

Put ω_1 for q_1/p_1, ω_2 for q_2/p_2, and so on; also A_1 for P_1/P_0, A_2 for P_2/P_0, and so on. Then the values of ω_1, ω_2, etc., ω_ν, are given as the roots of the equation

$$\omega^\nu-A_1\omega^{\nu-1}+\text{etc.}+(-1)^\nu A_\nu=0.$$

As A_1, A_2, etc., are essentially positive, the equation can have no negative roots.

* "Regular" is here used in a sense wider than "normal," to denote the *extended* law of error which has just been verified for the Binomial.

† The difference between the arrangements corresponding (1) to the supposition that $2a=2^\tau\Delta x$, or (2) to the supposition that $2a=2^{\tau-1}\Delta x$ may be neglected.

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First, let all the values of ω be real and therefore positive. Then the aggregate of which the frequency-locus is required may be considered as made up of ν parcels, each parcel consisting of m binomial elements with range Δx, for all the elements in all the parcels, and frequency the same for each element in any one parcel, but in general different for different parcels. By the preceding paragraphs the regular* law of error forms the approximate frequency-locus for each parcel and therefore for their sum. The mean square of deviation from the centre of gravity for the aggregate is the sum of the mean squares pertaining to the elements, viz.,

$$= \overset{r=\nu}{\underset{r=1}{S}} mp_r q_r \Delta x^2.$$

Next let some of the roots of the equation in ω be imaginary. A pair of imaginary roots $\alpha \pm \sqrt{-1}\beta$ corresponds to two successive binomial steps each of range Δx with the *imaginary frequency-distribution* (!), for the first step,

$$p_1 = \frac{1}{1+\alpha+\sqrt{-1}\beta}, \quad q_1 = 1 - p_1,$$

$$p_2 = \frac{1}{1+\alpha-\sqrt{-1}\beta}, \quad q_2 = 1 - p_2.$$

The result of two such steps taken in succession by a great number of particles may be considered as resulting in a distribution of this sort. The particles will be massed at the three points 0, Δx, $2\Delta x$ with the respective *real* proportionate frequencies

$$p_1 p_2 = \frac{1}{1+\alpha^2+\beta^2}; \quad p_1 q_2 + q_1 p_2 = \frac{2\alpha\{(1+\alpha)^2+\beta^2\} + 2\beta^2(1+\alpha)}{\{(1+\alpha)^2+\beta^2\}^2 + 4\beta^2(1+\alpha)^2}; \quad q_1 q_2 = \frac{\alpha^2+\beta^2}{(1+\alpha)^2+\beta^2}.$$

In other words the quadratic factor of the equation which yields two imaginary roots corresponds to a *trinomial step*, with *three* alternatives 0, Δx, $2\Delta x$, occurring with respective frequency, $1-P-Q$, P, Q; P and Q—unlike the p and q from which they are derived—being real fractions.

Following the analogy of the method proper to the binomial, let us investigate the probability that the sum of m such trinomials should differ from the most probable sum by just $r\Delta x$. The most probable sum is equal to m times the mean value of the (trinomial) element, viz.

$$m\{0 \times (1-P-Q) + \Delta x \times P + 2\Delta x \times Q\} = m\Delta x(P+2Q);$$

which may be considered as an integer multiple of Δx, with a loss of generality that is very easily repaired. Now the frequency with which out of the m steps $(mP+s)$ are of length Δx, while concurrently $(mQ+t)$ are of length $2\Delta x$ is, by well-known principles,

$$\frac{m!}{mP+s!\ mQ+t!\ m(1-P-Q)-s-t!} P^{mP+s} Q^{mQ+t} (1-P-Q)^{m(1-P-Q)-s-t}.$$

Proceeding as in the simpler case with the aid of Stirling's law we find as a first approximation to the above expression

$$\frac{1}{m2\pi\sqrt{P}\sqrt{Q}\sqrt{1-P-Q}} \exp.\left(-\frac{1}{2}\frac{s^2}{mP} - \frac{1}{2}\frac{t^2}{mQ} - \frac{1}{2}\frac{(s+t)^2}{m(1-P-Q)}\right).$$

* So far at least as verified in this section, i.e. up to the fourth approximation.

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We have now to introduce the condition $s+2t=r$. When the resulting value of t, viz. $\frac{1}{2}(r-s)$, is substituted in the index of the above expression, the index becomes

$$-\left[\frac{1}{2}\frac{s^2}{mP}+\frac{1}{2}\frac{(r-s)^2}{4mQ}+\frac{1}{2}\frac{(r+s)^2}{4m(1-P-Q)}\right].$$

Put $r\Delta x$ as before $=x$ and also $s\Delta x=y$, and we find for the required frequency an expression in terms of x and y, where y may have any value between extreme limits. To obtain an expression free from y it is proper to integrate the expression in terms of x and y, with respect to y between extreme limits. For this purpose rearrange the index in the form $\frac{(y-ax)^2}{2l}+\frac{x^2}{2k}$ with corresponding changes in the form of the constant outside the exponential. Put $y-ax=y'$; and integrating with respect to y' between $+\infty$ and $-\infty$ we obtain for the first approximation to the required frequency $\frac{1}{\sqrt{2\pi k}}\exp.-x^2/2k$; where

$$k=m\{p(1-p)+4q(1-q)-4pq\}\Delta x^2,$$

which is, as it ought to be, equal to m times the mean square of error for the trinomial element.

Thus for every pair of imaginary roots in the equation for ω a sum of m *trinomial* elements fluctuating according to the normal law of error is to be added to the sums of binomial elements corresponding to the real roots. Accordingly the sum of m elements whatever their laws of frequency (provided they are of the typical kind) fluctuates according to that law of error. The verification may be extended to the second, third, and further approximations for trinomial elements by an extension of the method which has been applied in this section to binomial elements.

The reasoning employed in the typical case having been thus verified, we proceed with the more confidence to extend that reasoning to the general case.

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III. *On Relations among Perpetuants.* By A. Young.

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Any covariant type of a system of quantics of infinite order can be expressed in terms of covariants of the form

$$(a_1a_2)^{\lambda_1}(a_1a_3)^{\lambda_2}\dots(a_1a_\delta)^{\lambda_{\delta-1}}{}^{*}, \quad\text{.....(i)}$$

or of the form

$$(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}, \quad\text{.....(ii)}$$

the sequence of the letters being fixed beforehand.

All such forms are linearly independent, for it is evident that no linear algebraical relation can exist between symbolical products of the form (i) when the sequence of letters is fixed.

The conditions for irreducibility of either (i) or (ii) are

$$\lambda_1 \nless 2^{\delta-2},\ \lambda_2 \nless 2^{\delta-3},\ \dots,\ \lambda_{\delta-1} \nless 1\dagger.$$

Again if all the letters are interchangeable, the conditions of irreducibility become

$$\left.\begin{array}{l}
\lambda_{\delta-1} = 1 + \xi_{\delta-1} \\
\lambda_{\delta-2} = 2 + \xi_{\delta-2} + \xi_{\delta-1} \\
\dots\dots\dots\dots\dots\dots \\
\lambda_r = 2^{\delta-r-1} + \xi_r + \xi_{r+1} + \dots + \xi_{\delta-1} \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
\lambda_2 = 2^{\delta-3} + \xi_2 + \xi_3 + \dots + \xi_{\delta-1} \\
\lambda_1 = 2^{\delta-2} + 2(\xi_2 + \xi_3 + \dots + \xi_{\delta-1}) + \xi_1
\end{array}\right\}, \quad\text{.....(iii)}$$

where the ξ's are positive integers or zeros‡.

It has been pointed out that this result, which was proved originally for perpetuants belonging to a single quantic (in which case ξ_1 must be even), also gives the conditions for perpetuant types when these are expressed in terms of products of either of the forms (i) and (ii), the sequence of the letters not being fixed§.

* In writing down symbolical products we shall omit factors of the form a_{1x}, a_{2x}, ...

† Grace, *Proc. Lond. Math. Soc.*, Vol. xxxv., p. 107.

‡ *Ibid.* p. 319.

§ Grace and Young, *Algebra of Invariants*, p. 379.

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The exact number of perpetuant types of degree δ and weight w is known to be

$$\binom{w-2^{\delta-1}+1+\delta-2}{\delta-2}.$$

But when the sequence of the letters is not fixed it will be found that the conditions (iii) give too many perpetuant types; it is the first object of this paper to determine what are the relations among these forms.

It will be convenient to make use of the notation of the theory of substitutions; to avoid confusion symbols which refer to substitutions are printed in Roman type.

The symbol $\{\mathrm{a\,b}\ldots\mathrm{k}\}$ is used to denote the sum of the substitutions of the symmetric group of the letters $a, b, \ldots k$.

The symbol $\{\mathrm{a\,b}\ldots\mathrm{k}\}'$ denotes the sum of the substitutions of the alternating group of the letters $a, b, \ldots k$ minus the sum of the substitutions of these letters which do not belong to the alternating group.

In the last part of the paper the reducibility of certain transvectants is deduced from the results obtained.

1. Consider a perpetuant of degree δ

$$P \equiv (a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\ldots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}.$$

All perpetuants such as P will be supposed arranged in order according to the indices of the different factors, the sequence of the letters not being fixed. Thus if

$$Q \equiv (b_1b_2)^{\mu_1}(b_2b_3)^{\mu_2}\ldots(b_{\delta-1}b_\delta)^{\mu_{\delta-1}}$$

where $b_1, b_2, \ldots b_\delta$ are the letters $a_1, a_2, \ldots a_\delta$ arranged in some order; then Q will precede P provided that the first of the differences

$$\mu_1-\lambda_1,\ \mu_2-\lambda_2,\ \ldots\ \mu_{\delta-1}-\lambda_{\delta-1}$$

which does not vanish is positive.

If all these differences are zero, P and Q belong to the same set, and take the same position in our arrangement.

To express that the sum of certain forms like P is equal to a linear function of perpetuants which precede them in the chosen arrangement and of products of forms of lower degree, it will be convenient to write

$$\Sigma P = R.$$

The symbol R is throughout used in this sense. Thus unless the indices λ satisfy the conditions (iii) we have

$$P = R.$$

2. *Covariants of degree three.*

All perpetuants of degree three can be expressed in terms of those of the form

$$(a_1a_2)^\lambda(a_2a_3)^\mu,\quad \lambda \nless 2\mu.$$

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If $\lambda = 2\mu$,

$$(a_1a_2)^{2\mu}(a_2a_3)^{\mu} - (a_1a_3)^{2\mu}(a_3a_2)^{\mu} = R.$$

If $\lambda = 2\mu + 1$,

$$(a_1a_2)^{2\mu+1}(a_2a_3)^{\mu} + (a_2a_3)^{2\mu+1}(a_3a_1)^{\mu} + (a_3a_1)^{2\mu+1}(a_1a_2)^{\mu} = R.$$

These facts are well known. They may be deduced at once from Stroh's series*.

The relations may be written

$$(a_1a_2)^{2\mu}(a_2a_3)^{\mu} = \tfrac{1}{6}\{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\}(a_1a_2)^{2\mu}(a_2a_3)^{\mu} + R,$$

$$\{\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\}'(a_1a_2)^{2\mu+1}(a_2a_3)^{\mu} = R.$$

When $\lambda > 2\mu + 1$, we have three independent forms $(a_1a_2)^{\lambda}(a_2a_3)^{\mu}$.

Hence the number of perpetuants of degree three and weight $w = 3k + 2$ is $3k$; for we may take $\mu = 1, 2, \ldots k$.

The number of perpetuants of weight $w = 3k + 1$ is $3(k-1) + 2 = 3k - 1$. And the number when $w = 3k$ is $3(k-1) + 1 = 3k - 2$.

In every case the number is $w - 2$; and this is known otherwise to be the exact number of perpetuants of degree three and weight w. Hence there can be no relations between these perpetuants other than those just enumerated.

3. Let $$P \equiv (a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\ldots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$$

be a perpetuant of degree δ, whose indices λ satisfy the conditions (iii): we proceed to prove that if $\xi_{r-1} = 0\,(r > 2)$, then

$$\{\mathfrak{a}_r\mathfrak{a}_{r+1}\}'P = R.$$

Let the symbol α refer to the perpetuant

$$(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\ldots(a_{r-2}a_{r-1})^{\lambda_{r-2}}$$

when considered as a single binary form of infinite order. Then

$$(\alpha a_r)^{\lambda_{r-1}}(a_ra_{r+1})^{\lambda_r}\ldots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}} - P = R.$$

Now $\lambda_{r-1} = 2^{\delta-r-1} + \lambda_r$, since $\xi_{r-1} = 0$, hence by Stroh's series

$$\Big\{-\sum_{=0}^{\lambda_r}\binom{\lambda_{r-1}+\lambda_r}{i}\binom{2\lambda_r - i}{\lambda_r}(\alpha a_r)^{\lambda_{r-1}+\lambda_r-i}(a_ra_{r+1})^i$$

$$+(-)^{\lambda_r}\sum_{i=0}^{2^{\delta-r-1}-1}\binom{\lambda_{r-1}+\lambda_r}{i}\binom{\lambda_r + 2^{\delta-r-1} - 1 - i}{\lambda_r}(a_ra_{r+1})^{\lambda_{r-1}+\lambda_r-i}(a_{r+1}\alpha)^i$$

$$+(-)^{\lambda_r}\sum_{i=0}^{\lambda_r}\binom{\lambda_{r-1}+\lambda_r}{i}\binom{\lambda_r + 2^{\delta-r-1} - 1 - i}{2^{\delta-r-1} - 1}(a_{r+1}\alpha)^{\lambda_r+\lambda_{r-1}-i}(\alpha a_r)^i\Big\}$$

$$\times(a_{r+1}a_{r+2})^{\lambda_{r+1}}(a_{r+2}a_{r+3})^{\lambda_{r+2}}\ldots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}} = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

But $$(a_ra_{r+1})^{\mu_1}(a_{r+1}\alpha)^{\mu_2}(a_{r+1}a_{r+2})^{\lambda_{r+1}}\ldots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$$

$$= (a_ra_{r+1})^{\mu_1}(a_{r+1}\alpha)^{\mu_2}(\alpha a_{r+2})^{\lambda_{r+1}}\ldots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}} + R;$$

and when $\mu_2 < 2^{\delta-r-1}$ the perpetuant on the right-hand side can be expressed in terms

* *Math. Ann.*, Bd. 31; *Algebra of Invariants*, p. 64.

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of forms which contain a greater number of factors involving a_r, a_{r+1}, α only. Thus we see that all the terms of the second sum in (iv) may be included in the symbol R; in fact this relation becomes

$$\{-(\alpha a_r)^{\lambda_{r-1}}(a_r a_{r+1})^{\lambda_r}+(-)^{\lambda_r}(a_{r+1}\alpha)^{\lambda_{r-1}}(\alpha a_r)^{\lambda_r}\}(a_{r+1}a_{r+2})^{\lambda_{r+1}}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}=R.$$

Whence $$\{\mathrm{a}_r\mathrm{a}_{r+1}\}'(\alpha a_r)^{\lambda_{r-1}}(a_r a_{r+1})^{\lambda_r}(a_{r+1}a_{r+2})^{\lambda_{r+1}}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}=R;$$

and therefore $$\{\mathrm{a}_r\mathrm{a}_{r+1}\}'P=R.$$

4. When $\xi_1=0$, we have $\lambda_1=2\lambda_2$.

In § 2 we saw that $$\{\alpha\beta\}'(a_1a_2)^{2\lambda_2}(a_2a_3)^{\lambda_2}=R$$

where α, β are any two of the letters a_1, a_2, a_3.

Hence also $$\{\alpha\beta\}'(a_1a_2)^{2\lambda_2}(a_2a_3)^{\lambda_2}(a_3a_4)^{\lambda_3}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}=R;$$

for $$(a_1a_2)^{2\lambda_2}(a_2a_3)^{\lambda_2}(a_3a_4)^{\lambda_3}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}-(a_1a_2)^{2\lambda_2}(a_3a_2)^{\lambda_2}(a_3a_4)^{\lambda_3}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}=R.$$

Therefore when $\xi_1=0$

$$\{\alpha\beta\}'P=R,\text{ where }\alpha,\ \beta\text{ are any two of }a_1,\ a_2,\ a_3.$$

Similarly from the fact that

$$\{\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3\}'(a_1a_2)^{2\lambda_2+1}(a_2a_3)^{\lambda_2}=R,$$

we deduce that if $\xi_1=1$

$$\{\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3\}'P=R.$$

The following relations have been obtained:

(a) $\xi_{r-1}=0$, $(r>2)$, $\{\mathrm{a}_r\mathrm{a}_{r+1}\}'P=R$.

(b) $\xi_1=0$, $\{\mathrm{a}_1\mathrm{a}_2\}'P=R$, $\{\mathrm{a}_2\mathrm{a}_3\}'P=R$.

(c) $\xi_1=1$, $\{\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3\}'P=R$.

(d) ξ_1 even, $\{\mathrm{a}_1\mathrm{a}_2\}'P=R$,

ξ_1 odd, $\{\mathrm{a}_1\mathrm{a}_2\}P=R$.

It remains to shew that there are no more relations.

5. Assuming that the relations just enumerated are all that exist between the forms which satisfy the conditions (iii), we proceed to prove that the number of these forms which are linearly independent is $\binom{w-2^{\delta-1}+1+\delta-2}{\delta-2}$; w being the weight and δ the degree.

Let $$\lambda_1=2^{\delta-2}+\mu_1,\quad \lambda_2=2^{\delta-3}+\mu_2,\dots\lambda_{\delta-1}=1+\mu_{\delta-1}.$$

The conditions (iii) become

$$\mu_2\nless\mu_3\nless\mu_4\dots\nless\mu_{\delta-1},\quad \mu_1\nless 2\mu_2$$

together with the fact that the μ's are positive integers.

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Consider first those forms for which

$$\mu_{r+1} = 0 = \mu_{r+2} = \ldots = \mu_{\delta-1},$$

and $\mu_1, \mu_2, \ldots \mu_r$ are all different from zero.

Let $$P = (a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \ldots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$$

be one of these forms; then by § 3

$$\{\mathrm{ab}\}' P = R,$$

when a, b are any two of the letters $a_{r+2}, a_{r+3}, \ldots a_\delta$.

The letters $a_1, a_2, \ldots a_{r+1}$ can be chosen in $\binom{\delta}{r+1}$ ways. When this set of letters has been selected, the number of forms corresponding to given values of $\mu_1, \mu_2, \ldots \mu_r$ depends (i) on what consecutive pairs of μ's are equal, and (ii) on whether $\mu_1 - 2\mu_2 = 0, 1$ or > 1. This number is then quite independent of δ, provided that $\delta \geqslant r+1$. Also the set of values which can be given to $(\mu_1, \mu_2, \ldots \mu_r)$ is independent of δ.

Hence if $\phi(r, \varpi)$ is the number of independent forms P of degree δ for which $\Sigma\mu = \varpi$, $\mu_r > 0$, and

$$\mu_{r+1} = \mu_{r+2} = \ldots = \mu_{\delta-1} = 0,$$

then $\phi(r, \varpi)$ is the number of independent forms of degree $r+1$, for which no μ is zero, and $\Sigma\mu = \varpi$.

Now if μ_r is not zero, the number of forms of degree $r+1$, corresponding to a given set of values $(\mu_1, \mu_2, \ldots \mu_r)$ of the μ's, is the same as the number of forms corresponding to the set of values $(\mu_1 - 2, \mu_2 - 1, \ldots \mu_r - 1)$. Hence $\phi(r, \varpi)$ is the total number of forms of degree $r+1$, for which $\Sigma\mu = \varpi - r - 1$.

Again, the total number of forms of degree δ is equal to the number of forms for which μ_1 is the last non-zero μ, together with the number of forms for which μ_2 is the last non-zero μ, and so on. Hence this number

$$= \phi(1, \varpi)\binom{\delta}{2} + \phi(2, \varpi)\binom{\delta}{3} + \ldots + \phi(\delta-2, \varpi)\binom{\delta}{\delta-1} + \phi(\delta-1, \varpi)\binom{\delta}{\delta}.$$

Now we shall assume that the total number of forms of degree $r < \delta$ and weight w is

$$\binom{w - 2^{r-1} + 1 + r - 2}{r-2} = \binom{\varpi' + r - 2}{r-2},$$

where $\varpi' = \Sigma\mu$: also that the number of forms of degree δ, for which $\Sigma\mu = \varpi'$, $\varpi' < \varpi$, is

$$\binom{\varpi' + \delta - 2}{\delta - 2}.$$

Then by hypothesis

$$\phi(r, \varpi') = \binom{\varpi' - 2}{r-1}$$

when $r + 1 < \delta$, and when $r + 1 = \delta$, but $\varpi' < \varpi + \delta$.

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Hence the number of forms of degree δ, for which $\Sigma\mu = \varpi$, is

$$\binom{\delta}{2} + \binom{\varpi-2}{1}\binom{\delta}{3} + \binom{\varpi-2}{2}\binom{\delta}{4} + \dots + \binom{\varpi-2}{\delta-3}\binom{\delta}{\delta-1} + \binom{\varpi-2}{\delta-2},$$

and this is the coefficient of x^{ϖ} in the expansion of $(1+x)^{\delta} \times (x+1)^{\varpi-2}$.

Therefore the number required

$$= \binom{\varpi+\delta-2}{\varpi} = \binom{\varpi+\delta-2}{\delta-2} = \binom{w-2^{\delta-1}+1+\delta-2}{\delta-2}.$$

Hence the number of independent forms of weight w and degree δ is

$$\binom{w-2^{\delta-1}+1+\delta-2}{\delta-2},$$

provided that this is true for weight $< w$ and degree δ, and also for degree $< \delta$ and any weight.

Now if $w < 2^{\delta-1}+1$, ϖ is negative, and there are no forms of weight w and degree δ; the formula is then true for weight $w < 2^{\delta-1}-1$ and degree δ; hence it is true for degree δ and any weight if it is true for degree $< \delta$ and any weight. But it is evidently true for degree 2, hence it is always true.

Thus on the assumption that there are no relations other than those of § 4, we find that the number of independent irreducible forms is the same as the exact number of perpetuants. It follows that there can be no other relations between the forms considered.

6. In consequence of the relations of § 4

$$(a_1 a_2)^{2^{\delta-2}} (a_2 a_3)^{2^{\delta-3}} \dots (a_{\delta-1} a_\delta)$$

can be written in the form

$$\{\mathrm{a}_1 \mathrm{a}_2 \dots \mathrm{a}_\delta\} P + R,$$

where P is a numerical multiple of the above form, and R is a sum of products of perpetuants.

Again, if

$$Q = (a_1 a_2)^{2^{\delta-2}+1} (a_2 a_3)^{2^{\delta-3}} \dots (a_{\delta-2} a_{\delta-1})^2 (a_{\delta-1} a_\delta),$$

we have

$$\{\mathrm{a}_1 \mathrm{a}_2\} Q = R, \qquad \{\alpha\beta\}' Q = R,$$

$$\{\mathrm{a}_1 \mathrm{a}_2 \mathrm{a}_3\}' Q = R,$$

where α, β are any two of the letters $a_3, a_4, \dots a_\delta$.

Now using the substitutional series

$$1 = \Sigma A_{a_1, a_2, \dots a_h} T_{a_1, a_2, \dots a_h}{}^{*},$$

in which the letters affected are $a_1, a_2, \dots a_\delta$; we have

$$Q = \Sigma A_{a_1, a_2, \dots a_h} T_{a_1, a_2, \dots a_h} Q.$$

* Young, *Proc. Lond. Math. Soc.*, Vol. XXXIII., p. 133, *et seq.*

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But from the above equations

$$Q = \{a_1 a_2\}' \{a_3 a_4 \dots a_\delta\} \frac{1}{2.(\delta-2)!} Q + R.$$

Hence every $T_{a_1, a_2, \dots a_\delta} Q$ is equal to R, except $T_{\delta-1,1}$ and $T_{\delta-2,1,1}$.

Now $$T_{\delta-1,1} \{a_1 a_2\}' \{a_3 a_4 \dots a_\delta\} = \{a_1 a_2\}' \{a_3 a_4 \dots a_\delta\} T_{\delta-1,1}$$
$$= \lambda \{a_1 a_2\}' [\{a_1 a_3 a_4 \dots a_\delta\} + \{a_2 a_3 a_4 \dots a_\delta\}].$$

Therefore $$T_{\delta-1,1} Q = \mu \{a_1 a_2\}' [\{a_1 a_3 a_4 \dots a_\delta\} + \{a_2 a_3 a_4 \dots a_\delta\}] Q + R$$
$$= \mu \{a_1 a_2\}' \{a_2 a_3 a_4 \dots a_\delta\} \{a_1 a_2\}' Q + R.$$

Again $$T_{\delta-2,1,1} Q = \frac{1}{2(\delta-2)!} \{a_1 a_2\}' \{a_3 a_4 \dots a_\delta\} T_{\delta-2,1,1} Q + R$$
$$= \nu \{a_2 a_4 \dots a_\delta\} \{a_1 a_2 a_3\}' \{a_3 a_4 \dots a_\delta\} Q + R.$$

But $$\{a_1 a_2 a_3\}' Q = R,$$

therefore $$\{a_1 a_2 a_3\}' . \{a_3 a_4 \dots a_\delta\} Q = (\delta-2)! \{a_1 a_2 a_3\}' Q + R = R;$$

and therefore $$T_{\delta-2,1,1} Q = R.$$

Hence $$Q = A \{a_1 a_2\}' \{a_2 a_3 a_4 \dots a_\delta\} Q + R,$$

where A is numerical; and R is a sum of products of perpetuants.

7. We proceed now to consider certain transvectants. The order of each of the quantics involved is supposed to be greater than the weight of the covariant in which it occurs; in this case theorems proved for perpetuants will be true.

(i) Consider first the transvectants

$$C \equiv ((a_1 a_2), a_{3x}^{n_3})^\lambda.$$

If $\lambda = 1$, the covariant is of degree three and weight two, and hence must be reducible,—for the minimum weight of an irreducible perpetuant of degree three is three.

If $\lambda = 2$, and the orders of the quantics represented by a_1, a_2 are the same,

$$C = (a_1 a_2)^2 (a_2 a_3) + R.$$

Also $$\{a_1 a_2\} C = 0.$$

But by §§ 4, 6 $$\{a_1 a_2\} (a_1 a_2)^2 (a_2 a_3) = 2 (a_1 a_2)^2 (a_2 a_3) + R;$$

therefore in this case C is reducible.

(ii) Transvectants $$C \equiv ((a_1 a_2), (a_3 a_4))^\lambda$$

are reducible when $\lambda = 1, 2, 3, 4$, owing to the fact that the minimum weight of an irreducible form of degree four is seven.

Let us suppose that the quantics concerned are

$$a_{1x}^{n_1}, \quad a_{2x}^{n_2}, \quad a_{3x}^{n_3}, \quad a_{4x}^{n_4}.$$

Then if $n_1 = n_2$ $$\{a_1 a_2\} C = 0.$$

Hence, if $\lambda = 5$, we have by § 6

$$C = \tfrac{1}{2} \{a_1 a_2\} C + R = R.$$

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Similarly, if $n_1 = n_2$ and $n_3 = n_4$,

$$\{a_1 a_2\} C = 0, \qquad \{a_3 a_4\} C = 0;$$

then if $\lambda = 6$ $\qquad C = \Sigma A \{ab\}' \{bcd\} C + R = R$

(where a, b, c, d are the letters a_1, a_2, a_3, a_4 in some order).

Again, if $n_1 = n_2 = n_3 = n_4$ and $\lambda = 7$,

$$\{a_1 a_2\} C = 0, \quad \{a_3 a_4\} C = 0, \quad [1 + (a_1 a_3)(a_2 a_4)] C = 0.$$

Also C can be expressed in terms of forms

$$(ab)^6 (bc)^3 (cd),$$

and of products of forms.

Hence $\qquad C = A_4 T_4 C + A_{3,1} T_{3,1} C + \Sigma A \{ab\}' \{cd\}' \{ac\} \{bd\} C + R.$

But from the above equations

$$T_4 C = 0, \qquad T_{3,1} C = 0;$$

and we have only to consider expressions like

$$\{a_1 a_2\}' \{a_3 a_4\}' \{a_1 a_3\} \{a_2 a_4\} C = \tfrac{1}{2} \{a_1 a_2\}' \{a_3 a_4\}' \{a_1 a_3\} \{a_2 a_4\} [1 + (a_1 a_3)(a_2 a_4)] C = 0.$$

And hence C is reducible in this case.

(iii) Transvectants $\qquad C \equiv ((a_1 a_2), \quad (a_3 a_4)^2)^\lambda.$

If $\lambda = 1, 2, 3$, C is reducible owing to the fact that the weight is less than seven.

If $\lambda = 4$, and $n_1 = n_2$, C is reducible since

$$\{a_1 a_2\} C = 0.$$

(iv) Transvectants $\qquad C \equiv ((a_1 a_2)^2, \quad (a_3 a_4)^2)^\lambda.$

If $\lambda = 1, 2$, C is reducible owing to the fact that the weight is less than seven.

If $\lambda = 3$, and $n_1 = n_2 = n_3 = n_4$, C is reducible since

$$[1 + (a_1 a_3)(a_2 a_4)] C = 0.$$

(v) Transvectants $\qquad C \equiv ((a_1 a_2), \quad (a_3 a_4)^3)^\lambda.$

C is reducible if $\lambda = 1, 2, 3, 4$, and $n_1 = n_2$, $n_3 = n_4$.

(vi) Transvectants $\qquad C \equiv ((a_1 a_2)^2, \quad (a_3 a_4)^3)^\lambda.$

C is reducible if $\lambda = 1, 2$ and $n_3 = n_4$.

(vii) Transvectants $\qquad C \equiv ((a_1 a_2)^3, \quad (a_3 a_4)^3)^\lambda.$

C is reducible if $\lambda = 1, 2, 3$, and $n_1 = n_2 = n_3 = n_4$.

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IV. *On certain Quintic Surfaces which admit of Integrals of the First Kind of Total Differentials.* Second Paper. By Arthur Berry, M.A., King's College, Cambridge.

[*Received* 31 October 1904.]

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§ 1. Introduction.

This paper is a continuation of one with the same title presented to the Society on January 4, 1902 and published in Volume XIX of the *Cambridge Philosophical Transactions*; it is closely connected with an earlier paper "On Quartic Surfaces which admit of Integrals of the first kind of Total Differentials," published in Volume XVIII of the same *Transactions*. I shall refer to these papers as "Quintics I" and "Quartics."

The problem considered is the discovery of quintic surfaces which admit of integrals of the first kind (everywhere finite) of total differentials. In "Quintics I" I dealt with cases in which the surface possesses a singular line of multiplicity greater than two, or a double curve of order greater than two, or two non-intersecting double straight lines. The present paper deals with cases where there is a double conic, which may in certain cases degenerate into two intersecting straight lines, distinct or coincident. The cases not dealt with generally in either paper are accordingly those in which the surface possesses a double straight line and no other

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singular curve, or merely multiple points but no singular curve; certain surfaces belonging to these latter types were however dealt with in a somewhat different connection in Parts II and III of "Quintics I."

In the present paper cases in which a surface has any further singular line in addition to the double conic, or in which the conic breaks up into two straight lines, one of which is a triple line, are ignored, as having been already dealt with. It is also unnecessary to consider cones. Surfaces linearly transformable into one another are not regarded as distinct, and, as a rule, specializations obtained by giving particular values to coefficients are ignored. Subject to these restrictions the paper professes to solve its problem completely, but owing to the complexity of the algebra I have very possibly missed some cases, though, I hope, not many.

I consider first the cases in which the quintic has in addition to the double conic a distinct triple point (§ 2) or double point (§ 3), in which cases the surface can be birationally transformed into one of lower order. I then consider the cases where there is no such point, distinguishing for convenience the cases when the double conic is non-degenerate (§ 4), or degenerates into two distinct straight lines (§ 5), or into two coincident straight lines (§ 6). The equations of the surfaces discovered are marked with Roman numerals in continuation of the enumeration given in "Quintics I." A summary of the results is given in § 7.

§ 2. QUINTICS WITH A DOUBLE CONIC AND A TRIPLE POINT.

Since a quintic with a quintuple point is a cone and a quintic with a quadruple point is rational and therefore cannot admit of an integral of the first kind, we begin with the case of a triple point not lying on the double conic.

The surface can be written

$$f \equiv q^2 z + q u_2 w + u_3 w^2 = 0,$$

where q, u_2, u_3 are homogeneous functions of x, y, z, of degrees 2, 2, 3 respectively; and the double conic is $q = w = 0$.

Transforming birationally, by taking x, y, z, q/w as new variables, we obtain the cubic

$$F \equiv w^2 z + w u_2 + u_3 = 0.$$

As a non-conical cubic is rational, this must reduce to a cone, and therefore F must be annihilated by a differential operator of the form

$$a \frac{d}{dx} + b \frac{d}{dy} + c \frac{d}{dz} + d \frac{d}{dw}.$$

Substituting and equating to zero the coefficient of w^2 we have $c = 0$; and since a, b obviously cannot simultaneously vanish we can by linear transformation of x, y reduce the operator to $\frac{d}{dx} + d \frac{d}{dw}$. Thus F is a function of y, z, $w - dx$, and hence

$$F \equiv (w - dx)^2 z + (w - dx) v_2 + v_3,$$

where v_2, v_3 are independent of x, w. Retransforming, we find for the quintic

$$f \equiv (q - dxw)^2 z + (q - dxw) v_2 w + v_3 w^2 = 0 \quad \text{.....(XXIII).}$$

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Let $$q \equiv ax^2 + (y, z)(x, y, z).$$
Then there is a triple point where $y = z = 0$, $ax - dw = 0$; this is a second distinct triple point, if neither a nor d vanishes.

If $d = 0$, $a \neq 0$, the two triple points coalesce into a triple point of a more complicated character, the tangent cone at which breaks up into the three planes $v_2 \equiv 0$; the surface assumes the simpler form

$$q^2 z + q v_2 w + v_3 w^3 = 0 \quad \text{.....(XXIV).}$$

If $a = 0$, $d \neq 0$, the second triple point becomes the point $y = z = w = 0$ which lies on the double conic; we have there a triple point of a complicated character, the tangent cone at which consists of 3 planes two of which coincide and moreover cut the surface in 5 straight lines.

If both a and d vanish, $y = z = 0$ is a triple straight line, so that we revert to a case already considered in "Quintics I."

The four fundamental quadrics θ (cf. "Quintics I," § 1) are easily found to be respectively

$$xq_1 - \tfrac{5}{2}q - dxw, \quad yq_1 - dyw, \quad zq_1 - dzw, \quad -\tfrac{3}{2}wq_1 + \tfrac{3}{2}dw^2,$$

where q_1 stands for $\dfrac{dq}{dx}$.

The general surface (XXIII) which we have obtained is the general quintic with a double conic and two triple points, a surface considered by Cayley*, and shewn by him to be rationally transformable into a cubic cone. The cases when a or d vanishes are limiting cases.

§ 3. QUINTICS WITH A DOUBLE CONIC AND A DOUBLE POINT.

Taking the conic to be $q \equiv (x, y, z)^2 = 0$, $w = 0$ and the double point to be $x = y = z = 0$, the surface is

$$f \equiv q^2 u_1 + q u_2 w + u_3 w^3 + v_2 w^3 = 0,$$

where u_1, u_2, u_3, v_2, are homogeneous functions of x, y, z of degree indicated by the suffix. Neither u_1 nor v_2 can vanish identically. We assume that there is no triple point off $w = 0$, as this would lead back to the case of § 2. Moreover, if there is more than one double point we assume that the most complicated one has been chosen to be $x = y = z = 0$, where we count a tacnode as more complicated than an ordinary uniplanar point, a uniplanar point than a biplanar point, a biplanar point than an ordinary conical point. Thus we may avoid duplication of cases by rejecting all cases in which the surface has elsewhere a double point of a more complicated character than the one at $x = y = z = 0$.

By the same quadric transformation as in § 2, we obtain the quartic

$$F \equiv q v_2 + u_3 w + u_2 w^2 + u_1 w^3 = 0.$$

We observe that from the nature of the quadric transformation F can only reduce to a cubic if $v_2 \equiv 0$, or if the conic split up into two straight lines, one of which is a triple line; both

* "On the Deficiency of certain Surfaces," *Math. Annalen*, Vol. III. (1871); *Coll. Math. Papers*, Vol. VIII. p. 397.

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these cases can be ignored. Moreover, we need not consider the case when F becomes a cone; for since the section by $w=0$ is two conics, either F degenerates, or these conics become pairs of generators; in this latter case by a linear transformation of x, y, z, F is reducible to a function of x, y, w and on retransformation we see that the quintic $f=0$ is also a cone.

We have therefore only to consider the possibility that F may coincide with one of the known non-conical quartic surfaces which admit of integrals of the first kind*.

If F be such a quartic it must satisfy the differential equation

$$\theta_1 \frac{dF}{dx} + \theta_2 \frac{dF}{dy} + \theta_3 \frac{dF}{dz} + \theta_4 \frac{dF}{dw} = 0,$$

where $\theta_i = a_i x + b_i y + c_i z + d_i w$, and $a_1 + b_2 + c_3 + d_4 = 0$. The functions θ have also to satisfy certain conditions at the singularities of F; in particular the conic $\theta_4 = 0$, $w=0$ must be an adjoint on the section of F by $w=0$; but this section is two conics, hence θ_4 divides by w, i.e. $a_4 = b_4 = c_4 = 0$. Any linear transformation of F which involves x, y, z only corresponds to an exactly similar transformation of the quintic f and can therefore be employed freely, but a linear transformation involving w corresponds to a rational quadric transformation of f. Transforming first x, y, z we know from the theory of bilinear forms that the quantities θ can be reduced to one of the three types

$$a_1 x + d_1 w, \quad b_2 y + d_2 w, \quad c_3 z + d_3 w, \quad d_4 w \text{ .....(i)};$$

$$a_1 x + d_1 w, \quad x + a_1 y + d_2 w, \quad c_3 z + d_3 w, \quad d_4 w \text{ .....(ii)};$$

$$a_1 x + d_1 w, \quad x + a_1 y + d_2 w, \quad y + a_1 z + d_3 w, \quad d_4 w \text{ .....(iii)};$$

where a_1, b_2, c_3, d_4; a_1, a_1, c_3, d_4; a_1, a_1, a_1, d_4; are now the roots of the fundamental determinant on which the integration of the differential equation depends. By the known results for quartics these roots taken in some order must be one of the four sets of quantities, $1, -1, 0, 0$; $1, 1, -1, -1$; $3, -3, 1, -1$; $0, 0, 0, 0$. In type (i) the roots cannot all vanish or we should have a cone; in types (ii), (iii) they must all vanish.

Remembering that w is essentially different from x, y, z, we thus have for a_1, b_2, c_3, d_4 one of the six sets of values

$$\begin{array}{rrrr} 1, & -1, & 0, & 0; \\ 0, & 0, & 1, & -1; \\ 1, & 1, & -1, & -1; \\ 3, & -3, & 1, & -1; \\ 1, & -1, & 3, & -3; \\ 0, & 0, & 0, & 0. \end{array}$$

We have now to reduce further by linear transformations which involve w; either quoting the known results for quartics, or directly, rejecting cases in which the integrals of the funda-

* "Quartics," p. 343, "Quintics I," p. 250 *footnote*, and De Franchis's paper "Le superficie irrazionali di 4° ordine di genere geometrico-superficiale nullo": *Rend. del Circolo Matematico di Palermo*, t. XIV. (1900), pp. 1—13.

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mental differential equation are logarithmic or conical, we get for the differential operator which annihilates F one of the nine forms:

$$\xi\frac{d}{d\xi}-\eta\frac{d}{d\eta}+0\,.\,\frac{d}{dz}+0\,.\,\frac{d}{dw}\quad\text{.....(A 1)},$$

$$0\,.\,\frac{d}{d\xi}+0\,.\,\frac{d}{d\eta}+\zeta\frac{d}{d\zeta}-w\frac{d}{dw}\quad\text{.....(A 2)},$$

$$\xi\frac{d}{d\xi}+\eta\frac{d}{d\eta}-z\frac{d}{dz}-w\frac{d}{dw}\quad\text{.....(B)},$$

$$3\xi\frac{d}{d\xi}-3\eta\frac{d}{d\eta}+\zeta\frac{d}{d\zeta}-w\frac{d}{dw}\quad\text{.....(E 1)},$$

$$\xi\frac{d}{d\xi}-\eta\frac{d}{d\eta}+3\zeta\frac{d}{d\zeta}-3w\frac{d}{dw}\quad\text{.....(E 2)},$$

$$0\,.\,\frac{d}{dx}+x\frac{d}{dy}+w\frac{d}{dz}-0\,.\,\frac{d}{dw}\quad\text{.....(C)},$$

$$w\frac{d}{dx}+x\frac{d}{dy}+0\,.\,\frac{d}{dz}+0\,.\,\frac{d}{dw}\quad\text{.....(D 1)},$$

$$0\,.\,\frac{d}{d\xi}+\xi\frac{d}{d\eta}+\eta\frac{d}{d\zeta}+0\,.\,\frac{d}{dw}\quad\text{.....(D 2)},$$

$$w\frac{d}{dx}+x\frac{d}{dy}+y\frac{d}{dz}+0\,.\,\frac{d}{dw}\quad\text{.....(F)},$$

where ξ, η, ζ stand for linear functions of the form $x+\lambda w$, $y+\mu w$, $z+\nu w$, and the reference letter corresponds to the notation which I have used for the several types of quartics which admit of integrals of the first kind ("Quartics," p. 343).

It is known that the most general quartics satisfying these equations actually possess integrals of the first kind, but may of course fail to do so if they are specialized. In our case the quartic is specialized by the conditions that the section by $w=0$ breaks up into two distinct conics, and that the point $x=y=z=0$ is an ordinary point on the surface.

We have now to discuss these nine cases:

(A 1). The fundamental integrals of the differential equation are $\xi\eta$, z, w and the most general quartic integral is:

$$F\equiv\xi^2\eta^2+\xi\eta\,(z,\,w)^2+(z,\,w)^4.$$

The section by $w=0$ breaks up into conics of the form $xy+k_1z^2$, $xy+k_2z^2$; the equation may be written

$$(\xi\eta+k_1z^2)\,(\xi\eta+k_2z^2)+\xi\eta w\,(z,\,w)^1+(z,\,w)^3\,w=0,$$

where the coefficients of the second and third terms have to be chosen so that the coefficient of w^4 may vanish.

Taking $q\equiv xy+k_1z^2$, $v_2=xy+k_2z^2$, the quintic is

$$f\equiv[\{(xw+\lambda q)\,(yw+\mu q)+k_1z^2w^2\}\,\{(xw+\lambda q)\,(yw+\mu q)+k_2z^2w^2\}/q$$
$$+(xw+\lambda q)\,(yw+\mu q)\,(zw,\,q)^1+(zw,\,q)^3]\div w=0.$$

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This has triple points where $x = z = w + \lambda y = 0$, $y = z = w + \mu x = 0$; we thus revert to a case treated in § 2, unless $\lambda = \mu = 0$; in this case the quintic is

$$f \equiv (xy + k_2 z^2)\, w^3 + xy\,(zw, q)^1\, w + (zw, q)^2\, z = 0 \quad \text{.....(XXV)},$$

the surface has triple points on the double conic, where $x = z = w = 0$, and $x = y = w = 0$, and may be regarded as a limiting case of Cayley's surface (§ 2), when the two triple points coalesce with the conic. The fundamental quadrics θ ("Quintics I," § 1), are easily found to be $xw, -yw$, 0, 0. The surface thus belongs to the class considered in "Quintics I," Part II, and is in fact a special case of the surface (XVII).

In general the conic is non-degenerate and the double point is an ordinary conical point. If k_1 vanishes the conic degenerates and if $k_2 = 0$ the double point becomes biplanar.

If $k_1 = k_2 = 0$ so that in the quartic $qv_2 \equiv x^2y^2$, we have the further possibility of taking $q = x^2$, $v = y^2$, giving rise to an essentially distinct quintic

$$f \equiv \{(w + \lambda x)^2 (yw + \mu x^2)^2 + x\,(w + \lambda x)\,(yw + \mu x^2)\,(zw, x^2)^1 + (zw, x^2)^2\} \div w = 0 \quad \text{...(XXVI)},$$

where as before the coefficients are so chosen that w divides out in the numerator.

The double point is uniplanar, the conic degenerates into the two coincident straight lines $x^2 = 0$, and there is a triple point on it where $x = z = 0$. In general there is no other multiple point.

The quadrics θ are:

$$-\lambda x^2 - xw, \quad 4\lambda xy + 5\mu x^2 + 9yw, \quad 4z\,(\lambda x + w), \quad -6w\,(\lambda x + w).$$

(A 2). The fundamental integrals are ξ, η, ζw, and the quartic is

$$F \equiv (\xi, \eta)^4 + (\xi, \eta)^2\, \zeta w + \zeta^2 w^2 = 0,$$

where the coefficients have to be so chosen that the coefficient of w^4 vanishes; q, v_2, u_2 are all 0^2 along $x = y = 0$, and therefore the quintic has an additional double line $x = y = 0$. This case has already been considered in "Quintics I."

(B). Three fundamental integrals are ξ/η, z/w, ξz, and

$$F \equiv (\xi, \eta \overset{2}{\between} z, w)^2 = 0,$$

where as before the coefficients must be chosen so as to make the coefficient of w^4 vanish.

The section by $w = 0$ is of the form $(x, y)^2 z^2$; this can be resolved into quadratic factors in three ways, giving three quadric transformations and therefore three quintics. If we take z to be common to q and v_2, since it is also common to u_2, it is a factor of the quintic, which accordingly degenerates; if we take $q = z^2$, $v_2 = (x, y)^2$, then since u_2 contains z as a factor, inspection of the form of the quintic shews that it has a triple line $z = w = 0$. We have now only to consider the case in which $q = (x, y)^2 \equiv ax^2 + 2hxy + by^2$, $v_2 = z^2$: it is convenient to make a linear transformation of x, y so as to remove one of the constants λ, μ, which occur in the θ's, say the former, so that ξ now reduces to x. We can now write F in the form

$$(a, h, b \between x, y + \mu w)^2\, z^2 + (x, y + \mu w)^2\, zw + (x, y + \mu w)^1\, xw^2 = 0,$$

where we have expressed the condition that the coefficient of w^4 vanishes. We observe that μ

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cannot vanish, or the quartic would have a double point at $x = y = z = 0$, which we have seen to be impossible, since u_1 cannot vanish identically.

The quintic is

$$f \equiv (a, h, b \between xw, yw + \mu q)^2 z^2 w/q + (xw, yw + \mu q)^2 z + (xw, yw + \mu q)^1 xq = 0.$$

This has a double line along $x = 0$, $w + b\mu y = 0$, which is distinct from the conic unless $b = 0$. If $b = 0$, the double line $x = w = 0$, which is now part of the conic, is cuspidal with the fixed tangent plane $w + 2\mu h x = 0$, and this expression is also a factor of the terms which are cubic in x, w; thus the double line is tacnodal, being the limit of two double lines which intersect. We revert to a case already considered in "Quintics I," § 6.

(E 1). The general integral is

$$F \equiv a\xi^2\eta^2 + b\zeta^2 w^2 + c\xi\eta\zeta w + d\xi w^3 + e\eta\zeta^3 = 0.$$

The section by $w = 0$ is $ax^2y^2 + eyz^3 = 0$, which can only break up into two conics if a or e vanishes. The birational transformation employed in "Quartics" § 5, shews that in each of these cases F is rational.

(E 2). The general integral is

$$F \equiv a\xi^2\eta^2 + b\xi^2\zeta + c\xi\eta\zeta w + d\eta^2 w + e\zeta^2 w^2 = 0,$$

where as before the coefficients must be chosen so that the coefficient of w^4 vanishes. The section by $w = 0$ is $ax^2y^2 + bx^3z = 0$; if $b = 0$ the quartic is rational as before, hence $b \neq 0$ and may conveniently be replaced by unity; a may or may not vanish. We now have two possibilities, $q \equiv x^2$, $v_2 \equiv xz + ay^2$ or $q \equiv xz + ay^2$, $v_2 \equiv x^2$.

In the former case the quintic becomes

$$f \equiv [a(w + \lambda x)^2 (yw + \mu x^2)^2 + x(w + \lambda x)^2 (zw + \nu x^2) + cx(w + \lambda x)(yw + \mu x^2)(zw + \nu x^2)$$
$$+ d(yw + \mu x^2)^3 + ex^2 (zw + \nu x^2)^2] \div w = 0,$$

where
$$a\lambda^2\mu^2 + \lambda^2\nu + c\lambda\mu\nu + d\mu^3 + e\nu^2 = 0.$$

There is a uniplanar point at $\lambda : \mu : \nu : -\lambda^2$; this is an isolated point and distinct from the double point at $x = y = z = 0$, provided that $\lambda \neq 0$; the surface thus possesses a double point of a more complicated kind than the one chosen as the principal point of the quadric transformation; hence as explained at the beginning of this paragraph the surface can be ignored. If $\lambda = 0$ this uniplanar point coalesces with a point on the conic; f assumes the simpler form

$$a(yw + \mu x^2)^2 w + x(zw + \nu x^2) w^3 + cx(yw + \mu x^2)(zw + \nu x^2)$$
$$+ d' \{\nu^2 (yw + \mu x^2)^3 - \mu^2 x^3 (zw + \nu x^2)^2\}/w = 0 \text{.....(XXVII)}.$$

The double conic consists of the two coincident straight lines $x^2 = 0$ and has a triple point on it where $y = 0$; the double point $x = y = z = 0$ is in general a conical point, but becomes biplanar if $a = 0$.

The four quadrics θ are

$$-xw, \quad 2\mu x^2, \quad 3\nu x^2 + zw, \quad 0.$$

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In the alternative case, $q \equiv xz + ay^2$, $v_2 \equiv x^2$, the quintic becomes

$$f \equiv [\{a(xw+\lambda q)^2(yw+\mu q)^2 + (xw+\lambda q)^2(zw+\nu q)\}/q + c(xw+\lambda q)(yw+\mu q)(zw+\nu q) + d(yw+\mu q)^2 + e(zw+\nu q)^2 q] \div w = 0 \quad \text{.....(XXVIII)},$$

the constants being connected by the same equation as before.

The double conic is in general non-degenerate, but reduces to two straight lines if $a = 0$; the double point at $x = y = z = 0$ is a uniplanar point, the tangent plane at which is $x = 0$, and therefore touches the conic; there is a second uniplanar point, with the same property, at $\lambda : \mu : \nu : -\lambda\nu - a\mu^2$.

To the generators of the cubic cone, into which the surface can be rationally transformed, correspond a family of rational quartic curves on the quintic, all passing through the two uniplanar points.

The four quadrics θ are

$$-2xu + 5\lambda q - 9xw, \quad -2yu + 10\mu q - 4yw, \quad -2zu + 15\nu q + zw, \quad 3wu + 6w^2,$$

where $$u \equiv 3\nu x + 4a\mu y + \lambda z.$$

(C). The fundamental integrals are x, w, $xz - yw$; the quartic is

$$F \equiv (xz - yw)^2 + (xz - yw)(x, w)^2 + x(x, w)^3 = 0,$$

where we have expressed the condition that the coefficient of w^4 vanishes.

The section by $w = 0$ is $$x^2(z, x)^2.$$

If we take $q \equiv x^2$, $v_2 \equiv (z, x)^2$, since x is also a factor of u_3, the quintic has a triple line along $x = w = 0$; if we take $q \equiv (z, x)^2$, $v_2 \equiv x^2$, then since u_3 is quadratic in x, z the quintic has a double line $x = z = 0$; lastly if we take x to be a common factor of q and v_2, the quintic divides by x and degenerates. Thus this case can be ignored altogether.

(D 1). The fundamental integrals are z, w, $x^2 - 2yw$; the quartic is

$$F \equiv (x^2 - 2yw)^2 + (x^2 - 2yw)(z, w)^2 + z(z, w)^3 = 0.$$

The section by $w = 0$ is of the form $(x^2, z^2)^2$; and the coefficient of $w(u_3)$ is at least quadratic in x, z; hence q, v_2, u_3 are all at least quadratic in x, z and the quintic has a double line along $x = z = 0$.

(D 2). If the constant λ does not vanish it is easy to see that μ can be removed by a linear transformation of x, y, z; if $\lambda = 0$ it is not generally possible to remove μ. Thus we have two slightly different cases to consider.

(I) $\lambda \neq 0$, $\mu = 0$.

The fundamental integrals are x, w, $y^2 - 2\xi z$; the quartic is

$$F \equiv (y^2 - 2\xi z)^2 + (y^2 - 2\xi z)(x, w)^2 + x(x, w)^3 = 0.$$

The section by $w = 0$ breaks up into two conics of the form

$$y^2 - 2xz + k_i x^2 = 0 \qquad (i = 1, 2);$$

by a linear transformation of z, x we can reduce one of these conics, which we take to be $q = 0$, to the form $y^2 - 2xz = 0$; the effect of this transformation is that the third term of F is $xw(x, w)^2$.

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The quintic is now

$$f \equiv qw\,(w - 2\lambda z)^2 + (w - 2\lambda z)\,(xw,\, q)^2 + x\,(xw,\, q)^2 = 0.$$

This has a triple point at $x = y = 0$, $w = 2\lambda z$ and therefore belongs to a type already considered in § 2.

(II) $\lambda = 0$. The quartic is reducible as before to

$$F \equiv (\eta^2 - 2xz)^2 + (\eta^2 - 2xz)\,(x,\, w)^2 + w\,(x,\, w)^2 = 0,$$

where the coefficients have to be chosen so that the coefficients of w^4 vanish, and $q \equiv y^2 - 2xz$. The corresponding quintic is

$$f \equiv [q\,(w^2 + 2\mu yw + \mu^2 q)^2 + (w^2 + 2\mu yw + \mu^2 q)\,(xw,\, q)^2 + (xw,\, q)^2] \div w = 0 \quad \text{...(XXIX).}$$

The double point $x = y = z = 0$ is an ordinary conical point; the double conic is non-degenerate, and has on it a triple point, where $x = y = 0$.

The four quadrics θ are: $4\mu x^2$, $4\mu xy + 5xw$, $4\mu xz + 5\mu q + 5yw$, $-6\mu xw$.

(F). Three fundamental integrals are w, $U \equiv x^2 - 2yw$, $V \equiv x^3 - 3xyw + 3zw^2$; from which can be deduced

$$\phi \equiv \frac{V^2 - U^3}{w^2} \equiv 6x^3z - 3x^2y^2 - 18xyzw + 8y^3w + 9z^2w^2.$$

The most general form of the quartic, subject to the condition that the coefficient of w^4 vanishes, is

$$F \equiv \phi + aU^2 + bVw + cUw^2 = 0.$$

The section by $w = 0$ is $6x^3z - 3x^2y^2 + ax^4$, so that we can take $q \equiv x^2$, $v_2 \equiv 6xz - 3y^2 + ax^2$, or *vice versâ*.

In the former case the quintic is

$$f \equiv (6xzw^2 - 3y^2w^2 - 18xyzw + 8y^3w + 9z^2x^2)\,w + ax^2\,(w - 2y)^2\,w$$
$$+ bx^2\,(w^2 - 3yw + 3xz) + cx^4\,(w - 2y) = 0 \quad \text{.....(XXX).}$$

The double point at $x = y = z = 0$ is a conical point, the double conic degenerates into the two coincident straight lines $x^2 = 0$, and there is a triple point on the conic at $y = 0$.

The quadrics θ are: x^2, $-4xy + 5xw$, $-4xz + 5yw$, $6xw$.

If we take $q \equiv 6xz - 3y^2 + ax^2$, $v_2 = x^2$, the quintic becomes

$$f \equiv \{6x^3zw^2 - 3x^2y^2w^2 - 18xyzqw + 8y^3qw + 9z^2q^2w + a\,(x^2w - 2yq)^2\}\,w/q$$
$$+ b\,(x^2w^2 - 3xyqw + 3zq^2) + c\,(x^2w - 2yq)\,q = 0 \quad \text{.....(XXXI).}$$

The double point is uniplanar, the tangent plane at it is $x = 0$, which touches the double conic; the double conic is non-degenerate. There do not appear to be any additional singular points or singular lines.

The quadrics θ are

$$-2xq_1 + 5q,\ -2yq_1 + 5xw,\ -2zq_1 + 5yw,\ 3wq_1,\ \text{where } q_1 \equiv \frac{dq}{dx} \equiv 6z + 2ax.$$

This completes the enumeration for the case of a double conic and a double point not lying on it. We have found seven surfaces, given by the equations (XXV)—(XXXI).

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§ 4. Quintics with a non-degenerate double conic but with no distinct multiple point.

We write the quintic in the form

$$f \equiv q^2 u_1 + q u_2 w + u_3 w^2 + v_2 w^3 + v_1 w^4 + \kappa w^5 = 0,$$

where v_1, κ, cannot both vanish, as we should then have a double point at $x = y = z = 0$.

In accordance with the general theory we have to find four quadrics θ_i such that f satisfies the fundamental differential equation

$$\theta_1 \frac{df}{dx} + \theta_2 \frac{df}{dy} + \theta_3 \frac{df}{dz} + \theta_4 \frac{df}{dw} = 0 \quad \text{.....(A)},$$

while the θ's are connected by the subsidiary equation

$$\frac{d\theta_1}{dx} + \frac{d\theta_2}{dy} + \frac{d\theta_3}{dz} + \frac{d\theta_4}{dw} = 0 \quad \text{.....(B)}.$$

The θ's have also to satisfy certain conditions at the singularities. Since the curve $\theta_4 = 0$, $w = 0$ must be adjoint on $f = 0$, $w = 0$ ("Quintics I," p. 265), and this latter curve is a double conic, θ_4 must vanish identically when $w = 0$, i.e. must divide by w. Further each of the six cubics $x\theta_2 - y\theta_1$, etc., pass through the double conic. We thus get

$$\theta_1 = xu + \alpha q + w\Theta_1, \quad \theta_2 = yu + \beta q + w\Theta_2, \quad \theta_3 = zu + \gamma q + w\Theta_3, \quad \theta_4 = wv + \tfrac{1}{2} d_4 w^2,$$

where u, v are linear in x, y, z and $\Theta_i = a_i x + b_i y + c_i z + d_i w$ $(i = 1, 2, 3)$. Equating to zero the term independent of w in the fundamental differential equation, we have

$$5uq^2 u_1 + 4q^2 q_a u_1 + q^3 \left(\alpha \frac{du_1}{dx} + \beta \frac{du_1}{dy} + \gamma \frac{du_1}{dz}\right) = 0,$$

where $2q_a$ is used to denote

$$\alpha \frac{dq}{dx} + \beta \frac{dq}{dy} + \gamma \frac{dq}{dz}.$$

It follows that $(5u + 4q_a) u_1$ must divide by q, but q is assumed non-degenerate, therefore $5u + 4q_a = 0$, and

$$\alpha \frac{du_1}{dx} + \beta \frac{du_1}{dy} + \gamma \frac{du_1}{dz} = 0 \quad \text{.....(1)}.$$

Applying now the subsidiary equation (B) we find that $v = \frac{8}{5} q_a$ and

$$a_1 + b_2 + c_3 + d_4 = 0 \quad \text{.....(2)}.$$

The quadrics θ now reduce to

$$-\tfrac{4}{5} x q_a + \alpha q + w\Theta_1, \quad -\tfrac{4}{5} y q_a + \beta q + w\Theta_2, \quad -\tfrac{4}{5} z q_a + \gamma q + w\Theta_3, \quad \tfrac{8}{5} w q_a + \tfrac{1}{2} d_4 w_1^2 \text{.....(3)},$$

subject to the condition (2).

The differential equation is much simplified if we can remove the constants d_1, d_2, d_3 in Θ_1, Θ_2, Θ_3. This can in general be effected by substitutions of the form $x + \lambda w$, $y + \mu w$, $z + \nu w$, for x, y, z. We suppose now that this reduction can be effected, postponing to the latter part of this article the discussion of the case when this reduction is impossible.

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Case I: d_1, d_2, d_3 all zero.

We proceed first to reduce Θ_1, Θ_2, Θ_3. For brevity let us denote by Δ the differential operator

$$\Theta_1 \frac{d}{dx} + \Theta_2 \frac{d}{dy} + \Theta_3 \frac{d}{dz}.$$

Equating to zero the coefficient of w' in the fundamental differential equation, we have

$$\Delta(q^2 u_1) + q^2 \left(\alpha \frac{d}{dx} + \beta \frac{d}{dy} + \gamma \frac{d}{dz}\right) u_2 = 0;$$

hence Δq must divide by q, say

$$\Delta q = 2kq \text{.....} (4),$$

where k is a constant which may vanish.

By the theory of bilinear forms we can reduce Δ to one of the three types:

$$a_1 x \frac{d}{dx} + b_2 y \frac{d}{dy} + c_3 z \frac{d}{dz} \text{ .....} (\text{i}),$$

$$a_1 x \frac{d}{dx} + (x + a_1 y) \frac{d}{dy} + c_3 z \frac{d}{dz} \text{ .....} (\text{ii}),$$

$$a_1 x \frac{d}{dx} + (x + a_1 y) \frac{d}{dy} + (y + a_1 z) \frac{d}{dz} \text{ .....} (\text{iii}).$$

Employing these forms in equation (4) and remembering that q is not to degenerate, we easily find that in

Case (i). $k =$ one of a_1, b_2, c_3, say a_1, and then $b_2 + c_3 = 2k$, so that $2a_1 - b_2 - c_3 = 0$.

Case (ii). q always degenerates.

Case (iii). $k = a_1$ and q is of the form $ax^2 + (y^2 - 2zx)$, which is reducible by a substitution, $z + \lambda x$ for z, to $y^2 - 2zx$.

Also from the coefficient of w^6 in the fundamental differential equation $\kappa d_4 = 0$. If $\kappa \neq 0$, $d_4 = 0$; if $\kappa = 0$ then from the coefficient of w^5, since v_1 cannot now vanish identically, one of $a_1 + 2d_4$, $b_2 + 2d_4$, $c_3 + 2d_4$ must vanish; thus in all cases we have the alternative conditions

$$d_4 = 0, \quad or \quad a_1 + 2d_4 = 0, \quad or \quad b_2 + 2d_4 = 0, \quad or \quad c_3 + 2d_4 = 0 \text{ .....} (5).$$

We now have enough equations to determine the ratios of the quantities a_1, b_2, c_3, d_4; we can obviously multiply them by any convenient numerical factor; solving we thus obtain four sets of values:

belonging to type (i)	0,	1,	-1,	0	.....(A),
	1,	6,	-4,	-3	.....(B),
(*or*	1,	-4,	-6,	-3)	
	0,	0,	0,	0	.....(C),
belonging to type (ii)	0,	0,	0,	0	.....(D).

In the first three cases we can take

$$q \equiv x^2 - 2yz \text{ and in (D) } q \equiv y^2 - 2zx.$$

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In cases (A) or (B) if we employ the transformation $2\beta z + (c_3 - \frac{1}{2}d_4)\,w = 2\beta z'$, which only fails if $\beta = 0$, Δ is transformed into an expression of the form

$$(a_1'x + b_1y)\frac{d}{dx} + b_2'y\frac{d}{dy} + (a_3x + c_3'z')\frac{d}{dz'},$$

where the new constants a_1', b_2', c_3', d_4' have respectively one of the two sets of values

$$\tfrac{2}{5},\quad -\tfrac{3}{5},\quad \tfrac{7}{5},\quad -\tfrac{6}{5},\quad or\quad 1,\quad -\tfrac{3}{2},\quad \tfrac{7}{2},\quad -3,$$

and the θ's are otherwise unchanged.

The same reasoning as before shews that these new constants must still satisfy the condition (5). On substitution of the numbers we see that this condition is not satisfied.

Similarly, if $\gamma \neq 0$, we can employ the substitution $2\gamma y + (b_2 - \frac{1}{2}d_4)\,w = 2\gamma y'$, leading to the sets of values

$$-\tfrac{2}{5},\quad \tfrac{3}{5},\quad -\tfrac{7}{5},\quad \tfrac{6}{5};\quad -2,\quad -\tfrac{3}{2},\quad \tfrac{7}{2},\quad 6,$$

which are similarly impossible.

Hence cases (A) and (B) are impossible unless $\beta = \gamma = 0$. In case (A) when $\beta = \gamma = 0$ we obtain from the coefficients of w^5 and w^4 in the fundamental differential equation

$$6\alpha x\,.\,\kappa + \left(y\frac{d}{dy} - z\frac{d}{dz}\right)v_1 = 0,$$

$$4\alpha x v_1 + \alpha x\frac{d}{dx}v_1 + \left(y\frac{d}{dy} - z\frac{d}{dz}\right)v_2 = 0.$$

From the first equation we see that v_1 is independent of x, y, z, and from the coefficient of x in the second equation we then see that $\alpha = 0$.

The whole differential equation now reduces to

$$\left(y\frac{d}{dy} - z\frac{d}{dz}\right)f = 0,$$

the integral of which is $\phi(x, w, yz)$ or, more conveniently $\psi(x, w, q)$, so that the quintic is

$$f \equiv q^2x + q\,(q,\ x^2)^1\,w + x\,(q,\ x^2)^1\,w^2 + (q,\ x^2)^1\,w^3 + \lambda x w^4 + \kappa w^5 = 0 \ldots\ldots\ldots(\text{XXXII}),$$

where $\qquad q \equiv x^2 - 2yz.$

The birational transformation $q = YW$, $x = X$, $z = Z$, $w = W$ converts this into the cubic cone

$$Y^2X + Y(YW,\ X^2) + X(YW,\ X^2) + (YW,\ X^2)\,W + \lambda X W^2 + \kappa W^3 = 0,$$

which is in general non-singular, so that an integral of the first kind effectively exists.

The surface has triple points at $x = y = w = 0$, $x = z = w = 0$. The surface satisfies the same differential equation as the surfaces (XVII) ("Quintics I," p. 277), and (XXV) of this paper (§ 3). It is a special case of the former and a slight generalisation of the latter: if we put $\lambda = \kappa = 0$ we get (XXV).

In case (B), when $\beta = \gamma = 0$, the equation to be integrated is

$$\left(-\frac{4}{5}\alpha x^2 + \alpha q + \varpi w\right)\frac{df}{dx} + \left(-\frac{4}{5}\alpha x + 6w\right)y\frac{df}{dy} + \left(-\frac{4}{5}\alpha x - 4w\right)z\frac{df}{dz} + \left(\frac{6}{5}\alpha x - \frac{3}{2}w\right)w\frac{df}{dw} = 0,$$

where as before $\qquad q \equiv x^2 - 2yz.$

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This admits the three independent integrals*

$$y^3z^3w^4, \quad U \equiv q^2z, \quad V \equiv (2aq - 5xw)\,yzw,$$

from which can be constructed

$$\phi \equiv (V^2 - 50y^3z^3w^4)^2 / Uy^3z^3w^4 \equiv y\,(4aq^2 - 20axw + 25w^2)^2.$$

The most general form of f is now

$$f \equiv \lambda q^2 z + \mu\phi + \nu V = 0 \quad \text{.....(XXXIII).}$$

If we put $2aq - 5xw = Xw$, $4aq^2 - 20axw + 25w = Ww$, $q = Yw$, $z = Z$, which is a birational transformation, we obtain the quartic

$$50\lambda Y^2Z^2 + (\mu W^2 + \nu XZ)(X^2 - YW) = 0.$$

This is a slightly specialised form of the surface E ("Quartics," p. 343), and as shewn there can be transformed into a non-singular cubic cone, so that an integral of the first kind effectively exists.

The quintic has tacnodes (close-points) on the conic at $x = y = 0$, $x = z = 0$.

In case (C), $\Delta \equiv 0$.

The coefficients of w^6 and w^5 in the fundamental differential equation give the equations

$$q_a\kappa = 0, \quad 4q_a v_1 + q\left(\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz}\right)v_1 = 0.$$

Since q_a cannot vanish identically when q is not degenerate, nor be a factor of q, the first equation shews that $\kappa = 0$ and the second that $v_1 = 0$; thus we have an impossible case.

In case (D),

$$\Delta \equiv x\frac{d}{dy} + y\frac{d}{dz}, \quad q \equiv y^2 - 2zx.$$

From the coefficient of w^5 in the fundamental differential equation, we have

$$6q_a\kappa + x\frac{dv_1}{dy} + y\frac{dv_1}{dz} = 0.$$

From the coefficient of z in this equation, we have $\kappa\alpha = 0$, so that κ or α vanishes. If $\kappa = 0$, v_1 is merely x, and the coefficient of w^4 in the fundamental differential equation then gives

$$4q_a \,.\, x + q\alpha + \left(x\frac{dv_2}{dy} + y\frac{dv_2}{dz}\right) = 0.$$

From the coefficients of y^2, zx

$$\alpha + 2f = 0, \quad -6\alpha + 2f = 0,$$

where $2f$ is the coefficient of yz in v_2, whence $\alpha = f = 0$.

Thus α always vanishes.

The θ's now reduce to

$$-\tfrac{4}{5}x(\beta y - \gamma x), \quad -\tfrac{4}{5}y(\beta y - \gamma x) + \beta q + xw, \quad -\tfrac{4}{5}z(\beta y - \gamma x) + \gamma q + yw, \quad \tfrac{8}{5}w(\beta y - \gamma x).$$

* In this and similar cases these integrals were by no means obvious, but, as when once discovered their verification is simple, I have not thought it worth while to give any indication of the process of discovery.

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If $\beta \neq 0$ the transformation $2\beta z - w = 2\beta z'$ does not affect θ_1, θ_4 but removes the terms xw, yw from θ_2, θ_3; we thus revert to case (C) which has already been shewn to fail.

Thus we must have $\beta = 0$.

The fundamental differential equation now has the integrals

$$x^2w^2, \quad xq^2, \quad (2\gamma y + w)/x,$$

and the quintic is

$$f \equiv q^2(x,\ 2\gamma y + w)^1 + q(x,\ 2\gamma y + w)^2 w + (x,\ 2\gamma y + w)^3 w^2 = 0 \quad \text{.....(XXXIV).}$$

The birational transformation $q = ZW$, $x = X$, $y = Y$, $w = W$ converts this into

$$Z^2(X,\ 2\gamma Y + W)^1 + Z(X,\ 2\gamma Y + W)^2 + (X,\ 2\gamma Y + W)^3 = 0,$$

a cubic cone which is in general non-singular.

The quintic has a triple point of a complicated kind on the conic at $x = y = 0$.

Case II: d_1, d_2, d_3 not all zero.

We have now to investigate the conditions that it should be impossible to remove d_1, d_2, d_3, by linear transformation. If we substitute

$$x = x' + \frac{\xi}{\omega} w, \quad y = y' + \frac{\eta}{\omega} w, \quad z = z' + \frac{\zeta}{\omega} w,$$

where ξ, η, ζ, ω are constants subject to the restriction $\omega \neq 0$, and x', y', z' are new variables, and express the fundamental differential equation in terms of $x'y'z'w$, the coefficient of w^2 in the new θ_1', i.e. the new d_1, becomes $\{\omega\theta_1(\xi,\ \eta,\ \zeta) - \xi\theta_4(\xi,\ \eta,\ \zeta)\}/\omega^2$, with similar expressions for d_2, d_3. Thus d_1, d_2, d_3 can be made to vanish if the three quadrics

$$S_1 \equiv \theta_1 - x\theta_4/w = 0, \quad S_2 \equiv \theta_2 - y\theta_4/w = 0, \quad S_3 \equiv \theta_3 - z\theta_4/w = 0,$$

have a common point not lying on $w = 0$. The process fails only if $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ meet only on $w = 0$, or in other words, if these equations, regarded as quadratic equations in the three non-homogeneous variables x/w, y/w, z/w, are inconsistent.

We take the θ's in the form given by (3), (p. 83). It is convenient to distinguish three cases according as (I) the point α, β, γ does not lie on the conic $q = 0$, (II) α, β, γ lies on $q = 0$, (III) α, β, γ all vanish.

(I) By linear transformation of x, y, z we can arrange that $\alpha = 1$, $\beta = \gamma = 0$, and $q = x^2 - 2yz$. The equation (4) then gives

$$x(a_1x + b_1y + c_1z) - y(a_2x + b_2y + c_2z) - z(a_3x + b_3y + c_3z) = 2k(x^2 - 2yz),$$

whence
$$b_3 = c_2 = 0, \quad a_3 = b_1, \quad a_2 = c_1, \quad 2a_1 - b_2 - c_3 = 0 \text{.....(6).}$$

By a substitution $(y + \mu w,\ y)$ we can now remove c_1, and by a substitution $(z + \nu w,\ z)$ we can remove b_1; these substitutions having been performed, a substitution $(x + \lambda w,\ x)$ removes d_1. The θ's now have the forms:

$$-\tfrac{1}{3}x^2 + q + a_1xw, \quad -\tfrac{1}{3}xy + b_2yw + d_2w^2, \quad -\tfrac{1}{3}xz + c_3zw + d_3w^2, \quad \tfrac{2}{3}xw + \tfrac{1}{2}d_4w^2.$$

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Accordingly

$$S_1 \equiv -x^2 - 2yz + a_1'xw, \quad S_2 \equiv -2xy + b_2'yw + d_2w^2, \quad S_3 \equiv -2xz + c_3'zw + d_3w^2,$$

where a_1', b_2', c_3' are written for brevity in place of $a_1 - \frac{1}{2}d_4$, $b_2 - \frac{1}{2}d_4$, $c_3 - \frac{1}{2}d_4$.

The three quadrics S meet on $w=0$ in the two points $x=y=0$, $x=z=0$, and nowhere else on $w=0$; hence, if the quadrics S are to have no common points off $w=0$, these two points must be multiple points of intersection together equivalent to 8 points.

At $x=z=w=0$ the tangent planes to S_1, S_2 are $z=0$, $2y-b_2'w=0$. Hence if this point is to count triply as an intersection of S_1, S_2, S_3, the line of intersection must be a generator of the cone S_3, therefore $d_3=0$. Similarly if $x=y=w=0$ counts triply as a point of intersection, $d_2=0$. But by hypothesis d_2, d_3 are not both zero. We must therefore have one of d_2, d_3 zero, say $d_3=0$, and the point $x=z=w=0$ must count as 6 points of intersection of the three surfaces. Now $x=z=0$ is a line on S_2, S_3 meeting S_2 where $b_2'y+d_2w=0$; this gives a finite value of y, contrary to hypothesis, unless $b_2'=0$. Again $x-a_1'w=z=0$ is a line on S_1, S_3 meeting S_2 where $-2a_1'y+d_2w=0$. Therefore also $a_1'=0$, and then from (6) $c_3'=0$, so that $a_1=b_2=c_3=d_4=0$.

The θ's now reduce to

$$-\tfrac{4}{3}x^2+q, \quad -\tfrac{4}{3}xy+d_2w^2, \quad -\tfrac{4}{3}xz, \quad \tfrac{8}{3}xw,$$

where $d_2 \neq 0$, and may conveniently be taken to be 1.

The fundamental differential equation now has the three integrals

$$z^2w^2, \quad U \equiv z(q^2+4xzw^2), \quad V \equiv q^2y + (6xyz - \tfrac{1}{3}x^3)w^2 - \tfrac{3}{2}zw^4.$$

Moreover these can be shewn to be the only quintic integrals, so that the surface is

$$f \equiv \lambda z^2w^2 + \mu U + \nu V = 0 \quad \text{.....(XXXV).}$$

The surface has a tacnode on the double conic at $x=z=0$, the tangent plane there being $z=0$.

The transformation

$$(q^3+6xzqw^2+6z^2w^4)/w^3 = Z^2W, \quad (q^2+4xzw^2)/w^2 = XZ, \quad q/w = Y, \quad z = Z,$$

which is birational, transforms f into

$$\lambda Z^3 + \mu XZ^2 + \tfrac{1}{24}\nu(X^3 - ZW^2) = 0,$$

a non-singular cubic cone.

(II) Let α, β, γ lie on $q=0$, then we can arrange axes of x, y, z so that $q \equiv x^2-2yz$, $\alpha=\gamma=0$, $\beta=1$. The condition that Δq divides by q shews as before that Δ can be reduced to the form

$$(a_1x+b_1y+c_1z)\frac{d}{dx} + (c_1x+b_2y)\frac{d}{dy} + (b_1x+c_2z)\frac{d}{dz},$$

where

$$2a_1 - b_2 - c_3 = 0 \quad \text{.....(7).}$$

Further by a substitution $(x+\lambda w, x)$ we can remove d_2, and by a substitution $(z+\nu w, z)$ we can remove d_3. The θ's are now

$$\tfrac{4}{3}xz + (a_1x+b_1y+c_1z)w + d_1w^2, \quad \tfrac{4}{3}yz + q + (c_1x+b_2y)w, \quad \tfrac{4}{3}z^2 + (b_1x+c_2z)w, \quad -\tfrac{8}{3}zw + \tfrac{1}{2}d_4w^2.$$

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The quadrics S accordingly become

$$S_1 \equiv 2xz + (a_1'x + b_1y + c_1z)\,w + d_1w^2 = 0, \quad S_2 \equiv x^2 + (c_1x + b_2'y)\,w = 0, \quad S_3 \equiv 2z^2 + (b_1x + c_3'z)\,w = 0,$$

where as before a_1', b_2', c_3' stand for $a_1 - \frac{1}{2}d_4$, $b_2 - \frac{1}{2}d_4$, $c_3 - \frac{1}{2}d_4$.

If $b_2' \neq 0$, from S_1 and S_2

$$b_2'(2xz + a_1'xw + c_1zw + d_1w^2) - b_1(x^2 + c_1xw) = 0.$$

If also $b_1 \neq 0$ we can eliminate x from S_3, obtaining a quartic in z, w in which the coefficient of z^4 does not vanish; we find accordingly at least one finite value of $z:w$; then S_3 gives a finite value of $x:w$ and S_1 a finite value of $y:w$, so that S_1, S_2, S_3 meet at least once off $w=0$, contrary to hypothesis.

Hence if $b_2' \neq 0$, $b_1 = 0$; S_3 then factorizes into $z.(2z + c_3'w)$. If $z=0$ then from S_1 $a_1'x + d_1w = 0$, so that we have a finite value of $x:w$ unless $a_1' = 0$, and hence from S_2 a finite value of $y:w$. The second factor of S_3 gives $z = -\frac{1}{2}c_3'w$ and S_1 then gives

$$1 - xc_3' + w(d_1 - \tfrac{1}{2}c_3'),$$

so that we get again a finite value unless $c_3' = 0$. If $a_1' = c_3' = 0$ then from (7) $b_2' = 0$ also. Thus we must always have $b_2' = 0$. $S_2 = 0$ and $S_3 = 0$ are now satisfied by $x = z = 0$, and S_1 gives then a finite value of $y:w$ unless $b_1 = 0$. The quadrics S are now merely functions of x, z, w and have in general no common solutions.

From $b_2' = 0$ and (7) we find $a_1 : b_2 : c_3 : d_4 = 2 : -3 : -7 : -6$; and we may assume ($\alpha$) that these constants have these actual values, or (β) that they all vanish.

In case (α) the θ's are

$$\tfrac{4}{5}xz + (2x + c_1z)\,w + d_1w^2, \quad \tfrac{4}{5}yz + q + (9x - 3y)\,w, \quad \tfrac{4}{5}z^2 + 7zw, \quad -\tfrac{6}{5}zw - 3w^2.$$

It is convenient to make a substitution $(x + \lambda w, x)$ so as to remove d_1 thereby in general reintroducing d_2; and then to make a substitution $(y + \lambda z, y)$, $(y + \lambda y + \frac{1}{2}\lambda^2 z, y)$ which can be so chosen as to remove c_1, without altering q. The θ's are now

$$\tfrac{4}{5}xz + 2xw, \quad \tfrac{4}{5}yz + q - 3yw + d_2w^2, \quad \tfrac{4}{5}z^2 + 7zw, \quad -\tfrac{6}{5}zw - 3w^2.$$

The terms independent of x in the quintic f must now satisfy the differential equation

$$(-\tfrac{6}{5}yz - 3yw + d_2w^2)\frac{df}{dy} + (\tfrac{4}{5}z^2 + 7zw)\frac{df}{dz} - (\tfrac{6}{5}zw + 3w^2)\frac{df}{dw} = 0.$$

Two independent integrals of this equation are

$$z^3(z + 5w)^3 w^4, \quad 10y/w + d_2 \log(z + 5w)/z.$$

It is evident that from these integrals no quintic integral can be constructed, d_2 being by hypothesis not zero, so that the quintic surface divides by x and therefore degenerates.

In case (β) the θ's are

$$\tfrac{4}{5}xz + c_1zw + d_1w^2, \quad \tfrac{4}{5}yz + q + c_1xw, \quad \tfrac{4}{5}z^2, \quad -\tfrac{6}{5}zw,$$

when $d_1 \neq 0$ by hypothesis.

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We now remove c_1 by a substitution (x, w), thereby reintroducing d_2; we can then remove d_2 by the substitution $(x+\lambda z, x)$, $(y+\lambda x+\frac{1}{2}\lambda^2 z, y)$ used in case (α). Thus we have removed c_1 from θ_1, θ_2, without otherwise altering the form of the θ's.

The fundamental differential equation now has the three independent integrals:

$$z^2w^2, \quad U \equiv (4xz + d_1w^2)\, zw^2, \quad V \equiv 6qz^2w + 6d_1zw^3 + d_1^2w^5.$$

From these we can construct also the quintic integral

$$\phi \equiv \{V^2 - d_1U^2/z^2w^2\}/z^2w^2 \equiv 4\,\{9q^2z + 18d_1qxw^2 - 16d_1x^2w^2 - 6d_1^2yw^4\}.$$

It can be verified that these are the only quintic integrals, so that our surface is

$$f \equiv \lambda z^2w^2 + \mu U + \nu V + \rho W\phi = 0 \quad\text{.....(XXXVI).}$$

The birational transformation

$$U = YZ^2W^2, \quad V = XZ^2W^2, \quad z = Z, \quad w = W,$$

converts the surface into

$$\lambda Z^3 + \mu YZ^2 + \nu XZ^2 + \rho\,(X^2Z - d_1Y^3) = 0,$$

a cubic cone which is in general non-singular.

The quintic has a triple point of a complicated type on the conic at $x = z = w = 0$.

This completes the case when the double conic does not degenerate and the surface has no singularity distinct from the conic. We have found five surfaces represented by (XXXII)—(XXXVI).

§ 5. QUINTICS WITH A DOUBLE CONIC, CONSISTING OF TWO DISTINCT INTERSECTING STRAIGHT LINES, BUT WITH NO DISTINCT MULTIPLE POINT.

We take the double lines to be $xy = 0$, $w = 0$, so that the quintic is of the form

$$f \equiv x^2y^2u_1 + xyu_2w + u_3w^2 + v_2w^3 + v_1w^4 + \kappa w^5 = 0.$$

As at the beginning of § 4 we reduce the θ's to the form

$$xu + \alpha xy + \Theta_1 w + d_1w^2, \quad yu + \beta xy + \Theta_2 w + d_2w^2, \quad zu + \gamma xy + \Theta_3 w + d_3w^2, \quad wv + \tfrac{1}{2}d_4w^2,$$

where $\Theta_i = a_ix + b_iy + c_iz$, and u, v, are also linear functions of x, y, z.

Now the conic $\theta_1 = x = 0$ must be adjoint along the section of f by $x = 0$, and as this section consists in part of the double straight line $x = w = 0$, the conic $\theta_1 = x = 0$ must reduce to $w^2 = 0$; i.e. $b_1 = c_1 = 0$; similarly $a_2 = c_2 = 0$. Similar reasoning shews that for all values of $\lambda : \mu$ the conic $\lambda\theta_1 + \mu\theta_4 = 0$, $\lambda x + \mu w = 0$ must be 0^2 along $x = w = 0$,

$$\therefore\ x(-v+u) + \alpha xy \text{ must be } 0^2 \text{ along } x = w = 0, \qquad \therefore\ v = u + \alpha y + \alpha' x,$$

and similarly $\qquad v = u + \beta x + b'y, \qquad \therefore\ v = u + \beta x + \alpha y.$

The subsidiary differential equation (B), then gives $4u + \beta x + \alpha y + v = 0$, whence

$$u = -\tfrac{2}{5}(\beta x + \alpha y), \qquad v = \tfrac{3}{5}(\beta x + \alpha y).$$

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As before we have

$$\alpha \frac{du_1}{dx} + \beta \frac{du_1}{dy} + \gamma \frac{du_1}{dz} = 0, \text{ and } a_1 + b_2 + c_3 + d_4 = 0 \quad \text{.....(8).}$$

The θ's are now:

$$-\tfrac{2}{3}x(\beta x + \alpha y) + \alpha xy + a_1 xw + d_1 w^2, \qquad -\tfrac{2}{3}y(\beta x + \alpha y) + \beta xy + b_2 yw + d_2 w^2,$$

$$-\tfrac{2}{3}z(\beta x + \alpha y) + \gamma xy + (a_3 x + b_3 y + c_3 z)\, w + d_3 w^2, \qquad \tfrac{2}{3}w(\beta x + \alpha y) + \tfrac{1}{2}d_4 w^2.$$

As before we can remove d_1, d_2, d_3 if the three quadrics

$$S_1 \equiv \theta_1 - x\theta_4/w = 0, \quad S_2 \equiv \theta_2 - y\theta_4/w = 0, \quad S_3 \equiv \theta_3 - z\theta_4/w = 0,$$

have a common point off $w = 0$. Now these quadrics are

$$S_1 \equiv -\beta x^2 + a_1' xw + d_1 w^2, \quad S_2 \equiv -\alpha y^2 + b_2' yw + d_2 w^2,$$

$$S_3 \equiv -z(\beta x + \alpha y) + \gamma xy + (a_3 x + b_3 y + c_3' z)\, w + d_3 w^2 = 0 \ldots(9),$$

where as before a_1', b_2', c_3' are written for $a_1 - \frac{1}{2}d_4$, $b_2 - \frac{1}{2}d_4$, $c_3 - \frac{1}{2}d_4$.

Hence we can remove d_1 in two ways if $\beta \neq 0$, and d_2 in two ways if $\alpha \neq 0$.

It is convenient at this stage to distinguish three cases according as (I) neither α nor β vanishes, (II) one only vanishes, say α, (III) both vanish; and to subdivide the third case again according as (III A) $\gamma \neq 0$, or (III B) $\gamma = 0$.

Case I: $\alpha \neq 0$, $\beta \neq 0$.

We can remove d_1, d_2. We suppose this done; then $x = y = 0$ lies on S_1, S_2 and meets S_3 where $c_3' z + d_3 w = 0$; thus we can remove d_3 unless $c_3' = 0$. If $c_3' = 0$, then $x = 0$, $-\alpha y + b_2' w = 0$, lies on S_1, S_2 and meets S_3 where $-b_2' z + (b_3 b_2'/\alpha + d_3)\, w = 0$; thus we can remove d_3 unless also $b_2' = 0$, and similarly unless $a_1' = 0$. Thus we can remove d_3 unless $a_1' = b_2' = c_3' = 0$, whence $a_1 = b_2 = c_3 = d_4 = 0$.

Take *first* the case, which proves to be simpler, in which $d_3 \neq 0$, and may conveniently be taken to be unity, and $a_1 = b_2 = c_3 = d_4 = 0$; we can further remove γ by a transformation $(z + \lambda x,\ x)$. The θ's are now:

$$-\tfrac{2}{3}x(\beta x + \alpha y) + \alpha xy, \quad -\tfrac{2}{3}y(\beta x + \alpha y) + \beta xy, \quad -\tfrac{2}{3}z(\beta x + \alpha y) + (a_3 x + b_3 y)\, w + w^2, \quad \tfrac{2}{3}w(\beta x + \alpha y).$$

The coefficients of w^6, w^5, w^4, w^3 in the fundamental equation now give the equations:

$$\frac{d}{dz} v_1 = 0,$$

$$3\kappa(\beta x + \alpha y) + (a_3 x + b_3 y)\frac{d}{dz} v_1 + \frac{d}{dz} v_2 = 0,$$

$$2(\beta x + \alpha y)\, v_1 + xy\left(\alpha \frac{d}{dx} + \beta \frac{d}{dy}\right) v_1 + (a_3 x + b_3 y)\frac{d}{dz} v_2 + \frac{d}{dz} u_3 = 0,$$

$$(\beta x + \alpha y)\, v_2 + xy\left(\alpha \frac{d}{dx} + \beta \frac{d}{dy}\right) v_2 + (a_3 x + b_3 y)\frac{d}{dz} u_3 + \frac{d}{dz} xyu_2 = 0.$$

From the first we infer that v_1 is independent of z, from the second and third that v_2 and u_3 are at most linear in z.

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In the last equation the coefficients of x^2z, y^2z, xyz give the equations

$$\bar{g}=0, \quad \bar{f}=0, \quad 2\alpha\bar{g}+2\beta\bar{f}+c=0,$$

where $2\bar{g}$, $2\bar{f}$ are the coefficients of xz, yz in v_2 and c is the coefficient of z^2 in u_2.

Hence $c=0$, so that the whole quintic is linear at most in z, and is accordingly rational or a cone.

We thus revert to the *second* case, in which d_3 can be removed.

By a substitution $(z+\lambda x, x)$ we can remove a_3 unless $c_3=a_1$, and similarly we can remove b_3 unless $c_3=b_2$.

Let us suppose *first* that we have removed a_3, b_3.

If we make the transformation $\beta x-(a_1-\frac{1}{2}d_4)w=\beta x'$, we do not reintroduce any of d_1, d_2, d_3; we may introduce a term $b_3'yw$ in θ_3, and the constants a_1, b_2, c_3, d_4 acquire new values, say a secondary system, viz. these are

$$a_1-\tfrac{7}{5}(a_1-\tfrac{1}{2}d_4), \quad b_2+\tfrac{3}{5}(a_1-\tfrac{1}{2}d_4), \quad c_3-\tfrac{2}{5}(a_1-\tfrac{1}{2}d_4), \quad d_4+\tfrac{6}{5}(a_1-\tfrac{1}{2}d_4).$$

Similarly by the transformation $\alpha y-(b_2-\frac{1}{2}d_4)w=\alpha y'$ we replace a_1, b_2, c_3, d_4, by a tertiary system

$$a_1+\tfrac{3}{5}(b_2-\tfrac{1}{2}d_4), \quad b_2-\tfrac{7}{5}(b_2-\tfrac{1}{2}d_4), \quad c_3-\tfrac{2}{5}(b_2-\tfrac{1}{2}d_4), \quad d_4+\tfrac{6}{5}(a_1-\tfrac{1}{2}d_4).$$

Now by considering the coefficients of w^5, w^6 in the fundamental differential equation, we obtain as before (equation (5) of § 4) the alternative conditions

$$a_1+2d_4=0, \text{ or } b_2+2d_4=0, \text{ or } c_3+2d_4=0, \text{ or } d_4=0 \quad\text{.....(10).}$$

Since this condition is independent of the existence of a_3 and b_3 it must be equally true for the secondary and tertiary systems of constants. For the primary system we need clearly only consider the *three* alternatives $a_1+2d_4=0$, $c_3+2d_4=0$, $d_4=0$; but for the secondary and tertiary systems we must consider all four. Applying the conditions (10) to the primary and secondary systems we obtain equations sufficient to determine the constants; on solving them we get nine possible systems, viz.:

$$\left.\begin{array}{rrrrl}
0, & 1, & -1, & 0 & \text{(i)}\\
1, & -3, & 2, & 0 & \text{(ii)}\\
1, & 1, & -2, & 0 & \text{(iii)}\\
4, & -11, & 9, & -2 & \text{(iv)}\\
2, & 2, & -3, & -1 & \text{(v)}\\
1, & -4, & 6, & -3 & \text{(vi)}\\
3, & -7, & 8, & -4 & \text{(vii)}\\
1, & 1, & -4, & 2 & \text{(viii)}\\
0, & 0, & 0, & 0 & \text{(ix)}
\end{array}\right\} \quad\text{.....(I).}$$

In the last two cases the secondary and tertiary systems are the same as the primary

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system. In the remaining seven cases the values of the tertiary constants (omitting common numerical factors) are respectively

$$\left.\begin{array}{rrrrl} 3, & -2, & -7, & 6 & \text{(i)} \\ 2, & -3, & -8, & 9 & \text{(ii)} \\ 4, & -1, & -6, & 3 & \text{(iii)} \\ 2, & -3, & -13, & 14 & \text{(iv)} \\ 7, & -3, & -8, & 4 & \text{(v)} \\ 1, & 1, & -14, & 12 & \text{(vi)} \\ 0, & 0, & 1, & 1 & \text{(vii)} \end{array}\right\} \quad \text{.....(II).}$$

We now see that the conditions (10) are only satisfied in cases (iii) and (v). In both these cases the tertiary constants are such that the constant a_3 if it has been introduced can be removed by a substitution $(z+\lambda x, z)$. Thus the tertiary system can be treated as the original primary system. Repeating the first transformation in these two cases we get the two sets of values 1, 1, -14, 12 and 0, 0, 1, -1 which obviously fail. Thus all the cases (i)—(vii) fail, and we have only to consider cases (viii) and (ix).

Now if the quintic is neither rational nor a cone it must contain a term at least quadratic in z. The terms highest in z in the quintic satisfy the fundamental differential equation with the constants γ, a_3, b_3 omitted, since the corresponding terms in the differential operator lower the degree in z; among the terms highest in z, those highest in w satisfy the equation

$$\left(a_1\frac{d}{dx}+b_2\frac{d}{dy}+c_3\frac{d}{dz}+\tfrac{1}{2}d_4\frac{d}{dw}\right)f=0 \quad \text{.....(11).}$$

The possible terms quadratic or cubic in z are from the form of f one or more of

$$xyz^2w, \quad xz^2w^2, \quad yz^2w^2, \quad z^2w^2, \quad z^2w^3;$$

and it is evident that in case (viii) none of these can satisfy (11).

Thus we are reduced to case (ix). In this case the coefficient of w^5 in the fundamental differential equation gives $\kappa=0$, and then the coefficient of w^4 gives

$$2(\beta x+\alpha y)\,v_1+xy\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)v_1=0,$$

which is impossible since the first term cannot divide by xy.

Thus if d_1, d_2, d_3 can be removed and also a_3, b_3 can be removed there is no solution.

Let us suppose *next* that one of a_3, b_3 can be removed, say a_3, but not b_3. Then $c_3=b_2$; we may still use the first transformation just employed, viz. $\{\beta x-(a_1-\tfrac{1}{2}d_4)w, \beta x\}$, so that the possible systems of values of the constants a_1, b_2, c_3, d_4 are the 9 just given in tabular form. But in none of these but the last is $c_3=b_2$; so that if $a_3=0$, $b_3\neq 0$ all the four constants must vanish. Similarly if $a_3\neq 0$, $b_3=0$. If neither a_3 nor b_3 can be removed then $c_3=a_1=b_2$; and then the conditions (10) shew that we must also have $a_1=b_2=c_3=d_4=0$. We can now employ a transformation $(z+\lambda w, z)$, to remove a_3 or b_3 without introducing d_3 and so reduce this case to the preceding one in which only one of a_3, b_3 does not vanish.

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In this case the terms in f which are independent of y satisfy the differential equation

$$\left(-2x\frac{d}{dx}-2z\frac{d}{dz}+3w\frac{d}{dw}\right)f=0,$$

of which the most general integral is $(x, z)^2 w^3$. Thus the term κw^5 vanishes, v_1, v_2 divide by y and the former may be taken to be simply y.

The coefficient of w^4 in the differential equation now gives

$$\left\{2(\beta x+\alpha y)+xy\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\right\}y+b_3y\frac{d}{dz}v_2=0.$$

Everything in this divides by y^2 except $3\beta xy$, therefore $\beta=0$ contrary to hypothesis.

The whole case when $\alpha\neq 0$, $\beta\neq 0$, accordingly leads to no solution.

Case II: $\alpha=0$, $\beta\neq 0$.

We can always remove d_1; we can remove d_2 unless $b_2-\frac{1}{2}d_4=0$.

Let us suppose *first* that we can remove d_2 as well as d_1.

The quadrics S are now, d_1 and d_2 having been removed,

$$S_1\equiv-\beta x^2+(a_1-\tfrac{1}{2}d_4)xw,\quad S_2\equiv(b_2-\tfrac{1}{2}d_4)yw,\quad S_3=-\beta xz+\gamma xy+\{a_3x+b_3y+(c_3-\tfrac{1}{2}d_4)z\}w+d_3w^2.$$

As before these meet on a point off $w=0$, so that d_3 can be removed, unless $a_1=c_3=\frac{1}{2}d_4$, whence $a_1:b_2:c_3:d_4=1:-4:1:2$.

Let us first suppose that d_3 cannot be removed, so that the four constants a_1, b_2, c_3, d_4 (α) are 1, -4, 1, 2, or (β) all vanish.

In case (α) the θ's are

$$-\tfrac{2}{3}\beta x^2+xw,\quad \tfrac{3}{3}\beta xy-4yw,\quad -\tfrac{2}{3}\beta yz+\gamma xy+(a_3x+b_3y+z)w+d_3w^2,\quad \tfrac{3}{3}\beta xw+w^2.$$

We can then remove b_3 by a substitution $(z+\lambda y, z)$ and a_3 by a substitution $(z+\lambda w, z)$. The terms in the quintic which are independent of y now satisfy the differential equation

$$\left\{(-\tfrac{2}{3}\beta x^2+xw)\frac{d}{dx}+(-\tfrac{2}{3}\beta xz+zw+d_3w^2)\frac{d}{dz}+(\tfrac{3}{3}\beta xw+w^2)\frac{d}{dw}\right\}f=0.$$

This has the two independent integrals $\beta\log x^2w^2-5w/x$, $2\beta z/x-d_3(w/x)^2$, from which no quintic integral can be constructed, so that the whole quintic divides by y and therefore degenerates.

In case (β) we can remove a_3 as before and we can remove γ by a substitution $(z+\lambda y, z)$. The θ's are now

$$-\tfrac{2}{3}\beta x^2,\quad \tfrac{3}{3}\beta xy,\quad -\tfrac{2}{3}\beta xz+b_3yw+d_3w^2,\quad \tfrac{3}{3}\beta xw.$$

The fundamental differential equation now has the two obvious integrals x/w, x^2y^3, and it can be verified that a third integral is $\{2\beta xz-(b_3y+d_3w)w\}y^2$. The only quintic integral which can be constructed from these is $(x, w)^2y^3$, so that f degenerates.

We thus revert to the case in which d_3 can be removed, if d_1, d_2 can; and consider it next.

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We can always remove a_3 by a substitution $(z+\lambda x, z)$ unless $c_3 = a_1$. Also since d_1, d_2, d_3 have been removed we have the conditions (10). Combining these with $c_3 = a_1$ we find that the constants a_1, b_2, c_3, d_4 have one of the four sets of values

$$\left.\begin{array}{lllll} 0, & 0, & 0, & 0 & \text{(ix)} \\ 1, & -2, & +1, & 0 & \text{(x)} \\ 2, & -3, & 2, & -1 & \text{(xi)} \\ 1, & -4, & 1, & 2 & \text{(xii)} \end{array}\right\} \quad \text{.....(III).}$$

If a_3 is removed we can use the first transformation employed in Case I., viz. $\{\beta x - (a_1 - \frac{1}{2}d_4)w, \beta x\}$, and by the same argument as before arrive at the values given in table (I). In Case I., however, we omitted the possibility $b_2 + 2d_4 = 0$, on account of symmetry; at present we no longer have symmetry and must therefore take this possibility into account. This leads to three new sets of values, viz.:

$$\left.\begin{array}{lllll} 1, & 6, & -4, & -3 & \text{(xiii)} \\ 3, & 8, & -7, & -4 & \text{(xiv)} \\ 1, & -4, & 1, & 2 & \text{(xv)} \end{array}\right\} \quad \text{.....(IV).}$$

Also as before one of the terms z^2xyw, z^2xw^2, z^2yw^2, z^2w^3, z^2w^3 must exist in the quintic and must satisfy the equation

$$\left(a_1x\frac{d}{dx} + b_2y\frac{d}{dy} + c_3z\frac{d}{dz} + \tfrac{1}{2}w_4\frac{d}{dw}\right)f = 0;$$

i.e. one of the five constants

$$a_1 + b_2 + 2c_3 + \tfrac{1}{2}d_4, \quad a_1 + 2c_3 + d_4, \quad b_2 + 2c_3 + d_4, \quad 3c_3 + d_4, \quad 2c_2 + \tfrac{3}{2}d_4,$$

must vanish, or more simply one of

$$2c_3 - d_4, \quad b_2 - c_3, \quad a_1 - c_3, \quad 3c_3 + d_4, \quad 4c_2 + 3d_4,$$

must vanish. This new condition is satisfied in all cases of table (III); otherwise only in (ix) of table (I) and (xv) of table (IV). Thus the only possible sets of values of the four constants are those given in table (III), where in cases (ix) and (xii) a_3 may or may not vanish; in cases (x) and (xi) a_3 cannot vanish.

In case (x) we can remove b_3 by a substitution $(z+\lambda y, z)$, and then the θ's are

$$-\tfrac{3}{5}\beta x^2 + xw, \quad \tfrac{3}{5}\beta xy - 2yw, \quad -\tfrac{3}{5}\beta xz + \gamma xy + a_3xw + zw, \quad \tfrac{3}{5}\beta xw.$$

The fundamental differential equation then admits of the three independent integrals

$$(\beta x - w)^3 w^2, \quad x^2y(\beta x - w)w, \quad 2z/x + \gamma y/w + a_3 \log y/w.$$

The letter z only occurs in the last integral, which is logarithmic; hence the only algebraic integral must be independent of z, and the quintic is a cone.

In case (xi) we have similarly two algebraic integrals which are independent of z and a logarithmic integral, viz.:

$$x^3y^2, \quad x(\beta x - \tfrac{5}{2}w)^2 w^3, \quad 5z/2x + \gamma y/w + a_3 \log y/w.$$

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In case (xii) it is simpler to consider the differential equation satisfied by the terms in the quintic independent of y, viz.:

$$\left\{(-\tfrac{3}{5}\beta x^2 + xw)\frac{d}{dx} + (-\tfrac{3}{5}\beta xz + a_3 xw + zw)\frac{d}{dz} + (\tfrac{3}{5}xw + w^2)\frac{d}{dw}\right\} f = 0.$$

This has the two independent integrals

$$(\beta z - a_3 w)/x, \quad -5w/x + \beta \log x^2 w^2.$$

From these evidently no quintic term can be formed, whether a_3 vanishes or not, so that the original quintic divides by y.

In case (ix) we can remove γ by a substitution $(z + \lambda y,\ z)$ and a_3 by a substitution $(z + \lambda w,\ z)$; the θ's then reduce to:

$$-\tfrac{3}{5}\beta x^2, \quad \tfrac{3}{5}\beta xy, \quad -\tfrac{3}{5}\beta xz + b_3 yw, \quad \tfrac{3}{5}\beta xw;$$

and the fundamental differential equation has the three independent integrals

$$x^2 y^2, \quad y/w, \quad U \equiv (2\beta xz - b_3 yw)/x^2.$$

If $b_3 \neq 0$ the only quintics which can be constructed from these three integrals are $x^3(y, w)^2$, $Ux(y, w)^2$, so that the quintic degenerates.

If $b_3 = 0$, the general quintic integral is $(x,\ z \overset{3}{\between} y,\ w)^2$, so that $x = z = 0$ is a triple line, a case previously considered.

This completes the case when $\alpha = 0$, $\beta \neq 0$, and d_1, d_2 can be removed; and there is in this case no solution of our problem.

We have now to consider *secondly* the case when $\alpha = 0$, $\beta \neq 0$, $d_1 = 0$ but $d_2 \neq 0$; then as we have seen $b_2 - \frac{1}{2}d_4 = 0$.

We make no assumption for the present about d_3, and express first the condition that a term quadratic or cubic in z exists. The highest term in z as we have seen must satisfy the fundamental differential equation modified by the omission of the coefficients γ, a_3, b_3, d_3. If $z^3 w^2$ exists we have at once $3c_3 + d_4 = 0$, whence $a_1 : b_2 : c_3 : d_4 = 7 : -3 : 2 : -6$.

If $z^3 w^2$ does not exist there must be quadratic terms of the form

$$cxyz^2 w + (\kappa_1 x + \kappa_2 y)\, z^2 w^2 + \bar{c} z^2 w^3.$$

Applying the differential equation and equating coefficients we obtain the four equations

$$c(b_2 - c_3) - \beta\kappa_2 = 0,$$

$$cd_1 - \kappa_1(b_2 - c_3) + \beta\bar{c} = 0,$$

$$(3b_2 + 2c_3)\kappa_2 = 0,$$

$$d_1\kappa_2 + (3b_2 + 2c_3)\bar{c} = 0.$$

These are easily seen to be consistent only if $b_2 = c_3$, or $3b_2 + 2c_3 = 0$, whence

$$a_1 : b_2 : c_3 : d_4 = -4 : 1 : 1 : 2, \ \textit{or} \ = 3 : -2 : 3 : -4.$$

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We thus have altogether 4 possible systems of values of the constants a_1, b_2, c_3, d_4, viz.:

$$\begin{array}{rrrrl} 7, & -3, & 2, & -6 & \text{(i)}, \\ -4, & 1, & 1, & 2 & \text{(ii)}, \\ 3, & -2, & 3, & -4 & \text{(iii)}, \\ 0, & 0, & 0, & 0 & \text{(iv)}. \end{array}$$

In case (i), since none of $c_3 - b_2$, $c_3 - \frac{1}{2}d_4$, $c_3 - a_1$ vanishes we can successively remove b_3, d_3, a_3; the θ's then reduce to

$$-\tfrac{2}{5}\beta x^2 + 7xw, \quad \tfrac{3}{5}\beta xy - 3yw + d_2w^2, \quad -\tfrac{2}{5}\beta xz + \gamma xy + 2zw, \quad \tfrac{3}{5}\beta xw - 3w^2.$$

The fundamental differential equation has a logarithmic integral $d_3 \log \dfrac{\beta x - 10w}{x} - 10y/w$, and two algebraic integrals $U \equiv x^3(\beta x - 10w)^3 w^4$, $V \equiv (\gamma xy - zw - \frac{1}{5}d_2\gamma xw)\, xw\, (\beta x - 10w)$. The quintic must be a function of U, V only; if $\gamma \neq 0$, V is the only possible quintic integral and the quintic accordingly degenerates. If $\gamma = 0$, V reduces to $xzw^2(\beta x - 10w)$, and from this and V we deduce z^2w^2, but no other quintic terms, so that again the quintic degenerates.

Cases (ii) and (iii) can be interchanged by the transformation $\beta x - (a_1 - \frac{1}{2}d_4)\, w = \beta x'$, already employed more than once; so that we need only consider one of them, say case (ii). We cannot in general remove d_3 or b_3, but can remove a_3 by a substitution $(z + \lambda x, z)$. The terms independent of x in the quintic f satisfy the simplified differential equation

$$(y + d_2w)\frac{df}{dy} + (z + b_3y + d_3w)\frac{df}{dz} + w\frac{df}{dw} = 0.$$

This has the two independent integrals

$$y/w - d_2 \log w, \quad b_3(y/w)^2 + 2d_3y/w - 2d_2z/w,$$

of which the first is logarithmic and the second is of zero dimensions. Hence no quintic integral can be constructed from them, and the whole quintic degenerates by having x as a factor.

It remains to consider case (iv) in which the constants a_1, b_2, c_3, d_4 all vanish. In this case we can remove d_3 by a substitution $(z + \lambda y, z)$, and then a_3 by a substitution $(z + \mu w, z)$; and the θ's are then:

$$-\tfrac{2}{5}\beta x^2, \quad \tfrac{3}{5}\beta xy + d_2w^2, \quad -\tfrac{2}{5}\beta xz + \gamma xy + b_3yw, \quad \tfrac{3}{5}\beta xw.$$

The corresponding fundamental differential equation has the three independent integrals

$$y^2w^2, \quad U \equiv (\beta xy - d_2w^2)\, xw, \quad V \equiv \{\beta^2x^2z - \beta\gamma x^2y - \tfrac{1}{2}\beta b_3xyw + \tfrac{1}{2}\gamma d_2xw^2 + \tfrac{1}{6}b_3d_2w^3\}\, w^2.$$

From these we can in general construct no quintic integral which is quadratic in z, so that the surface is rational or a cone. But, if $\gamma = b_3 = 0$, we have an integral z/x and the general form of the quintic is then

$$f \equiv (\beta xy - d_2w^2)^2 (x, z)^1 + (\beta xy - d_2w^2)(x, z)^2 w + (x, z)^3 w^2 = 0 \ldots\ldots \text{(XXXVII)}.$$

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The transformation $\beta xy - d_2 w^2 = YW$, $x = X$, $z = Z$, $w = W$ converts this into

$$Y^2 (X, Z)^1 + Y (X, Z)^2 + (X, Z)^3 = 0,$$

a cubic cone, which is in general non-singular.

The quintic has a triple point at $x = z = w = 0$; and is further characterized by the existence on it of a family of pairs of conics, which are cut out by planes of the family $x + \lambda z = 0$. These conics correspond to the generators of the cubic cone into which the surface has been transformed.

We have now completed the discussion of the case $\alpha = 0$, $\beta \neq 0$, and have found merely the surface (XXXVII) just considered.

Case III A: $\alpha = \beta = 0$; $\gamma \neq 0$.

We may conveniently take γ to be unity.

From the terms independent of w in the differential equation we have $\frac{du_1}{dz} = 0$, so that u_1 is independent of z.

From the coefficient of w in the differential equation we have

$$xy \frac{du_2}{dz} + \left\{ a_1 x \frac{d}{dx} + b_2 y \frac{d}{dy} + (a_3 x + b_2 y + c_3 z) \frac{d}{dz} \right\} x^2 y^2 u_1 = 0,$$

so that u_2 is at most linear in z; and similarly from the coefficient of w^2 we infer that u_3 is at most quadratic in z.

Now we have seen that at least one term in the quintic must be quadratic or cubic in z; hence the only possible terms of this type are $\kappa_1 x z^2 w^2$, $\kappa_2 y z^2 w^2$, $\bar{c} z^2 w^3$. Hence as before we must have one of the three alternative conditions

$$a_1 + 2c_3 + d_4 \equiv c_3 - b_2 = 0, \quad c_3 - a_1 = 0, \quad 4c_3 + 3d_4 = 0 \quad \text{.....(12).}$$

We now consider as before the possibility of removing the constants d_1, d_2, d_3 by substitutions $(x + \lambda w, x)$, $(y + \mu w, y)$, $(z + \nu w, z)$.

First: let d_1, d_2 be removed. Then the four fundamental constants a_1, b_2, c_3, d_4 cannot vanish, since we should then have $\frac{df}{dz} = 0$, giving a cone.

The coefficient of z^2 in the fundamental differential equation gives

$$(c_3 - b_2) \kappa_1 = 0, \quad (c_3 - a_1) \kappa_2 = 0, \quad (4c_3 + 3d_4) \bar{c} = 0 \quad \text{.....(13).}$$

Also the coefficient of $xyzw^3$ in the differential equation gives

$$\bar{c} + a_3 \kappa_2 + b_3 \kappa_1 = 0.$$

The last equation shews that if κ_1, κ_2 both vanish $\bar{c}$ vanishes also: therefore one of κ_1, κ_2 must not vanish, say κ_1, and therefore from (13)

$$b_2 = c_3 \quad \text{.....(14).}$$

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We can remove d_3 by a substitution $(z+\nu w, z)$ unless $c_3 = \frac{1}{2}d_4$. If this equation holds, then the four fundamental constants are $-4, 1, 1, 2$; since $c_3 \neq a_1$ we can remove a_3 and the θ's are then:

$$-4xw, \quad yw, \quad xy + (b_3 y + z)\,w + d_3 w^2, \quad w^2.$$

The fundamental equation now has the three independent integrals

$$xw^4, \quad y/w, \quad z/w + \tfrac{1}{6}xy/w^2 - (b_3 y/w + d_3)\log w.$$

The first two are independent of z; hence the third must occur in the quintic, and must therefore be non-logarithmic, i.e. $b_3 = d_3 = 0$, and the third integral reduces to $(5zw + xy)/w^2$. Any quintic integral constructed from these three integrals, of which the last two are of zero dimensions, must have x as a factor, so that the quintic degenerates.

We thus revert to the case $c_3 \neq \frac{1}{2}d_4$, in which d_3 can always be removed. The coefficients of w^5 and w^6 in the fundamental differential equation then give as before the alternative conditions

$$a_1 + 2d_4 = 0, \text{ or } b_2 + 2d_4 = 0, \text{ or } c_3 + 2d_4 = 0, \text{ or } d_4 = 0 \quad \text{.....(15).}$$

The first, with (14), gives $c_3 = \frac{1}{2}d_4$ and can therefore be rejected; the second or third gives for the four fundamental constants the values (α) $3, -2, -2, 1$ and the last gives (β) $2, -1, -1, 0$.

In case (α) we can remove a_3 by a substitution $(z+\lambda x, z)$, and the θ's are

$$3xw, \quad -2yw, \quad xy + (b_3 y - 2z)\,w, \quad \tfrac{1}{2}w^2.$$

We now have the three integrals of the fundamental differential equation

$$x^2y^3, \quad yw^4, \quad -zw^4 + \tfrac{2}{5}xyw^3 + 2b_3 yw^4 \log w.$$

As before, since z must occur in the quintic, the last integral must be non-logarithmic, so that $b_3 = 0$, and the quintic is then

$$f \equiv x(2xy - 5zw)^2 + y(xy, w^2)^2 + (2xy - 5zw)(xy, w^2)^1 w = 0 \text{.....(XXXVII A).}$$

The quintic is quadratic in x, w^2; hence the double line $x = w = 0$ is tacnodal and can be regarded as the limit of two coincident double lines; we therefore revert to the case of three intersecting double lines, two of which coincide, a case already treated in "Quintics I," § 6; the surface which we have found is in fact reducible by the linear substitution

$$2x = X, \quad y = Z, \quad 5z = W, \quad w = Y,$$

to the surface (VII) of that article.

In case (β) we can remove a_3 as before, so that the θ's are

$$2xw, \quad -yw, \quad xy + (b_3 y - z)\,w, \quad 0.$$

The differential equation has the integrals xy^2, w, $-\frac{1}{2}x/w + z/y + b_3 \log y$. As before b_3 must vanish; and we now obtain

$$f \equiv ax(xy - 2wz)^2 + bxy(xy - 2wz)\,w + cxy^2w^2 + \kappa w^5 = 0,$$

so that $x = w = 0$ is a triple line, a case already considered.

This completes the case when $\alpha = \beta = 0$, $\gamma = 1$ and d_1, d_2 can be removed.

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Secondly, let one of d_1, d_2 be removable and the other not: let us take $d_1 = 0$, $d_2 \neq 0$, then $b_2 = \frac{1}{2}d_4$.

We can always remove a_3 by a substitution $(y + \lambda w,\ y)$.

The conditions (12) now give three sets of values for a_1, b_2, c_3, d_4, viz., (α) -4, 1, 1, 2, (β) 3, -2, $+3$, -4, (γ) 0, 0, 0, 0.

In case (α) we can remove d_3 by a substitution $(z + \lambda y,\ z)$; the terms independent of x in the differential equation then satisfy the equation

$$\left\{(y + d_2 w)\frac{d}{dy} + (b_3 y + z)\frac{d}{dz} + w\frac{d}{dw}\right\} f = 0\,;$$

this equation has the two independent integrals

$$y/w - d_2 \log w, \quad 2d_2 z/w - b_3 y^2/w^2.$$

From these no quintic integral can be constructed, so that the original quintic divides by x and degenerates.

In case (β), since $c_3 \neq b_2$ or $\frac{1}{2}d_4$ we can remove b_3, d_3 by substitutions $(z + \lambda y,\ z)$, $(z + \mu w,\ z)$, and the θ's then reduce to

$$3xw, \quad -2yw + d_2 w^2, \quad xy + 3zw, \quad -2w^2.$$

The fundamental differential equation now has the integrals

$$x^2 w^3, \quad 2y/w + d_2 \log w, \quad xw\,(xy^2 - 2d_2 z w^2).$$

The only quintic integrals which can be formed from these are simply the first and third of these, so that the quintic degenerates.

In case (γ) we can remove d_3 by a substitution $(z + \lambda y,\ z)$, and b_3 by a substitution $(x + \mu w,\ x)$, so that the θ's are

$$0, \quad d_2 w^2, \quad xy, \quad 0.$$

The integrals of the fundamental differential equation are now

$$x, \quad w, \quad xy^2 - d_2 z w^2,$$

from which no quintic integral which is quadratic in z can be constructed.

Thirdly, let neither d_1 nor d_2 be removable; then $a_1 = b_2 = \frac{1}{2}d_4$, whence by (12) a_1, b_2, c_3, d_4 all vanish. We can remove a_3, b_3 by substitutions $(x + \lambda w,\ x)$, $(y + \mu w,\ y)$ and then d_3 by a substitution $(z + \nu x,\ z)$. The θ's are now

$$d_1 w^2, \quad d_2 w^2, \quad xy, \quad 0.$$

The fundamental differential equation has the integrals

$$d_2 x - d_1 y, \quad w, \quad 6d_1^2 z w^2 - 3d_1 x^2 y + d_2 x^3,$$

from which no quintic integral quadratic in z can be constructed; so there is no solution of our problem.

This completes the case in which $\alpha = \beta = 0$, $\gamma \neq 0$. We have found no new solution of our problem.

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Case III B: $\alpha = \beta = \gamma = 0$.

The θ's all divide by w, and we may for convenience omit this factor.

The terms independent of w in the differential equation give the equation

$$\left\{2a_1 + 2b_2 + a_1 x \frac{d}{dx} + b_2 y \frac{d}{dy} + (a_3 x + b_3 y + c_3 z) \frac{d}{dz}\right\} u_1 = 0,$$

whence we obtain the alternative conditions

$$3a_1 + 2b_2 = 0, \text{ or } 2a_1 + 3b_2 = 0, \text{ or } c_3 + 2d_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

We have as before the conditions necessary for the existence of a term, quadratic or cubic in z,

$$2c_3 - d_4 = 0, \text{ or } c_3 = a_1, \text{ or } c_3 = b_2, \text{ or } 4c_3 + 3d_4 = 0, \text{ or } 3c_3 + d_4 = 0 \ldots\ldots\ldots (17).$$

Solving these two systems of equations we get ten possible systems of values of a_1, b_2, c_3, d_4, viz.:

6,	− 9,	1,	2	(i),
− 9,	6,	1,	2	(ii),
2,	− 3,	2,	− 1	(iii),
− 3,	2,	2,	− 1	(iv),
2,	− 3,	− 3,	4	(v),
− 3,	2,	− 3,	4	(vi),
4,	− 6,	− 1,	3	(vii),
− 6,	4,	− 1,	3	(viii),
1,	− 1,	0,	0	(ix),
0,	0,	0,	0	(x),

where the systems with the even numbers (ii), (iv), (vi), (viii) only differ from the immediately preceding systems by an interchange of x and y.

We can now consider the possibility of removing the constants d_1, d_2, d_3.

(I) Let d_1, d_2, d_3 be removable; then from consideration of the coefficients of the two highest powers of w in the differential equation we have as before the alternative conditions

$$a_1 + 2d_4 = 0, \text{ or } b_2 + 2d_4 = 0, \text{ or } c_3 + 2d_4 = 0, \text{ or } d_4 = 0.$$

On inspection we see that these are only satisfied in cases (iii) (*or* iv), (vii) (*or* viii), (ix), (x). But the last leads to $\frac{df}{dz} = 0$, giving a cone.

Case (iii). We can remove b_3 by a substitution $(z + \lambda y, z)$; the θ's are now

$$2x, \quad -3y, \quad a_3 x + 2z, \quad -\tfrac{1}{2} w.$$

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The fundamental differential equation has the three independent integrals

$$x^2y^2, \quad xw^4, \quad a_3 \log x - 2z/x.$$

In order that z may occur the last integral must be non-logarithmic, so that $a_3 = 0$, and the integral is z/x.

The most general quintic is now a sum of terms of the form $(x, z)^3 y^2$, $(x, z)^2 yw^2$, $(x, z)^1 w^4$; expressing the condition for the existence of the double conic we have therefore for our quintic

$$f \equiv x^2y^2 (x, z)^1 + y (x, z)^2 w^2 + (x, z)^1 w^4 = 0.$$

This surface, like (XXXVII A), has a tacnodal line $y = w = 0$.

If the coefficient of x^2y^2 is not merely x it can be taken to be z, and we have the slightly simplified form

$$f \equiv x^2y^2z + y (x, z)^2 w^2 + (x, z)^1 w^4 = 0 \text{.....(XXXVIII A)}.$$

This is easily seen to be identical with surface (VII), of "Quintics I.," § 6. But if this coefficient happens to be x we have the special case

$$f \equiv x^3y^2 + y (x, z)^2 w^2 + (x, z)^1 w^4 = 0 \text{ .....(XXXVIII)},$$

which was omitted by error in "Quintics I.," § 6.

The birational transformation $x = X$, $z = Z$, $w = W$, $y = W^2/Y$, converts this last surface into

$$X^3 + Y (X, Z)^2 + Y^2 (X, Z)^1 = 0,$$

a cubic cone which is in general non-singular.

Case (vii). We can remove a_3, b_3 in the usual way, so that the θ's are

$$4xw, \quad -6yw, \quad -zw, \quad \tfrac{3}{2}w^2.$$

We have the three independent integrals x^2y^2, yw^4, z^3w^2, and the quintic is

$$f \equiv x^2y^2 + (axy + bz^2) zw^2 + cyw^4 = 0 \text{ .....(XXXIX)}.$$

The surface has a triple point at $x = z = w = 0$.

The birational transformation $x = X$, $z = Z$, $w = W$, $y = YW/X$ converts it into

$$X^2Y^2 + aXYZW + bXZ^3 + cYW^3 = 0.$$

This is a special case of the quartic which I call (E) ("Quartics," p. 343); and the transformation there employed converts it into a cubic cone which is in general non-singular.

Case (ix). We can remove a_3, b_3 and the θ's are merely

$$xw, \quad -yw, \quad 0, \quad 0.$$

The general integral is $\phi (xy, z, w)$ so that

$$f \equiv x^2y^2z + xy (xy, z^2)^1 w + z (xy, z^2)^2 w^2 + \lambda zw^4 + \kappa w^5 = 0 \text{ .....(XL)}.$$

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This surface is a specialization of (XXXII), of § 4 [see p. 85]; the double conic which was there assumed non-degenerate being in this case a pair of straight lines.

(II) Let d_1, d_2 be removable, but d_3 not removable. Then $c_3 = \frac{1}{2}d_4$, and we have one of cases (i) (*or* ii), (ix), (x): and the last fails as before.

Case (i). We can remove a_3, b_3 so that the θ's are proportional to

$$6x, \quad -9y, \quad z + d_3 w, \quad w.$$

The differential equation now has two algebraic integrals involving x, y, w only, and a logarithmic integral involving z, so that we have a failure as before.

Case (ix). We can remove a_3, b_3 so that the θ's are proportional to

$$x, \quad -y, \quad d_3 w, \quad 0.$$

This fails for the same reason as case (i).

(III) Let one of d_1, d_2 be removable, the other not; say, $d_2 = 0$, $d_1 \neq 0$, then $a_1 = \frac{1}{2}d_4$, and we have to consider cases (v) and (x).

Case (v). We can remove a_3 by a substitution $(z + \lambda x, z)$, thereby changing d_3 and then we can remove d_3 by a substitution $(z + \nu w, z)$. The θ's are now proportional to

$$2x + d_1 w, \quad -3y, \quad b_3 y - 3z, \quad 2w.$$

The differential equation has the two algebraic integrals $d_1 z/y - b_3 x/w$, $y^2 w^3$, and a logarithmic integral $2x/w - d_1 \log w$. The only quintic integrals which can be constructed from these are of the form

$$y^2 w^3, \quad (d_1 zw - b_3 xy)^2 w, \quad (d_1 zw - b_3 xy)\, yw^3,$$

all of which divide by w, whether b_3 vanish or not.

Case (x). We can remove d_3 by a substitution $(z + \lambda x, z)$. The θ's are now proportional to

$$d_1 w, \quad 0, \quad a_3 x + b_3 y, \quad 0.$$

The differential equation has the integrals y, w, $U \equiv 2d_1 zw - a_3 x^2 - 2b_3 xy$.

The most general quintic which can be constructed from these is

$$f \equiv U^2(\lambda y + \mu w) + U(y, w)^3 + (y, w)^5 = 0.$$

But since $y = w = 0$ is to be a double line either $\lambda = \mu = 0$, or $a_3 = 0$, and in either case $y = w = 0$ is at least a triple line.

(IV) Let neither of d_1, d_2 be removable; then $a_1 = b_2 = \frac{1}{2}d_4$, and the only possibility is case (x). We can remove d_3 as before by a substitution $(z + \lambda x, z)$. We may conveniently replace d_1, d_2 which do not vanish, each by unity. The θ's are now

$$w^2, \quad w^2, \quad (a_3 x + b_3 y)\, w, \quad 0,$$

where either of a_3, b_3 may vanish, but not both, as we should then have a cone. Let us suppose $a_3 \neq 0$, so that (modifying z) we can replace it also by unity.

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The fundamental differential equation has the three independent integrals:

$$x-y, \quad w, \quad U \equiv 2zw - 2xy + (1-b_3)\,y^2.$$

From these we can construct the quintic integral

$$f \equiv U^2\{\lambda(x-y)+\mu w\} + U(abcd\between x-y,\, w)^3 + (a',\, b',\, c',\, d',\, e',\, f'\between x-y,\, w)^5 = 0.$$

When $w=0$, this reduces to

$$\{-2xy+(1-b_3)\,y^2\}^2\lambda(x-y) + a\{-2xy+(1-b_3)\,y^2\}(x-y)^3 + a'(x-y)^5.$$

This to divide by x^2y^2, leading to $a=a'=0$, $\lambda(1-b_3)=0$; but, if $\lambda=0$, the surface degenerates, therefore $b_3=1$; the further condition that $xy=0$, $w=0$ should be double gives $b'=0$, and the surface then reduces to

$$f \equiv (xy-zw)^2(x-y,\, w)^1 + (xy-zw)(x-y,\, w)^3 w + (x-y,\, w)^5 w^2 = 0 \quad \text{.....(XLI).}$$

The birational transformation $xy-zw=ZW$, $x=X$, $y=Y$, $w=W$ converts it into

$$Z^2(X-Y,\ W)^1 + Z(X-Y,\ W)^3 + (X-Y,\ W)^5 = 0,$$

a cubic cone, in general non-singular.

The quintic has a triple point at $x=y=w=0$.

The pencil of planes $x-y+\lambda w=0$ cuts the surfaces in two families of conics given by the equation of the plane and an equation of the form $xy-zw+\mu w^2=0$.

We have now completed the case where $\alpha=\beta=\gamma=0$, finding four new surfaces (XXXVIII)—(XLI), of which (XXXVIII) properly belongs to the case of three intersecting double lines treated in "Quintics I.," § 6. Since when $\alpha=\beta=\gamma=0$ the θ's all divide by w, all the surfaces obtained should belong to the class treated in "Quintics I.," § 10. On inspection we see at once that (XXXVIII) is a special case of (XVIII), (XXXIX) is a special case of (XXI) (the coefficients ρ, σ of that equation being replaced by zero), (XL) is a special case of (XVII). If in (XLI) we write $x-y=2X$, $x+y=2Z$, $z=W$, $w=Y$ we obtain a surface which is a special case of (XVI).

The results of this section (III B), might have been obtained by assuming the result of "Quintics I.," § 10, and examining which of the surfaces there obtained had, or could have, double conics on them; but I have preferred to carry out the independent investigation of this paper, as a check on the work of the earlier paper.

We have now completed the case when the double conic breaks up into two distinct straight lines.

§ 6. QUINTICS WITH A DOUBLE CONIC, CONSISTING OF TWO COINCIDENT STRAIGHT LINES, BUT WITH NO DISTINCT MULTIPLE POINT.

The conic being $x^2=0$, $w=0$, the quintic can be written in the form

$$f \equiv x^4u_1 + x^2u_2w + u_3w^3 + v_2w^2 + v_1w^4 + \kappa w^5 = 0.$$

As at the beginning of § 5, we reduce the θ's to the form

$$xu + \alpha x^2 + a_1xw + d_1w^2, \quad yu + \beta x^2 + \Theta_2 w + d_2w^2, \quad zu + \gamma x^2 + \Theta_3 w + d_3w^2, \quad wv + \tfrac{1}{2}d_4w^2,$$

where $\Theta_i = a_ix + b_iy + c_iz\ (i=2,\,3)$ and u, v are also linear functions of x, y, z.

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From the term independent of w in the fundamental differential equation we have as before

$$5uu_1 + 4axu_1 + x^2\left(\alpha\frac{du_1}{dx} + \beta\frac{du_1}{dy} + \gamma\frac{du_1}{dz}\right) = 0.$$

Hence (i), if u_1 does not divide by x, we have $5u + 4ax = 0$, whence as before $v = \frac{8}{5}ax$, and

$$\alpha\frac{du_1}{dx} + \beta\frac{du_1}{dy} + \gamma\frac{du_1}{dz} = 0, \quad a_1 + b_2 + c_3 + d_4 = 0 \text{ .....(18)},$$

or (ii), if u_1 is a multiple of x, say x itself, we have

$$5u + 4ax + ax = 0\,; \text{ or } u = -ax.$$

The secondary differential equation then gives $v = 2ax$.

The coefficient of w^2 in the fundamental differential equation then gives

$$axu_3 + x^2\left(\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz}\right)u_3 + \left\{a_1x\frac{d}{dx} + \Theta_2\frac{d}{dy} + \Theta_3\frac{d}{dz} + \tfrac{1}{2}d_4\right\}x^2u_3 + 5d_1x^4 = 0.$$

The second, third and fourth terms of this equation divide by x^2, so that axu_3 must do the same; but u_3 cannot divide by x since $x = w = 0$ would then be a triple line, a case previously considered; hence α must vanish, and we revert to a special case of case (i).

Thus we have only to consider case (i), in which the θ's are

$$-\tfrac{4}{5}ax^2 + ax^2 + a_1xw + d_1w^2, \quad -\tfrac{4}{5}axy + \beta x^2 + \Theta_2w + d_2w^2, \quad -\tfrac{4}{5}axz + \gamma x^2 + \Theta_3w + d_3w^2, \quad \tfrac{8}{5}axw + \tfrac{1}{2}d_4w^2,$$

subject to the conditions (18).

Now u_3 must contain one or more terms of the form $(y, z)^2$, since otherwise $x = w = 0$ would be a triple line. By substitution in the differential equation we see that these terms are annihilated by the operator

$$(b_2y + c_2z)\frac{d}{dy} + (b_3y + c_3z)\frac{d}{dz} + d_4.$$

By linear transformation of y and z we can always remove c_2, and can further remove b_3 unless $c_3 = b_2$. Applying the operator in this reduced form to $(y, z)^2$ we must have one of the alternatives:

$$3b_2 + d_4 = 0, \textit{ or } 2b_2 + c_3 + d_4 \equiv b_2 - a_1 = 0, \textit{ or } c_3 - a_1 = 0, \textit{ or } 3c_3 + d_4 = 0 \text{ .....(19)};$$

if b_3 is not removable we have the further condition $b_2 = c_3$, whence:

$$a_1 = b_2 = c_3 = 1, \quad d_4 = -3, \textit{ or } a_1 = b_2 = c_3 = d_4 = 0 \text{ .....(20)}.$$

We have seen (Case II. of § 4), that the constants d_1, d_2, d_3 can be removed by linear substitution if the three quadrics

$$S_1 \equiv \theta_1 - x\theta_4/w = 0, \quad S_2 \equiv \theta_2 - y\theta_4/w = 0, \quad S_3 \equiv \theta_3 - z\theta_4/w = 0$$

meet in a point not lying on $w = 0$. In our case

$$S_1 \equiv -\alpha x^2 + (a_1 - \tfrac{1}{2}d_4)xw + d_1w^2, \quad S_2 \equiv -\alpha xy + \beta x^2 + a_2xw\,(b_2 - \tfrac{1}{2}d_4)yw + d_2w^2,$$
$$S_3 \equiv -\alpha xz + \gamma x^2 + a_3xw + b_3yw\,(c_3 - \tfrac{1}{2}d_4)zw + d_3w^2.$$

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Hence we can remove d_1 in two ways if $\alpha \neq 0$; supposing this reduction effected so that $d_1 = 0$, the quadrics meet where $x = 0$, $(b_2 - \frac{1}{2}d_4)\,y + d_2 w = 0$, $b_2 y + (c_3 - \frac{1}{2}d_4)\,z + d_3 w = 0$, so that we can remove d_2 unless $b_2 - \frac{1}{2}d_4 = 0$, and then d_3 unless $c_3 - \frac{1}{2}d_4 = 0$. If $b_2 - \frac{1}{2}d_4 = 0$, we take the other factor of S_1, viz., $-\alpha x + (a_1 - \frac{1}{2}d_4)\,w$, and we see that we can remove d_2 unless $a_1 - \frac{1}{2}d_4 = 0$ and d_3 unless $c_3 - a_1 = 0$. We can proceed similarly if $c_3 - \frac{1}{2}d_4 = 0$.

Thus, if $\alpha \neq 0$, we can remove d_2 unless

$$a_1 = b_2 = \tfrac{1}{2}d_4 \quad \text{.....(21)},$$

and we can remove d_3 unless

$$a_1 = c_3 = \tfrac{1}{2}d_4 \quad \text{.....(22)}.$$

It is now convenient to distinguish two cases according as α does not or does vanish.

Case I: $\alpha \neq 0$.

As we have just seen we can always remove d_1.

Firstly, let us suppose that we can also remove d_2, d_3.

The coefficients of w^6 and w^5 in the fundamental differential equation give as before the alternative conditions

$$a_1 + 2d_4 = 0, \text{ or } b_2 + 2d_4 = 0, \text{ or } c_3 + 2d_4 = 0, \text{ or } d_4 = 0 \quad \text{.....(23)}.$$

As before at least one term quadratic or cubic in z must occur in our quintic, i.e. one of the terms x^2z^2w, xz^2w^2, yz^2w^2, z^2w^3, z^2w^5 must exist. As before we deduce the alternative conditions

$$4b_2 + 3d_4 = 0, \text{ or } c_3 - a_1 = 0, \text{ or } c_3 - b_2 = 0, \text{ or } 3c_3 + d_4 = 0, \text{ or } 4c_3 + 3d_4 = 0 \text{.....(24)}.$$

The only values of a_1, b_2, c_3, d_4 which satisfy (19), (23) and (24) are (α) 0, 0, 0, 0; (β) 1, 0, -1, 0; (γ) 1, -1, 0, 0; of which we need not consider both (β) and (γ), as they are obtained from one another by a mere interchange of y and z.

If in case (β) we make the transformation $\alpha x - w = \alpha x'$, a_1, b_2, c_3, d_4 are respectively changed into $\frac{1}{5}$, $-\frac{4}{5}$, $-\frac{8}{5}$, $\frac{12}{5}$; the constants d_2, d_3 reappear but can be removed again by substitutions $(y + \mu w,\ y)$, $(z + \nu w,\ z)$; we can then apply the conditions (23), which are obviously not satisfied.

Thus we have only to consider case (α): the θ's are now

$$\tfrac{1}{5}\alpha x^2, \quad -\tfrac{4}{5}\alpha xy + \beta x^2 + a_2 xw, \quad -\tfrac{8}{5}\alpha xz + \gamma x^2 + (a_2 x + b_2 y)\,w, \quad \tfrac{6}{5}\alpha xw.$$

We can further remove β by a substitution $(y + \mu x,\ y)$, γ by a substitution $(z + \nu x,\ z)$, and then a_2 by a substitution $(z + \lambda w,\ z)$; it is convenient further to remove a_2 by a substitution $(y + \varpi w,\ y)$, thereby reintroducing d_2. The θ's are now

$$\tfrac{1}{5}\alpha x^2, \quad -\tfrac{5}{5}\alpha xy, \quad -\tfrac{4}{5}\alpha xz + b_2 yw + d_2 w^2, \quad \tfrac{6}{5}\alpha xw.$$

The fundamental differential equation has the three independent integrals

$$x^4 y, \quad y^2 w^2, \quad x^2 U,$$

where

$$U \equiv \alpha xz - b_2 yw - \tfrac{1}{3}d_3 w^2.$$

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From these we can construct the quintic

$$f \equiv y\,(x^2,\ yw)^2 + xU\,(x^2,\ yw)^1 + U^2 w = 0 \qquad \text{(XLII)}.$$

This is the most general quintic integral if U does not factorize, but if $b_3 = d_3 = 0$ we have an integral of a more general form

$$x^4\,(y,\ z)^1 + x^3\,(y,\ z)^2\,w + (y,\ z)^3\,w^2 = 0,$$

which has a triple point at $x = y = z = 0$, contrary to our present hypothesis.

The transformation $x = X,\ y = Y,\ w = W,\ U = YZ$ converts (XLII) into

$$(X^2,\ YW)^2 + XZ\,(X^2,\ YW) + YZ^2 W = 0.$$

This is a slightly specialized form of the quartic (A) ("Quartics," p. 343), which is known to possess an integral of the first kind.

The quintic surface has a triple point at $x = y = w = 0$.

Secondly, let us suppose that after d_1 has been removed, either or both of d_2 and d_3 cannot be removed by substitutions $(y + \lambda w,\ y)$, $(z + \mu w,\ z)$. If d_2 cannot be removed, we have seen that the conditions (21) must hold, so that either (α) $a_1 = 1,\ b_2 = 1,\ c_3 = -4,\ d_4 = 2$, or ($\beta$) $a_1,\ b_2,\ c_3,\ d_4$ all vanish.

In case (α) since the conditions (20) are not satisfied, $b_3 = 0$; the terms highest in z are annihilated by

$$(\tfrac{1}{5}\alpha x^2 + xw)\frac{d}{dx} + (-\tfrac{4}{5}\alpha xy + \beta x^2 + a_2 xw + yw + d_2 w^2)\frac{d}{dy} + (-\tfrac{4}{5}\alpha xz - 4zw)\frac{d}{dz} + (\tfrac{6}{5}\alpha xw + w^2)\frac{d}{dw}.$$

The terms highest in z must be either z^2w^2, or one of $x^2z^2w,\ xz^2w^2,\ yz^2w^2,\ z^2w^3$; on substitution and arranging by powers of w we see that none of these terms can exist; so that this case fails.

Since $b_3 = 0$ there is in this case no essential difference between y and z; so that the same reasoning shews that if d_3 is not removable, the case corresponding to (α), viz. that in which $a_1 = c_3 = 1,\ b_2 = -4,\ d_2 = 2$, fails also. Hence if either d_2 or d_3 or both cannot be removed we have only to consider case (β) in which $a_1,\ b_2,\ c_3,\ d_4$ all vanish. The θ's are then:

$$\tfrac{1}{5}\alpha x^2,\quad -\tfrac{4}{5}\alpha xy + \beta x^2 + a_2 xw + d_2 w^2,\quad -\tfrac{4}{5}\alpha xz + \gamma x^2 + (a_3 x + b_3 y)\,w + d_3 w^2,\quad \tfrac{6}{5}\alpha xw.$$

By substitutions of the form $(y + \lambda_1 w,\ y)$, $(y + \lambda_2 x,\ y)$, $(z + \lambda_3 w,\ z)$, $(z + \lambda_4 x,\ z)$ we can successively remove $a_2,\ \beta,\ a_3,\ \gamma$; finally by a substitution $(z + \mu y,\ z)$ we can remove d_3, unless d_2 vanishes. The θ's are now:

$$\tfrac{1}{5}\alpha x^2,\quad -\tfrac{4}{5}\alpha xy + d_2 w^2,\quad -\tfrac{4}{5}\alpha xz + b_3 yw + d_3 w^2,\quad \tfrac{6}{5}\alpha xw,$$

where one of d_2, d_3 vanishes and the other does not. These two cases need only be distinguished if $b_3 \neq 0$.

Case (i). $d_2 = 0,\ d_3 \neq 0$.

The fundamental differential equation is the same as the form to which it was finally reduced in the preceding case, viz. when $d_2,\ d_3$ were removable, but it was found convenient to reintroduce d_3. The integral is accordingly (XLII), as before

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Case (ii). $d_2 \neq 0$, $d_3 = 0$, $b_3 \neq 0$.

The terms cubic in y, z which must exist in u_3 are annihilated by $b_3 \frac{d}{dz}$, so that they reduce to a single term λy^3, where $\lambda \neq 0$.

The coefficient of w^5 in the fundamental differential equation, gives

$$6ax\kappa + b_3 y \frac{dv_1}{dz} + d_2 \frac{d}{dy} v_2 = 0;$$

hence v_2 contains no term of the form $2fyz$.

The coefficient of w^4 now gives

$$\left(ax^2 \frac{d}{dx} + 4ax\right) v_1 + b_3 y \frac{d}{dz} v_2 + d_2 \frac{d}{dy} \lambda y^3 = 0.$$

The only term in this equation containing y^3 is $3\lambda d_2$, which must accordingly vanish, contrary to hypothesis.

This completes the whole case when $\alpha \neq 0$.

Case II: $\alpha = 0$.

First, let us suppose that d_1, d_2, d_3 can be removed; then the analysis employed in the corresponding sub-division of Case I. ($\alpha \neq 0$), shews that the constants a_1, b_2, c_3, d_4 either (α) all vanish or (β) have the values 1, 0, -1, 0.

In case (α), the θ's are

$$0, \quad \beta x^2 + a_2 xw, \quad \gamma x^2 + (a_3 x + b_3 y) w, \quad 0.$$

The fundamental differential equation has the three independent integrals

$$x, \quad w, \quad U \equiv \gamma x^2 y + (a_3 xy + \tfrac{1}{2} b_3 y^2) w - (\beta x^2 + a_2 xw) z.$$

If U does not divide by a function of x and w any quintic function made up of these integrals can only contain U linearly, so that z also only occurs linearly, a case of failure.

(i) *If* $b_3 = 0$, U divides by x and can be replaced by the simpler integral

$$V \equiv \gamma xy + a_3 yw - \beta xz - a_2 zw;$$

if V cannot be further reduced by the removal of a factor; from x, w, V we can construct no term cubic in y, z conjointly; so that we revert to the case of a triple line; if a further factor linear in x and w divides out, then V reduces to a linear function of y, z say $\lambda y + \mu z$, and then our quintic is a function of x, w, $\lambda y + \mu z$, i.e. a cone.

(ii) *If* $b_3 \neq 0$, but $\beta = \gamma = 0$, U divides by w and can be replaced by the simpler integral $W \equiv a_3 xy + \frac{1}{2} b_3 y^2 - a_2 xz$; but from x, w, W we can construct no quintic containing terms of the type $(y, z)^3 w^2$; so that we have a failure as before.

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In case (β) we can remove a_2, a_3, b_3 in the usual way, so that the θ's are:

$$xw, \quad \beta x^2, \quad \gamma x^2 - zw, \quad 0.$$

The fundamental differential equation has now the three independent integrals

$$w, \quad U \equiv \beta x^2 - 2yw, \quad V \equiv x(\gamma x^2 - 3zw).$$

From these we can construct

$$\phi \equiv (\beta^3 V^2 - \gamma^2 U^3)/w \equiv 6\beta^2\gamma x^4(\gamma y - \beta z) + 9\beta^3 x^2 z^2 w - 12\beta\gamma^2 x^3 y^2 w + 8\gamma^2 y^3 w^2.$$

The most general quintic integral that can be constructed is

$$f \equiv a\phi + bUV + cU^2 w + dVw^2 + eUw^3 + \kappa w^5 = 0 \text{.....(XLIII)},$$

where a, b, c, d, e, κ are arbitrary coefficients.

The birational transformation $x = X$, $w = W$, $2yw - \beta x^2 = YW$, $x(\gamma x^2 - 3zw) = ZW^2$ converts the surface into

$$a(\beta^3 Z^2 W - \gamma^2 Y^3) + bYZW + cY^2 W + dYW^2 + eZW^2 + \kappa W^3 = 0,$$

a cubic cone, which is in general non-singular.

The quintic has a triple point at $x = y = w = 0$.

Secondly, let us suppose that d_1 can be removed but that one or both of d_2, d_3 cannot be removed; suppose first that d_2 cannot be removed; then $b_2 - \frac{1}{2}d_4 = 0$. Combining this with conditions (19) we have five possible sets of values for a_1, b_2, c_3, d_4; viz.

1,	0,	-1,	0	(i),
1,	1,	-4,	2	(ii),
3,	-2,	$+3$,	-4	(iii),
7,	-3,	2,	-6	(iv),
0,	0,	0,	0	(v).

In cases (i)—(iv) b_3 vanishes, and we can remove d_2 by a substitution $(z + \lambda w, z)$. The conditions for the existence of a term quadratic or cubic in z readily shew that cases (i) and (ii) are impossible; the corresponding conditions for y shew that case (iv) is impossible. In case (iii) the terms independent of x in the quintic satisfy the equation

$$\left\{(-2yw + d_2 w^2)\frac{d}{dy} + 3zw\frac{d}{dz} - 2w^2\frac{d}{dw}\right\} f = 0.$$

This equation has the two independent integrals $z^2 w^3$, $2y/w + d_2 \log w$; from these no quintic terms of the type $(yz)^3 w^2$ can be constructed.

Thus cases (i)—(iv) all fail. In all these cases b_3 vanishes, so that there is no essential difference between y and z; thus if we had assumed d_3 not removable we should have found four cases of failure exactly corresponding to (i)—(iv). Hence whether it be d_2 or d_3 that is irremovable we have only to consider case (v), in which a_1, b_2, c_3, d_4 all vanish. The θ's are now:

$$0, \quad \beta x^2 + a_2 xw + d_2 w^2, \quad \gamma x^2 + (a_3 x + b_3 y)w + d_3 w^2, \quad 0.$$

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The fundamental differential equation has the three integrals

$$x, \quad w, \quad U \equiv \gamma x^2 y + a_2 xyw + \tfrac{1}{2} b_3 y^2 w + d_3 w^2 y - (\beta x^2 + a_2 xw + d_2 w^2)\, z.$$

In order that our quintic should contain terms cubic in y, z conjointly U must occur raised to the power 2 at least; hence it must divide out by a factor depending only on x, w. If $b_3 \neq 0$, the only factor which can divide out is w; and then $\beta = \gamma = 0$ and U can be replaced by the simpler integral $V \equiv a_2 xy + \frac{1}{2} b_3 y^2 + d_3 w^2 y - a_2 xz - d_2 zw$; now V^2 contains y^4, which cannot occur in our quintic; hence V still cannot occur raised to the second or higher power; and we have a case of failure. If $b_3 = 0$, U is linear in y, z; hence in order that terms cubic in y, z conjointly should occur, U^3 must occur; hence U must divide by a quadratic factor depending only on x and w, and the residual factor is of the form $V \equiv \lambda y + \mu z$, where λ and μ are constants. Our quintic is accordingly a function of three variables only, viz. x, w, $\lambda y + \mu z$, and is a cone.

This completes the case when d_1 can be removed.

Thirdly, let us suppose that we cannot remove d_1: then $a_1 = \frac{1}{2} d_4$; combining this with the conditions (19) we have for the fundamental constants five possible sets of values, viz.:

3,	− 2,	− 7,	6	(i),
3,	− 7,	− 2,	6	(ii),
1,	1,	− 4,	2	(iii),
1,	− 4,	1,	2	(iv),
0,	0,	0,	0	(v).

In all cases except the last $b_3 = 0$, so that there is no essential distinction between y and z; hence it is unnecessary to consider cases (ii) and (iv), if we consider cases (i), (iii), and (v).

Case (i). We can remove in the usual way d_2, d_3, a_2, a_3, reducing the θ's to

$$3xw + d_1 w^2, \quad \beta x^2 - 2yw, \quad \gamma x^2 - 7zw, \quad 3w^2.$$

It can then at once be verified, as on previous occasions, that no term quadratic or cubic in z can occur in our quintic.

Case (iii). We can remove d_2 by a substitution $(y + \lambda x, y)$, a_3 by a substitution $(z + \mu x, z)$ and then d_3 by a substitution $(z + \nu w, z)$; so that the θ's are

$$xw + d_1 w^2, \quad \beta x^2 + a_2 xw + yw, \quad \gamma x^2 - 4zw, \quad w^2.$$

As before no term quadratic or cubic in z can exist.

In case (v), we can remove d_2, d_3 by substitutions $(y + \lambda x, y)$, $(z + \mu x, z)$; so that the θ's reduce to:

$$d_1 w^2, \quad \beta x^2 + a_2 xw, \quad \gamma x^2 + (a_2 x + b_3 y)\, w, \quad 0.$$

The fundamental differential equation now has the three independent integrals

$$w, \quad \tfrac{1}{3}\beta x^3 + \tfrac{1}{2} a_2 x^2 w - d_1 yw^2, \quad \tfrac{1}{4}\beta b_3 x^4 + \tfrac{1}{3}(a_2 b_3 - \gamma d_1)\, x^3 w - (\tfrac{1}{2} a_2 x + b_3 y)\, xw^2 + d_1^2 z^2 w.$$

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From these we can construct no integrals quadratic in y or z unless at least one of the last two integrals divides by w, so reducing. This can only happen if $\beta = 0$, or $b_3 = 0$. If $b_3 = 0$ we can remove β from the differential equation and therefore from its integrals by a substitution $(y + \lambda z,\ y)$; hence whether b_3 vanishes or not, β must vanish.

We now have the simpler integrals

$$w, \quad U \equiv a_2 x^2 - 2d_1 yw, \quad V \equiv 2(a_2 b_3 - \gamma d_1)x^3 - 3a_2 d_1 x^2 w - 6b_3 d_1 xyw + 6d_1^2 zw^2.$$

From these we can construct the quintic integral $\phi \equiv (\lambda U^3 + \mu V^2)/w$, where λ and μ are constants so chosen that

$$\lambda a_2^3 + 4\mu(a_2 b_3 - \gamma d_1)^2 = 0.$$

The most general quintic is now

$$f \equiv \phi + (U,\ w^2)^2 w + \kappa_1 V w^2 + \kappa_2 UV = 0 \quad \text{.....(XLIV)},$$

where κ_1, κ_2 are arbitrary coefficients.

The birational transformation $U = YW$, $V = ZW^2$, $x = X$, $w = W$ transforms the surface into

$$\lambda Y^3 + \mu Z^2 W + (Y,\ W)^2 W + \kappa_1 Z W^2 + \kappa_2 YZW = 0,$$

a cubic cone which is in general non-singular.

The form of the quintic requires some modification if $a_2 = \gamma = 0$, as in that case the integral ϕ which we have used becomes indeterminate. The simplest integrals are now

$$y, \quad w, \quad a_3 x^3 + 2b_3 xy - 2d_1 zw.$$

If from these we construct the most general quintic with $x^2 = 0$, $w = 0$ double on it we find that w must be a factor of the whole expression, unless $b_3 = 0$. If $b_3 = 0$, a_3 cannot vanish or we should have a cone; we may conveniently take each of d_1, a_3 to be unity and the quintic now becomes

$$f \equiv (x^2 - 2zw)^2 (y,\ w)^1 + (x^2 - 2zw)(y,\ w)^2 w + (y,\ w)^3 w^2 = 0 \quad \text{.....(XLV)},$$

the θ's being now w^2, 0, xw, 0.

If we use the transformation $x^2 - 2zw = ZW$, $x = X$, $y = Y$, $w = W$, we get the cubic cone

$$Z^2 (Y,\ W)^1 + Z(Y,\ W)^2 + (Y,\ W)^3 = 0,$$

which is in general non-singular.

This completes the case of $\alpha = 0$, so also the whole case when the double conic degenerates into two coincident straight lines; we have found four surfaces represented by equations (XLII)—(XLV).

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§ 7. CONCLUSION.

We have now given the equations of a number of quintics satisfying the conditions of our problem as defined in the introduction. Each surface possesses at least one integral of the first kind of a total differential, and has a double conic but no further double line; cones have not been taken into account. According to the system of classification adopted there are twenty-three surfaces, the equations of which are marked by Roman numbers in continuation of the enumeration given in "Quintics I." *Three* of these surfaces possess a triple point not lying on the double conic; they are Cayley's quintic with a double line and two triple points, and two limiting forms of it; they are represented by equations (XXIII) (with a modified form) and (XXIV). *Seven* surfaces represented by equations (XXV)—(XXXI) have a double point not lying on the conic. The remaining *thirteen* surfaces have no double or multiple point distinct from the double conic; in the case of *five* of these thirteen surfaces, represented by the equations (XXXII)—(XXXVI), the conic is non-degenerate, in the case of *four* surfaces, represented by equations (XXXVII) and (XXXIX)—(XLI), the conic degenerates into two intersecting straight lines which are distinct, while in the remaining *four* cases, represented by equations (XLII)—(XLV), the conic degenerates into two coincident straight lines. The surface (XXXVIII) should have been given in "Quintics I," § 6, and only differs slightly from the surface (VII) of that article. Six of the twenty-three surfaces belong to a class considered in "Quintics I," Part II, viz.: those for which the fundamental quadrics θ have a common linear factor; three, viz. (XXV), (XXXII) and (XL), are special cases of (XVII), two, viz. (XLI) and (XLV), are special cases of (XVI), while (XXXIX) is a special case of (XXI).

The surfaces of this paper share with all other known quartic and quintic surfaces, which admit of integrals of the first kind, the property that they can be transformed birationally into cones; so that in all cases the geometrical genus (p_g) is zero and the numerical genus (p_n) is negative. All the surfaces of this paper can be transformed into cubic cones, so that $p_n = -1$. Each surface possesses a family of rational curves corresponding to the generators of the cone. I have not in general worked out the equations of these curves, but they can be immediately deduced from the transformations, which I have always given, by means of which the surface can be transformed into cones either directly or *via* quartics. None of the surfaces of this paper are ruled, a result which is in agreement with Schwarz's enumeration * of ruled quintics.

* "Ueber die geradlinigen Flächen fünften Grades": *Crelle's Journal*, t. 67 (1866); reprinted in *Gesammelte Abhandlungen*, t. 2, pp. 25—64.

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V. *The Law of Error.* Part II. By Professor Edgeworth.

ANALYSIS OF CONTENTS.

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The conclusions which have been proved for the "typical" case are now to be extended to more general cases. These may be reached by removing or relaxing the conditions which form the attributes of the typical case*; beginning with the one which was last in the original enumeration.

Removal of the eleventh condition. The condition that the finite difference $\Delta\xi (= \Delta x)$, being the interval between adjacent values of an element, shall be the same for all the elements may be dispensed with by taking Δx as the greatest common measure of all the intervals (between adjacent values) presented by all the elements (in the long run). Thus if an element which (measured from its lower extremity) presents the values

$$0, \quad \Delta x, \quad 2\Delta x, \ldots \nu\Delta x$$

is to be combined with a binomial element which presents only the values 0 and $\nu\Delta x$, it is allowable to regard the latter element as presenting the values

$$\Delta x, \quad 2\Delta x, \ldots (\nu - 1)\,\Delta x$$

with zero frequency.

* See Part I. p. 38.

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Substitution for the tenth, ninth, and eighth conditions. The three conditions which (in the reverse order) come next may be reduced to one: namely that a descending series should be formed by the coefficients k, k_1, $k_2 \ldots k_t$*, each divided by k (or $2k$) raised to a corresponding power, viz., respectively $\frac{1}{2}(0+2)$, $\frac{1}{2}(1+2)$, $\frac{1}{2}(2+2), \ldots \frac{1}{2}(t+2)$. For thereby it will be secured that the mean powers of the compound x will be broken up into groups forming successive approximations. Thus†

$$x^{(2p)} = \frac{2p\,!\,k^p}{p\,!\,2^p}\left(1 + \frac{p(p-1)}{1\,.\,3\,!}\,\frac{k_2}{k^2}\right) + \ldots,$$

$$x^{(2p+1)} = \frac{2p+1\,!\,k^{(p-1)}}{p-1\,!\,2^{p-1}}\left\{\frac{k_1}{3\,!\,k^{\frac{3}{2}}} + \frac{2(p-1)}{5\,!}\,\frac{k_3}{k^{\frac{5}{2}}}\right\} + \ldots.$$

Also the expression for the ordinate‡ can be exhibited as a descending series, viz.,

$$\mathrm{y}_0,\ -\mathrm{k}_1\frac{1}{3\,!}\,\frac{\mathrm{d}^3\mathrm{y}_0}{\mathrm{dx}^3},\ +\mathrm{k}_2\frac{1}{4\,!}\,\frac{\mathrm{d}^4\mathrm{y}_0}{\mathrm{dx}^4} + \tfrac{1}{2}\mathrm{k}_1^2\,\frac{1}{3\,!\,3\,!}\,\frac{\mathrm{d}^6\mathrm{y}_0}{\mathrm{dx}^6},\ + \ldots,$$

where $$\mathrm{x} = x/\sqrt{2k},\quad \mathrm{k}_r = k_r/(2k)^{\frac{1}{2}(r+2)},$$

$$\mathrm{y}_0 = \frac{1}{\sqrt{2\pi}}\,e^{-\mathrm{x}^2},\ \text{ and accordingly } \frac{\mathrm{d}^r\mathrm{y}_0}{\mathrm{dx}^r} = (2k)^{\frac{1}{2}r}\,\frac{d^r y_0}{dx^r}.$$

The convergency of the series is thus secured, provided that x is not too large with respect to $2k$§.

This convergence tends to be impaired by the inequality between the elements in respect of mean deviation against which the tenth condition is directed. This may be seen by considering the case in which some of the component elements have a mean-square-of-deviation which is very small in comparison with the corresponding coefficient for the other elements. Thus suppose the system of elements to consist of two classes, m_1 elements assuming the values 0, 1, 2, 3, ... 18 with the same frequency as the number of pips on a domino taken at random, and m_2 elements assuming the values 0 or 1 according as a tossed coin turns up tail or head. We have for k the mean-square-of-deviation for the compound $m_1 \times 16{\cdot}5 + m_2 \times 0{\cdot}25$. For k_2 we have $m_1 \times -83{\cdot}325 + m_2 \times -0{\cdot}125$. Unless m_2 is very great in comparison with m_1 it will not contribute much to the condition that k_2/k^2 should be small. It must be supposed that the elements are not‖ regular, otherwise the inequality of their dispersion would not impair the regularity of the compound.

Another circumstance unfavourable to the fulfilment of the essential condition is extreme asymmetry on the part of the elementary locus; which is guarded against by the eighth condition. The effect of asymmetry is illustrated by the case of the binomial when one of the alternatives is very improbable. The quantity mpq which affects all the coefficients k_1, k_2, ... as a denominator then tends to be small, and accordingly those coefficients tend to be large¶. Analogously in general when an element has a very

* See Part I. p. 44.

† *Ibid.* p. 43.

‡ *Ibid.* p. 45.

§ See Appendix, Note 2.

‖ It will be remembered that the term "regular" is applied to the law of error as generalised in this paper; the term "normal" being confined to the first approximation.

¶ See Part I. Sect. 5.

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asymmetrical locus its mean powers will be small, say $\epsilon\gamma^2$, $\epsilon'\gamma^3$, $\epsilon''\gamma^4$, &c., where ϵ, ϵ', ϵ'' are small fractions of the same order. Accordingly the k coefficients will not form a descending series (unless m is very large). E.g. $k_2 = \Sigma(\epsilon''\gamma^4 - 3\epsilon^2\gamma^4)/4\epsilon^2\gamma^4 m^2$; where ϵ and ϵ'' are presumably of the same small order.

In general, whatever the abnormality of the elements, whether it be due to asymmetry or other incidents, and whether or not it be aggravated by inequality between the elements in respect of dispersion, the regular law of error will be generated, provided that the number of the elements is sufficiently great. The number need not be great when the elements are nearly regular (themselves perhaps compounds). If they are perfectly regular the sum of any *two* elements will obey the regular law of error.

Removal of the seventh condition. In what precedes it has been taken for granted that the mean powers of the elements are finite; as follows from the attribute that the range is finite. This attribute is only required in order to afford that inference; if the method of proof recommended in Sect. 1 of Part I. is adopted. That proof is not affected by the range of the elements becoming infinite, provided that their mean powers remain finite.

But in employing the method originated by Professor Morgan Crofton, in cases where the range is infinite we are met with a difficulty when we proceed to expand the expression for the frequency of the compound in terms of the mean powers of the elements*. Some of the constituents of these means now become infinite; and accordingly we are employing Taylor's theorem for the expansion of $f(x+h)$ in ascending powers of h, in cases where h is infinite. A similar difficulty is presented by Laplace's method when it is attempted to expand $\chi(\alpha)$ (in our notation) in ascending mean powers of an element†. As Laplace himself says, "si les erreurs peuvent s'étendre à l'infini, l'application de la méthode précédente peut offrir quelques difficultés‡."

These difficulties may be removed by the following theorem. An element with indefinitely great range may be reduced, without sensible effect on the locus of the compound, to an element with range finite and small in comparison with the "standard deviation"§ of the compound; provided that the mean powers (up to the tth) of deviations in excess of the centre of gravity, and likewise the mean powers of deviations in defect—each set being taken in absolute quantity—are finite and not very great, so that the tth root of the mean tth power is a small quantity‖ relatively to the standard deviation of the compound. For consider a case in which the area of the element above its centre of gravity is p, and below the centre of gravity q $(p+q=1)$. Let OP, measured from the centre of gravity to the right, represent the tth root of the mean tth power of deviations in excess; and likewise OQ the tth root of the mean tth power of deviations in defect. And let us endeavour to arrange that as large as possible a portion of the element should lie above a limit considerably greater than OP, say $3OP$; and likewise as large as possible a proportion below $3OQ$. The arrangement most favourable to this result would be that the portions of the area not outside these limits should be huddled up in the neighbourhood of the centre

* See Part I. Sect. 2.

† See Part I. p. 52.

‡ *Théorie analytique des Probabilités*, Bk II. ch. IV. § 18.

§ Professor K. Pearson's convenient term for the quantity here denoted by $\sqrt{k}$.

‖ A quantity of the order $1/\sqrt{m}$ according to the scale of units recommended in this paper. See Part I. Sect. 1, and Appendix, Note 1.

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of gravity so as not to contribute to the mean powers of the element. Let ρp be the proportion of the area outside $3OP$; and likewise $\rho' q$ the proportion outside $3OQ$. If all the portion outside $3OP$ were indefinitely near that limit, $\rho p\,(3OP)^{10} = pOP^{10}$. But if the portion outside $3OP$ is more dispersed, $p\,.\,OP^{10} = \rho p\,(OP')^{10}$, where OP' is greater than $3OP$. Accordingly $\rho \not> (\frac{1}{3})^{10}$. Likewise it may be shown that ρ' is very small. *À fortiori* when the element has a less peculiar form of frequency-locus than that which has been supposed*. Therefore (it being granted that the mean powers of the elements are finite) all the elements may be considered as ranging within limits sufficiently small to permit of the expansions practised in the methods originated by Laplace and Professor Morgan Crofton. The proportion of observations involving values of elements outside those limits must be so small as to be insensible when distributed over an area (bounded by the frequency-locus of the observations compounded of elements) which is nearly equal to unity.

Removal of the sixth condition. Beginning with the case of two dimensions, let us represent the frequency-locus of the rth element by the equation $\zeta_r = \phi_r(\xi, \eta)$; signifying that the probability, or comparative frequency in the long run, of the concurrence of the values ξ and η is for the rth element $\phi_r(\xi, \eta)\,\Delta\xi\Delta\eta$. Let the pair of values contributed by the rth element to the sth observation be $({}_s\xi_r,\ {}_s\eta_r)$. The sth observation consists of a pair of values for the compound quantity $({}_sx,\ {}_sy)$ such that

$$ {}_sx = \sum_{r=1}^{r=m} {}_s\xi_r; \quad {}_sy = \sum_{r=1}^{r=m} {}_s\eta_r. $$

Beginning with the method placed first in our First Part† we have to find a representative surface whose mean powers of deviation from the centre of gravity are approximately equal to those of the actual locus; not only mean powers for one of the variables, such as the mean of x^p or of y^q, but also the mean of products such as $x^t \times y^{t'}$. The latter mean may be designated thus $(x,\ y)^{(t),\,(t')}$. If $f(xy)$ is a surface approximating in form to the actual locus $[\iint x^t y^{t'} f(xy)\,dxdy]$—the square brackets denoting integration between extreme limits—

$$ = (x,\ y)^{(t),\,(t')}. $$

Take the centre of gravity as before for the origin, and transform to *principal* axes by putting $x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$, where θ is taken so that $(X,\ Y)^{(1),\,(1)}$, the mean product of the coordinates pertaining to the compound, may vanish. This will be effected if

$$ \tan 2\theta = 2\,(x,\ y)^{(1),\,(1)} / (x,\ y)^{2,\,0} - (x,\ y)^{0,\,2}\ddagger. $$

Following the analogy of the simpler case§ let us express the generating function of $\frac{1}{t!\ t'!}(X,\ Y)^{(t),\,(t')}$ in a form suited to our purpose. The generating function of

$$ \frac{1}{t!\ t'!}\,X^t Y^{t'} $$

is $e^{\alpha X + \beta Y}$; the coefficient of $\alpha^t \beta^{t'}$ in the expansion of this function being $\frac{1}{t!\ t'!}\,X^t Y^{t'}$.

* See an example in Appendix, Note 5. † Section 1.

‡ Compare "The Compound Probability Curve," *Phil. Mag.* 1896, vol. XLI. p. 208, by the present writer.

§ Part I. p. 42.

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And $e^{\alpha X+\beta Y} \equiv e^{\alpha \Sigma \Xi+\beta \Sigma H}$ (if Ξ_r, H_r are the coordinates for the rth element referred to the principal axes X and Y)

$$\equiv (1+\alpha\Xi_1+\beta H_1)+\frac{1}{2!}(\alpha^2\Xi_1^2+2\alpha\beta\Xi_1 H_1+\beta^2 H_1^2)+\dots$$

$$\times(1+\alpha\Xi_2+\beta H_2)+\frac{1}{2!}(\alpha^2\Xi_2^2+\dots\dots\dots\dots\dots\dots)+\dots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\times(1+\alpha\Xi_m+\beta H_m)+\frac{1}{2!}(\alpha^2\Xi_m^2+\dots\dots\dots\dots\dots\dots)+\dots.$$

Therefore, taking averages on both sides of the identity,

$$1+\alpha X^{(1)}+\beta Y^{(1)}+\tfrac{1}{2}(\alpha^2 X^{(2)}+2\alpha\beta(X,\ Y)^{(1),(1)}+\beta^2 Y^{(2)})+\dots$$

is identical with the mean of the last written product; that is, as the elements vary independently of each other

$$e^{\alpha\Xi_1+\beta H_1+\frac{1}{2!}(\alpha^2{}_{2,0}\kappa_1+2\alpha\beta{}_{1,1}\kappa_1+\beta^2{}_{0,2}\kappa_1)+\frac{1}{3!}({}_{3,0}\kappa_1\alpha^3+3{}_{2,1}\kappa_1\alpha^2\beta)+\dots}$$

$$\times e^{\alpha\Xi_2+\beta H_2+\frac{1}{2!}(\alpha^2{}_{2,0}\kappa_2+2\alpha\beta{}_{1,1}\kappa_2+\beta^2{}_{0,2}\kappa_2)+\frac{1}{3!}({}_{3,0}\kappa_2+\dots)+\dots}$$

$$\vdots \qquad \vdots$$

where ${}_{p,q}\kappa_r$ is the coefficient of $\frac{1}{p!\,q!}\alpha^p\beta^q$ in the expansion of

$$\log\left(1+\alpha\Xi_r^{(1)}+\beta H_r^{(1)}+\frac{1}{2!}(\alpha^2\Xi_r^{(2)}+2\alpha\beta(\Xi_r H_r)^{(1),(1)}+\beta^2 H_r^{(2)}+\dots)+\dots\right).$$

Put ${}_{2,0}K$, or more shortly K, for $\sum_{r=1}^{r=m}{}_{2,0}\kappa_r^{(2)}$, ${}_{0,2}K$, or more shortly K', for $\sum_{r=1}^{r=m}{}_{0,2}\kappa_r^{(2)}$; and generally ${}_{p,q}K$ for $\sum_{r=1}^{r=m}{}_{p,q}\kappa_r$; and we obtain for the generating function of $\frac{1}{t!\,t'!}(X,\ Y)^{(t),(t')}$—remembering that $X^{(1)}$, $Y^{(1)}$, $(X,\ Y)^{(1),(1)}$ all vanish—

$$e^{\frac{1}{2}(\alpha^2 K+\beta^2 K')+\frac{1}{3!}({}_{3,0}K\alpha^3+3{}_{2,1}K\alpha^2\beta+3{}_{1,2}K\alpha\beta^2+{}_{0,3}K\beta^3)+\dots};$$

where ${}_{p,q}K$ is of an order inversely proportional to the $(p+q-2)$th power of $\sqrt{m}$ (if the elements are supposed to be typical*, except in that they are two-dimensioned).

The values of $(X,\ Y)^{(t),(t')}$ thus given may be expressed as integrals of a representative surface, if we convert the generating function into an operator by omitting the first pair of terms in the index, viz. $\frac{1}{2}K\alpha^2+\frac{1}{2}K'\beta^2$, substituting in the remaining terms for α, $-\frac{d}{dX}$, and for β, $-\frac{d}{dY}$, and putting for operand the (normal) function

$$\frac{1}{2\pi\sqrt{KK'}}\ \mathrm{Exp}-\left(\frac{X^2}{2K}+\frac{Y^2}{2K'}\right),\ \text{say } Z_0.$$

For it will be found that the expression given by the generating function

$$\frac{1}{t!\,t'!}(X,\ Y)^{(t),(t')}$$

* It is often convenient in enouncing the relaxation of one typical attribute to treat the others as in force, where no new difficulty is introduced by their relaxation.

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is identical with

$$\frac{1}{t!\,t'!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X^t Y^{t'} Z dX\, dY;$$

where Z is the function which results from the performance on Z_0 of the operation which has been indicated. For by analogy with the case of one dimension* it will be found that any term in one of the compared expressions, e.g. a term which involves K^p and K'^q, is equal to a corresponding term in the other expression. The comparison may be effected by the equations

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X^{2p} Y^{2q} Z_0 = \frac{2p!}{2^p p!}\,\frac{2q!}{2^q q!} K^p K'^q;$$

$$\frac{1}{t!\,t'!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X^{2p+r} Y^{2q+s} \frac{d^{r+s} Z_0}{dX^r dY^s}(-1)^r(-1)^s\, dX dY = \frac{K^p K'^q}{2^p p!\; 2^q q!};$$

where $$t = 2p + r, \quad t' = 2q + s.$$

It remains to transform the expression which has been found back to the original coordinates. This is effected by substituting in the operand Z_0 for X and Y their values in terms of x, y, and θ, for K and K' their values in terms of $(x, y)^{(2),(0)}$, $(x, y)^{(1),(1)}$, $(x, y)^{(0),(2)}$, say k, l, and k' respectively, and for $\tan\theta$ also its value in terms of those coefficients. There results for the operand, and as the first approximation to the required law of error,

$$z_0 = \frac{1}{2\pi\sqrt{kk' - l^3}}\,\mathrm{Exp} - (k'x^2 - 2lxy + ky^2)/2\,(kk' - l^2).$$

For the operator it suffices to change each coefficient of the type ${}_{p,q}K$ to the corresponding symbol in *lower case*, viz. ${}_{p,q}k$ where ${}_{p,q}k$ has the same relation to z that ${}_{p,q}K$ has to Z; that is, ${}_{p,q}k$ is the difference between $(x, y)^{(p),(q)}$ as it actually is, and as it would be if z_{p+q-2} the $(p+q-1)$th approximation were correct†. This may be shown by considering the group of terms which are of the order $(p+q-2)/\sqrt{m}$‡, as having the character of an *emanant*. For instance, in the case of $p+q=3$ the group (of the order $1/\sqrt{m}$) is

$$\frac{1}{3!}\left({}_{3,0}K\frac{d^3Z_0}{dX^3} + 3\,{}_{2,1}K\frac{d^3Z_0}{dX^2dY} + 3\,{}_{1,2}K\frac{d^3Z_0}{dXdY^2} + {}_{0,3}K\frac{d^3Z_0}{dY^3}\right).$$

This expression is the mean, that is, the quotient of a sum, of constituents each of which is of the form

$$\frac{1}{3!}\left(X'^3\frac{d^3Z_0}{dX^3} + 3X'^2Y'\frac{d^3Z_0}{dX^2dY} + \dots\right),$$

where $X'Y'$ is a particular pair of values assumed by the compound quantity. But by the theory of emanants§ this constituent is identical with

$$\frac{1}{3!}\left(x'^3\frac{d^3z_0}{dx^3} + 3x'^2y'\frac{d^3z_0}{dx^2dy} + \dots\right),$$

* Part I. p. 44.

† Or the sum of the corresponding differences for the elements. Cp. Part I. p. 44.

‡ On the supposition of *typical*—not in general *regular*—elements.

§ Salmon's *Higher Algebra*, Art. 125.

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where $x'y'$ is that pair of coordinates referred to the original axes which corresponds to $X'Y'$. Whence it follows that the group

$$\frac{1}{3!}\left({}_{3,0}K\frac{d^3Z_0}{dX^3}+3\,{}_{2,1}K\frac{d^3Z_0}{dX^2dY}+\ldots\right)$$

is identical with the group

$$\frac{1}{3!}\left({}_{3,0}k\frac{d^3z_0}{dx^3}+3\,{}_{2,1}k\frac{d^3z_0}{dx^2dy}+\ldots\right).$$

Like reasoning may be applied to any other group.

With the aid of premisses furnished by the foregoing proof of the law of error in two dimensions, the same conclusion may be obtained by the proof which is based on partial differential equations*. When a new element is taken in we have now for the first approximation:

$$(1)\quad \partial z=[SS\xi^2\phi_m(\xi,\eta)\,\Delta\xi\Delta\eta]\frac{d^2z}{dx^2};$$

$$(2)\quad \partial z=[SS\xi\eta\phi_m(\xi,\eta)\,\Delta\xi\Delta\eta]\frac{d^2z}{dxdy};$$

$$(3)\quad \partial z=\tfrac{1}{2}[SS\eta^2\phi_m(\xi,\eta)\,\Delta\xi\Delta\eta]\frac{d^2z}{dy^2};$$

the square brackets indicating that the summation extends over the whole range of the new element. By first principles† the three bracketed quantities may be replaced respectively by Δk, Δl, $\Delta k'$, the additions made by taking in the new element to k, l, k' (respectively representing $x^{(2)}$, $(x, y)^{(1),(1)}$, $y^{(2)}$). We have thus the system of partial differential equations:

$$(1)\quad \frac{dz}{dk}=\frac{1}{2}\frac{d^2z}{dx^2};\qquad (2)\quad \frac{dz}{dl}=\frac{d^2z}{dxdy};\qquad (3)\quad \frac{dz}{dk'}=\frac{1}{2}\frac{d^2z}{dy^2}.$$

If we transform to principal axes l disappears. There must be satisfied the two partial differential equations

$$(1)\quad \frac{dZ}{dK}=\frac{1}{2}\frac{d^2Z}{dX^2};\qquad (2)\quad \frac{dZ}{dK'}=\frac{1}{2}\frac{d^2Z}{dY^2}.$$

For subsidiary equations we obtain, by parity with Professor Crofton's reasoning‡ in the case of a single dimension,

$$(\text{I})\quad Z+X\frac{dZ}{dX}+2K\frac{dZ}{dK}=0;$$

$$(\text{II})\quad Z+Y\frac{dZ}{dY}+2K'\frac{dZ}{dK'}=0.$$

Combining the subsidiary with the leading system, we obtain the first integrals:

$$(1)\quad ZX+\frac{dZ}{dX}=C_1;\qquad (2)\quad ZY+\frac{dZ}{dY}=C_2;$$

where C_1 does not involve X, nor C_2, Y. Further, as by the fundamental theory the odd

* See Part I. Sect. 2. † Cp. Part. I. p. 46.

‡ *Encyclopædia Britannica*, 9th edition, Art. "Probability," p. 781.

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powers of X and Y in Z may be neglected in a first approximation*, when $X=0$, $\frac{dZ}{dX}=0$, and therefore $C_1=0$. By parity of reasoning $C_2=0$. Whence by a second integration—utilising the condition $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z dX dY = 1$—we obtain the normal surface.

There is no difficulty in extending this method to further approximations on the lines indicated by the present writer in a former paper†; with the aid of the fundamental theory stated in this paper.

The proof of the law of error for two dimensions on the lines traced by Laplace for one dimension‡ presents no additional difficulty; so short is the step from the generating function (or the operator) which has been above written to that function in α for one dimension§, in α and β for two dimensions, which is proper to Laplace's method.

Nor does the use of the method which is based on the condition of reproductivity present peculiar difficulty in the case of two dimensions‖.

Nor need we be detained by the extension of the theory to dimensions more than two; nor by relaxations of the typical conditions, such as have been already considered with reference to one dimension.

Slight relaxation of the fifth condition. So far we have supposed the method of compounding the elements to be of the simplest sort, namely, summation. To this species may be reduced the case in which the compound is the sum, not indeed of the m elements, but of m magnitudes each of which is a given function of a different element, say

$$x' = \psi_1(\xi_1) + \psi_2(\xi_2) + \ldots + \psi_m(\xi_m).$$

Here if ξ_r obeys the law of frequency $\eta = \phi_r(\xi_r)$, the statistical quantity which we may call $[\psi_r]$ obeys the law

$$\eta = \phi(\xi_r) \Big/ \frac{d\psi_r}{d\xi_r},$$

where for ξ_r is to be substituted its value in terms of $[\psi_r]$ obtained from the equation

$$[\psi_r] = \psi_r(\xi_r).$$

The compound quantity is thus shown to be the sum of m elements $[\psi_1], [\psi_2], \ldots [\psi_m]$, each of which fluctuates independently. Accordingly (if the new elements satisfy the essential conditions) the law of error will be set up. This variety includes the important case in which the function whereby the elements are compounded is linear¶.

Functions other than linear are not *in general* amenable to any of the proofs which have been given. Nor is there any universal form towards which the frequency-locus of such aggregates converges. If we are free to employ *any* combination of the elements we

* See Part I. p. 47.

† *Phil. Mag.*, 1896, vol. XLI. p. 208.

‡ See Mr Burbury's extension of Laplace's method, in *Phil. Mag.* 1894.

§ See Part I. p. 52.

‖ See Appendix, Note 6.

¶ Another important simple case belonging to this variety is where $\psi(\xi) = \xi^a$, a being a positive integer; the case discussed by Todhunter, *History of Probabilities*, Art. 1006. Among known and simple cases of relations other than summation by which the law of error is set up may be mentioned here the *median* and *percentiles* of a large set of elements; though they cannot be classed as *functions* of the elements.

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can arrange that any assigned function shall be the limiting form for the frequency-locus of that combination when the number of the elements is increased without limit. For consider the frequency-locus of a statistical quantity, x', which is some function of the sort of compound which we have hitherto been considering, viz. a sum of m independent elements; say $x' = \phi(x)$, where $x = \xi_1 + \xi_2 + \ldots + \xi_m$*. Then, if $\theta(x)$ denote the normal error-function,

$$\frac{1}{\sqrt{2\pi k}} \text{Exp.} - x^2/2k,$$

we have for the frequency-locus of x', $\theta(x)\dfrac{dx}{dx'}$, if for x is substituted its value in x' derived from the equation $x' = \phi(x)$. The resulting expression in x' is to be identified with some assigned function, say $\mathrm{f}(x')$. We have thus $\theta(x)\dfrac{dx}{dx'} \equiv \mathrm{f}(x')$. Whence $\Theta(x) + C \equiv F(x')$, if capital letters are used to denote the respective indefinite integrals. Whence

$$x' = F^{-1}(C + \Theta(x)).$$

For example, let $\mathrm{f}(x')$ be (for positive values of x) $(a - x')/a^2$. Then

$$F(x') = (ax' - \tfrac{1}{2}x'^2)/a^2;$$

which is to be equated to $C + \Theta(x)$. Thus

$$x'^2 - 2ax' + 2a^2(C + \Theta(x)) = 0,$$

$$x' = a\{1 \pm \sqrt{1 - 2(C + \Theta(x))}\}.$$

Remarking that Θ is a quantity, which increases from zero to $\frac{1}{2}$, as x increases from zero to infinity, let us arrange that x' should increase from 0 to a while x increases from zero to infinity. This is effected by taking the lower, the negative, sign in the expression for x', and putting $C = 0$. The resulting frequency-locus for positive values of x' will therefore be of the prescribed form $y = (a - x')/a^2$, if $x' = a(1 - \sqrt{1 - 2\Theta(x)})$†. By a proper change of signs the result becomes applicable to negative values of x and x'. The larger the number of elements the more closely will the actual frequency-locus of x' approximate to the right line (corresponding to the assigned function) as a limiting form.

A fortiori there is no unique limiting form to be expected when the compound involves the elements, not implicitly as a function of their sum, but in any other way.

Of course if the function can be expanded in ascending powers of the elements which after the first may be neglected, the case is reduced to that of a linear function. Between this simple case and the most general one—both of which have been now considered—occurs an intermediate, probably extensive, case: where the compound is expanded‡ in

* The writer has employed the form $x' = ax + bx^2 + cx^3$ for the representation of abnormal groups (*Journal of the Statistical Society*, 1900).

† Laplace has expressed $\Theta(x)$ in several forms; discussed by Dr Burgess in the *Trans. Roy. Soc. of Edinburgh*, vol. XXXIX. 1900.

‡ Quantities of the order $(1/\sqrt{m})^i$ in relation to the first term of the compound magnitude being retained.

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ascending powers (and products) of the elements, which are not neglected, but retained down to a certain order of magnitude, say the tth. It may be enquired whether the regular law of error is applicable to such expansible functions of elements.

The answer to this question may be found by showing that there is a generating function for the mean powers of such a combination of elements essentially resembling the generating function which has been found for the mean powers of a compound formed by a sum of elements*.

The following method of determining the law of frequency for expansible functions has the advantage of affording a fresh proof of the law of error for sums of elements. If x' is the given function of the elements, and y' its sought law of frequency, $y'\Delta x'$ may be considered as a certain strip of the locus

$$w = \phi_1(\xi_1) \times \phi_2(\xi_2) \dots \phi_m(\xi_m)$$

which (multiplied by $\Delta\xi_1\Delta\xi_2 \dots \Delta\xi_m$) represents the probability that any particular system of values for the m elements should concur ($\eta = \phi_r$ being the law of frequency for the rth element). The slice of this "solid" which contains all the "points," or systems of $\xi_1, \xi_2 \dots \xi_m$, for which the given combination of these variables is between† x' and $x' + \Delta x'$, is equal to $y'\Delta x'$. Change the system of independent variables from $\xi_1, \xi_2, \xi_3 \dots \xi_m$ to $x', \xi_2, \xi_3 \dots \xi_m$. The required strip may be written $[\iint \dots \Delta x'\,{}_1w\,d\xi_2 d\xi_3 \dots d\xi_m]$; where the integrations are to extend between the extreme limits of the variables; ${}_1w$ is what w becomes when for ξ_1 is substituted its value in $x', \xi_2, \xi_3 \dots$, and this form of w is multiplied by (the determinant proper to a change in the system of variables, here) the differential of that value of ξ_1 with respect to x', $\frac{d\xi_1}{dx'}$.

The handling of this integral is facilitated by two propositions which have been above established. (α) The equation for the locus of any element may be put, without loss of generality, in the form

$$\eta_r = \frac{1}{\sqrt{2\pi\kappa}} e^{-\xi^2/2\kappa}\left[1 - \frac{\kappa_1}{(2\kappa)^{\frac{3}{2}}}\frac{1}{3\,!}(12\xi/\sqrt{2\kappa} - 8\xi^3/(2\kappa)^{\frac{3}{2}}) + \frac{\kappa_2}{(2\kappa)^2}\frac{1}{4\,!}(12 - \dots) + \dots\right];$$

the form of the regular law of error‡, where $\kappa, \kappa_1, \kappa_2$, &c. are determined in the usual way by the mean powers of the element under consideration. For the mean powers of the element as determined from the proposed locus are the same (up to any assigned power) as the given mean powers; and it has been postulated that when one locus has the same mean powers as another they may be considered as identical§. (β) The portion of any element which lies outside a limit equateable to κ multiplied by a small numerical

* See Appendix, Note 7.

† Or rather *at* x', if the function designated by x' varies with its variables $\xi_1, \xi_2, \dots$ discontinuously by very small steps.

‡ In the *form*, but not in general presenting the essential character of the law of error, viz. a series of approximating terms.

§ Part I. p. 40.

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coefficient (small with reference to $\sqrt{m}$) is an inconsiderable fraction*. By these propositions w, the locus of concurrent elements, may be put in the form†

$$\frac{1}{Q}\, e^{-\xi_1^2/2_1\kappa_0 - \xi_2^2/2_2\kappa_0 \dots} \left[1,\ -\frac{\Sigma_r\kappa_1}{(2_r\kappa_0)^{\frac{3}{2}}}\frac{1}{3\,!}\left(\frac{12\xi_r}{\sqrt{2_r\kappa_0}} - \frac{8\xi_r^3}{(2_r\kappa_0)^{\frac{3}{2}}}\right),\ +\Sigma_r\kappa_2\frac{1}{4\,!}(12-\dots)\right];$$

where the portion within square brackets outside the exponential breaks up into groups forming a descending series (the variables, by proposition (β), being restricted to the order of the corresponding power of ${}_r\kappa_0$‡).

These theorems are now to be employed in cases where x', the given function of the elements, may be expanded in ascending powers thereof, say

$$x' = \alpha_1\xi_1 + \alpha_2\xi_2 + \dots \alpha_n\xi_n,\ +\beta_{11}\xi_0^2 + 2\beta_{12}\xi_1\xi_2 + 2\beta_{13}\xi_1\xi_3 + \dots,\ +\gamma_{111}\xi_1^3 + 3\gamma_{21}\xi_1^2\xi_2 + 6\gamma_{123}\xi_1\xi_2\xi_3 + \dots,$$

where the α's, β's, and γ's are respectively of the order 1, $1/\sqrt{m}$, $1/m$..., as in general (with respect to the typical case§) it will be found necessary to postulate. Let us begin with the simple case in which the β's, γ's, &c. vanish, and the α's are all equal to unity, the case already treated by other methods, in which the compound x' is equal to the simple sum of the elements. By the present method we have for the locus of x', when in w (as above directed) there is substituted for ξ_1 its value $-\xi_2 - \xi_3 \dots - \xi_m + x'$, say $-\xi_2 + R$, $\left(\frac{d\xi_1}{dx'}\right.$ being in this case unity$\left.\right)$, an expression which may be put in the form

$$\frac{1}{Q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots e^{-\frac{({}_1\kappa_0+{}_2\kappa_0)}{2_1\kappa_0\,{}_2\kappa_0}\left(\xi_2 - R\frac{{}_2\kappa_0}{({}_1\kappa_0+{}_2\kappa_0)}\right)^2 - \frac{R^2}{2({}_1\kappa_0+{}_2\kappa_0)} - \frac{\xi_3^2}{2_0\kappa_3} - \frac{\xi_4^2}{2_0\kappa_4} - \dots}$$

$$\times\left[1 + \frac{{}_1\kappa_1}{(2_1\kappa_0)^{\frac{3}{2}}}\frac{1}{3\,!}\left(\frac{12(R-\xi_2)}{\sqrt{2_1\kappa_0}} - \frac{8(R-\xi_2)^3}{(2_1\kappa_0)^{\frac{3}{2}}}\right) - \frac{{}_2\kappa_1}{(2_2\kappa_0)^{\frac{3}{2}}}\frac{1}{3\,!}\left(\frac{12\xi_2}{\sqrt{2_2\kappa_0}} - \dots\right) + \dots\right] d\xi_2 d\xi_3 \dots d\xi_m$$

—to confine ourselves at first to a first correction. Change the variable ξ_2 by substituting for it $\xi_2' + R_2\kappa_0/({}_1\kappa_0 + {}_2\kappa_0)$, and integrate with respect to ξ_2' between extreme limits—for ξ_2', as for ξ_2, $+\infty$ and $-\infty$. There results an expression cleared of ξ_2 (as well as ξ_1), in which it will be observed that $\xi_3, \xi_4, \dots \xi_m, x'$, enter the exponent only in (powers and products of) the second dimension; and enter the parenthesis on the right of the exponential only in the first and third dimensions. Also the (inverse) coefficient on the left of the exponential, Q, which consisted originally of the continued product

$$\sqrt{2\,{}_1\kappa_0\pi},\quad \sqrt{2\,{}_2\kappa_0\pi}\dots\sqrt{2\,{}_m\kappa_0\pi}$$

is now transformed by substituting for the first two factors $\sqrt{2\,({}_1\kappa_0 + {}_2\kappa_0)\,\pi}$ (one $\sqrt{\pi}$ being ejected). Rearranging the index properly we may similarly get rid of ξ_3, the parenthesis on the right still involving only terms of the first and third dimensions of the variables, while in the coefficient on the left instead of the first *three* original factors we have now

$$\sqrt{2\,({}_1\kappa_0 + {}_2\kappa_0 + {}_3\kappa_0)\,\pi}$$

* Above, p. 116.

† The number of the element to which any κ coefficient belongs is designated by a subscript on the *left*; the place on the right being already devoted to the order of the coefficient.

‡ $\xi_r/\sqrt{2_r\kappa_0}$ being of the order unity, as we may say; small compared with $\sqrt{m}$.

§ See note on p. 117.

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(another $\sqrt{\pi}$ having been ejected). Continuing the process we reach ultimately the form

$$\frac{1}{\sqrt{2k\pi}} e^{-x'^2/2k} [1 + B_1 x' + B_3 x'^3].$$

We may be sure, in the present case for which the regular law of error has been otherwise demonstrated, that the constants B_1 and B_3 are identical with those pertaining to that law.

Next, let there be retained in the expansion of x' terms of the order $1/\sqrt{m}$ involving second powers (and products) of $\xi_1, \xi_2 \ldots$. We have then

$$\xi_1 = -\xi_2 - \xi_3 \ldots - \xi_m + x' - \beta_{11}(-\xi_2 - \xi_3 - \ldots + x')^2 - 2\beta_{12}(-\xi_2 - \xi_3 - \ldots + x')\xi_2 - \ldots$$

(with the degree of accuracy required for a second approximation) from which the expression for $\frac{d\xi_1}{dx'}$ can be derived. Substituting these values in the expression to be integrated we obtain an expression similar to that of the last paragraph; except that the exponent now contains, besides terms of the second dimension in $\xi_2, \xi_3 \ldots \xi_m, x'$, terms of the third dimension: such as $-2\beta_{11}x'^2\xi_2/2_1\kappa_0$ or $+2\cdot 2\beta_{12}\xi_2\xi_3\xi_4/2_1\kappa_0$. But as each of these terms is affected with a small coefficient, the variables also being (in effect) confined to a limit of the order $1/\sqrt{m}$, we are entitled to bring down these adventitious terms from the exponent to the external parenthesis; thus obtaining for the expression to be integrated a form similar to that which was dealt with in the last paragraph; the form not being impaired by the factor

$$\frac{d\xi_1}{dx'} (= 1 - 2\beta_{11}x' - 2\beta_{12}\xi_2 - 2\beta_{13}\xi_3 - \ldots).$$

Proceeding therefore, as before, we obtain a final result of the form

$$\frac{1}{\sqrt{2\pi k}} e^{-x'^2/2k} [1 + B_1 x' + B_3 x'^3].$$

But we have not now the same *a priori* assurance that this form is identical with the regular law of error. Indeed we may be sure that it is not identical with the law of error in the form which we have mostly employed—referred to the centre of gravity. For the mean first power is not zero, but $\beta_{11}\xi_1^{(2)} + \beta_{22}\xi_2^{(2)} + \ldots$ a quantity of the order $1/\sqrt{m}$, say β_1 (+ quantities of lower orders). Put $x' = x'' + \beta$, and, expanding in powers of β and neglecting small quantities, we shall obtain for the locus of x'' an expression of the form

$$y = \frac{1}{\sqrt{2\pi k}} e^{-x''^2/2k} [1 + B_1' x'' + B_3' x''^3].$$

This expression must satisfy the conditions that

(1) the area $\int_{-\infty}^{\infty} y dx \, (x''^{(0)}$, as we may say$) = 1$,

(2) $\int_{-\infty}^{\infty} x'' y dx \, (= x''^{(1)}) = 1$,

(3) $\int_{-\infty}^{\infty} x''^2 y dx \, (= x''^{(2)}) = k$,

(4) $\int_{-\infty}^{\infty} x''^3 y dx \, (= x''^{(3)}) = x'^{(3)} - 3x'^{(2)}\beta + 3x'^{(1)}\beta^2 - \beta^3$,

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(quantities of the order $1/m\,(\beta^2)$ being neglected). Equations (1) and (3) are already satisfied. We thus obtain, on performing the integration, two simple equations for B_1' and B_3', affording for the solution a unique system of values. But the same equations are satisfied by the corresponding coefficients in the form

$$\frac{1}{\sqrt{2\pi k}}e^{-\frac{x''^2}{2k}}\left[1-\frac{1}{3\,!}x''^{(3)}\frac{d^3}{dx''^3}\right].$$

Therefore the compared loci are identical as to their constants, as well as in form.

We have so far supposed that the α coefficients in the first term of the expansion of x' are each = unity. But it is easy to pass from the simple case which has been treated to that in which the α coefficient of ξ_r is α_r by observing that wherever ${}_r\kappa_0$ occurs in the preceding paragraphs $\mathrm{a}_r{}^2{}_r\kappa_0$ should now be written; and (more generally) wherever ${}_r\kappa_t$ occurs $\mathrm{a}_r{}^{t+2}{}_r\kappa_t$ should now be written.

By parity of reasoning we may obtain the frequency-locus of x' when account is taken of those terms in the expansion of x' which are affected with the γ's. Some difficulty in identifying the locus thus found with the regular law of error may be caused by the incident that the denominator in the exponent, viz.—as in the preceding paragraphs—$2k$, is not now (as might perhaps be expected) identical with the mean square of deviation for x'. That mean square

$$= \Sigma\xi^{(2)} + 2\Sigma\,(3\gamma_{112}\xi_1{}^{(2)}\xi_2{}^{(2)} + 3\gamma_{122}\xi_1{}^{(2)}\xi_2{}^{(2)} + 3\gamma_{231}\xi_3{}^{(2)}\xi_1{}^{(2)} + \ldots)$$

(up to the required degree of approximation) $= k + (\text{say})\,\gamma$. In order to compare the locus which has just been found with the law of error in the form that has been given, it is proper to substitute in the former, for k, $k'-\gamma$ (where k' is the mean square of error for x', and also—quantities below the order $1/m$ being neglected—for x''). Expanding in ascending powers of γ we shall not introduce any new powers of x''*, in addition to the first and third which come in by the first correction, and the second, fourth, and sixth powers which the second correction introduces. As before it may be proved that the frequency-locus of x'' must be identical as to its constants as well as in its form with the third approximation of the regular law of error.

By parity of reasoning we may proceed to another and another approximation, observing that the coefficient of any power of x'' outside the exponential, e.g. x, is now to be treated as made up of parts of different orders, e.g. $B_1x + D_1x$, B_1 being of the order $1/\sqrt{m}$, D_1 of the order $1/m^{\frac{3}{2}}$†.

When we go on to the case of several dimensions the question arises, how are we to understand an expansible frequency-function of several variables pertaining to a compound of numerous elements? The proper conception appears to be that the locus pertaining to each element, e.g. the rth, is of the form $\zeta = \phi_r(\xi, \eta, \ldots)$, assigning the frequency with

* As may be seen by observing that $\frac{dy_0}{dx} = \frac{1}{2}\frac{d^2y_0}{dx^2}$ where y_0 is the normal law of error (Part I. Sect. 2), and so the expansion in powers of γ will involve only even powers of x.

† The small quantities β and γ defined in the last paragraphs will be made up of different orders.

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which particular values of ξ and η are concurrently enjoyed by the rth element. The values of $\xi, \eta, \ldots$ are assumed by the rth element independently of all other elements, but not in general independently of each other, the variables for each element being in general correlated. There is sought a locus of the form

$$z = F(x', y', \ldots),$$

assigning the frequency with which particular values are concurrently enjoyed by the compound magnitudes x' and $y' \ldots$, each a (different) function of the elements, say,

$$x' = \psi(\xi_1, \xi_2, \ldots; \quad \eta_1, \eta_2, \ldots; \quad \ldots),$$
$$y' = \chi(\xi_1, \xi_2, \ldots; \quad \eta_1, \eta_2, \ldots; \quad \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

these functions admitting of being expanded in ascending powers of the ξ's and η's. Proceeding by analogy we obtain, by multiplying together the ϕ's, an expression for the frequency with which m particular systems of values will be concurrently enjoyed by the m elements. Substituting for ξ_1 and η_1 their values in terms of the other elements and $x', y', \ldots$, attending also to the determinant proper to the change of variables for ξ_1, η_1 to x', y'*, ..., we obtain as before an expression, with terms of the second dimension in the exponent and a descending series of groups in the external parenthesis. The exponent being put in the proper form, ξ_2 and η_2 may be got rid of by integration, and finally a result obtained which may be shown to be identical with the regular law of error both as to form and constants.

Slight relaxation of the fourth condition. The modification of the fifth attribute which has just been obtained suggests an analogous modification of the fourth condition, that the variations of the elements contributing to a compound observation should be independent†. Consider the exponent in the expression for ${}_1w$ in the preceding paragraphs, when first it was disturbed by our having put for ξ_1, not as in the simplest case‡ a linear function of $\xi_2, \xi_3, \ldots x'$, but one involving ascending powers of those variables. Just the same effect will be produced if we suppose that ${}_1w$—or rather what it becomes when the x' therein is replaced by $\xi_1 + \xi_2 + \xi_3 + \ldots + \xi_m$—now denotes the *law of interdependence* between the values of the elements, and the problem is to find the locus of x the sum of $\xi_1, \xi_2, \ldots \xi_m$ elements no longer perfectly independent, but correlated in such wise that ${}_1w$ represents the probability that a particular system of values is enjoyed by the elements. As the compound need not be a perfectly linear function of the elements, so the elements need not be perfectly independent of each other. The two kinds of imperfection may coexist. But the divergence from the typical ideal must in both cases be small. The suggestion that there is a general form of frequency-curve for a compound of elements interdependent in *any* way is met by

* Above, p. 122.

† It need hardly be said that the fourth condition is not infringed when the m elements are interdependent in such wise that the set can be broken up into a large number (n) of groups, the members of each group being correlated *inter se* but independent with respect to the members of all other groups. For then the sum of the elements in each group may be regarded as a single element obeying a definite frequency-law; so that, n being sufficiently large, the law of error will be fulfilled without further relaxation of the typical conditions.

‡ Above, p. 124.

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the same reasoning as that which has been applied to the supposition that the compound is *any* function of independent elements. We saw* that if we are free to take x' as any function of the elements, we can enable the frequency-locus to take any assigned form. Suppose that the relation of x' to the elements has been properly determined for this purpose; and that (according to the last method) the values of ξ_1 and $\frac{d\xi_1}{dx'}$ having been thence derived, we obtain ${}_1w$ in the form

$$\chi(x', \xi_2, \xi_3, \dots \xi_m);$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots {}_1w d\xi_2 d\xi_3 \dots d\xi_m$$

results in a function of x' which is of the assigned form, say $f(x')$. Now substitute in χ, for x', $\xi_1 + \xi_2 + \dots + \xi_m$; and let χ (thus modified) represent the law of interdependence between the elements. If there is sought the frequency-locus of x the sum of such elements, it is found to be $f(x)$!

Considerable relaxation of the third condition. The divergence from the type which has been discussed in the last section is to be distinguished from the interdependence between values assumed by the same elements in different observations. Such interdependence, if complete, would of course be fatal to the fulfilment of the law of error. If each of the elements had exactly the same value for all the observations there would be only one observation! But a less complete degree of such interdependence is not fatal. Other things being favourable, there will still be fulfilled the approximate equality between the mean powers of the actual and those of the representative locus. Only the long run by which the mean powers of the actual locus are to be ascertained will now be longer. The Pearsonian criterion applied incautiously to such a system would yield a too unfavourable answer to the enquiry whether a given set of observations should be regarded as having emanated from—being a sample of—a distribution which in the long run would present the law of error. For the number, n', which forms one of the data for that test is not now the number $(+1)$ of the observations in a proposed set†; as may be seen in the simple case where the observations are supposed to enter in pairs, the value of any element entering into the first, or any *odd* observation being the same as the value of that element which enters into the second, or generally the next, observation $({}_r\xi_1 = {}_r\xi_2;\ {}_r\xi_{p+1} = {}_r\xi_{p+2})$. The observations forming each pair being thus identical we should regard $\frac{1}{2}m$ as the effective number of independent observations.

The interdependence between the observations is more serious when they are considered, not as in this paper, with reference to the law of frequency according to which they are grouped, but as constituents of an average affording the measurement of an object under observation. In this case each observation—divided by the total number of observations—stands in the relation of what is here called an "element" to the average. Accordingly a considerable interdependence between these observations will (by the last section) impair the law of error, and the rules which are based on it.

* Above, p. 121.

† Compare Mr Bowley's remarks on the application of the criterion to statistics of wages, *Journ. Stat. Soc.* 1902, p. 338.

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Relaxation of the second condition. So far it has been taken for granted that throughout the long run formed by a series of observations the frequency-loci of the elements remain unaltered. But the conclusions which have been reached would not be affected by the relaxation of this condition, provided that the functions of the mean powers which have been utilised, $k_0, k_1, k_2, \ldots k_t$, remain undisturbed, however the individual elements change their form and κ coefficients. For example, the first approximation for the frequency of an aggregate made up of m_1 elements, each formed by the number of pips on a domino taken at random, and m_2 elements, each formed by the number of pips turned up by a die thrown at random—the constant k in the normal function which forms the first approximation to this law of frequency would be unaffected if in the course of a series of observations there were substituted for 198 dies, or any multiple thereof, the same multiple of 35 dominoes*.

The average, indeed, of the total number of pips, presented by the $m_1 + m_2$ elements, would be altered by the change; but even this disturbance would be avoided if there were substituted, for the 198 elements formed by the casting of dies, 35 elements, each consisting of the number of pips on a domino *plus* the constant 1·8†.

Laxity of the first condition. The condition that the varying values of the elements should be assumed at random limits the domain of our law less than might be supposed; first, because the condition is fulfilled far beyond the sphere of aleatory phenomena from which examples of randomness are usually selected, and secondly because the condition need not be fulfilled perfectly.

(1) Randomness is here understood as an objective‡ property of things, implying, *inter alia*, (α) that there is a "genus" or "series"§ of instances such that the proportion of instances possessing a certain attribute—belonging to a certain species—tends, as the number of instances is increased, to approach a fixed limit; (β) that a like series may be formed by taking as the genus each successive pair (or triplet, quartette...) of the original instances, the species being defined by the characters of the two (or more) constituent instances; (γ) that like propositions hold good however a set of instances is arranged so as to form new series. As these properties do not depend on our "ignorance" or feeling of uncertainty, so they are presented by phenomena which admit of the most exact knowledge and power of prediction. Such is the sequence of decimals in mathematical constants; the random character of which, in the case of π, has been shown by Dr Venn. There are doubtless in the region of physics an infinite number of examples like the following:—The dates in minutes of high-water in the morning at London Bridge, given in the *Nautical Almanack*, have been taken for January 1st, April 1st, and July 1st, 1903; and according as the number of minutes in the date fell in the first, second, third, or fourth quarter of the hour, the number 1, 2, 3, or 4 has been put down. The

* The mean square of deviation is for the domino (the sum of two digits) 16·5, for the die 2·91$\dot{6}$.

† The average number of pips turned up by 198 dies $= 3\frac{1}{2} \times 198 = 693$; the average number of pips on 35 dominoes $= 2 \times 4{\cdot}5 \times 35 = 315$; $693 - 315 = 378$; $378/35 = 1{\cdot}8$.

‡ Cp. Venn, *Logic of Chance*; and Leslie Ellis, in *Camb. Phil. Trans.*, vol. IX. p. 605, on the "realistic" nature of Probabilities.

§ *loc. cit.*

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three numbers thus obtained have been added together. A like triplet has been obtained from January 2nd, April 2nd, and July 2nd; and so on (cases in which no morning tide is given for one of the dates belonging to the triplet being omitted). Other triplets are formed by similarly combining the dates of February, May, August, 1903. There are thus obtained fifty figures of which the ideal distribution is deduced from the respective probabilities that a single triplet should have the values 3, 4, 5 ... 11, 12 on the supposition that high tide is as likely to occur at one part of an hour as another. The actual being compared with the ideal distribution by the Pearsonian criterion, the probability of random origin is found to be considerable, about ·7. A not very different result is obtained from the like treatment of afternoon tides.

(2) The illustration may be varied so as to illustrate imperfect randomness. If the triplets are selected—not from days separated by three months, but—from adjacent days, the periodicity of the tides impairs the random character of the result. But there are many methods of selecting triplets (or more composite elements)—from days that are neither very distant, not yet quite adjacent—of which it may be affirmed that they will probably result in an approximation to the ideal distribution. The character of periodicity—often consistent with, but sometimes fatal to, randomness—is illustrated by the digits of a recurring decimal with a long period. The writer has formed 180 decades out of the first 1800 figures in the period of the decimal which represents the fraction $\frac{1}{1861}$. These figures are given in the *Messenger of Mathematics* for 1864 (Vol. II. p. 1) in eighteen rectangular blocks, each block having ten rows and ten columns. For the first half of the series, the sum of the ten figures forming each column of a block has been taken; and for the second* half of the series the sum of the ten figures forming each row. If the figures dealt with obeyed a random law of frequency, the probability of any assigned digit occurring being $\frac{1}{10}$, it is to be expected that sums of ten digits would range according to an approximate law of error for which the mean square of deviation is $2\frac{1}{10}(0{\cdot}5^2 + 1{\cdot}5^2 + 2{\cdot}5^2 + 3{\cdot}5^2 + 4{\cdot}5^2) = 8{\cdot}25$. To test this the observed sums have been tabulated in a form convenient for the purpose of verification. In the annexed table the first row presents limits demarcating the total possible range for a sum of ten digits, between 0 and 100, into seven compartments. The second row shows the number of observed decades in each compartment. The third row gives the corresponding calculated number, that is the mean of the numbers that would be given in the long run by batches of ideally random digits. The fourth, fifth, and sixth rows give the data for Professor Karl Pearson's criterion of the probability that the observed set of statistics forms a random deviation from the theoretical frequency distribution assigned by the normal law of error. Employing the test we obtain for P, the probability that the given set is a sample of observations obeying the law of error, about 0·9. It is interesting to compare this result with one obtained from a set of 180 decades in other respects similar to the above, except that the digits were taken "at random" in the ordinary usage of the term—from mathematical and statistical tables†. The probability of this set having

* The better to secure that the random distribution of the digits should not be vitiated by the *complementary* character (*loc. cit.*, p. 39) of the digits forming the latter half of the period.

† Part of the figures referred to in the *Journal of the Statistical Society*, 1885, Jubilee volume, p. 186.

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Table showing the distribution of decades formed by adding successive digits in the periods of the decimal which represents $\frac{1}{1861}$.

Limits of compartments	0 — 35·5	40·5	43·5	46·5	49·5	54·5	100
Observed number of decades	26	28	22	23	22	36	23
Calculated number of decades (m)	26·604	29·223	22·367	23·611	22·367	29·223	26·604
e	0·604	1·223	0·367	0·611	0·367	6·777	3·604
e^2	0·365	1·44	0·135	0·37	0·135	45·928	12·98
e^2/m	0·01	0·05	0·006	0·015	0·006	1·57	0·48

$$\chi^2 = Se^2/m = 2{\cdot}1. \qquad P = 0{\cdot}9.$$

resulted from a random distribution proves to be about 0·2. The figures of a recurring decimal—the subject of exactest science—present the appearance of randomness in a higher degree than digits taken really at random! But if we vary the experiment by taking the first five digits of the decimal together with the five which occupy the 931st, 932nd, ... 935th places to form the first decade, and likewise for the second decade the five digits in places 6—10 together with the five in places 936—940, and so on, then the law of error entirely breaks down: there will be obtained a series of 186 compound observations, every one of which = 9*! Yet though the condition of randomness above designated (γ)† is thus imperfectly fulfilled, there are a great many ways of parcelling these digits which will probably result in a more or less perfect fulfilment of the law.

The laxity of the conditions which give rise to the law of error may explain its prevalence over an immense range of phenomena—from the movements of molecules to the actions of men.

* For a reason ably stated in the paper referred to (p. 39 *et seq.*). † p. 128.

F. Y. EDGEWORTH.

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APPENDIX.

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1. *Orders of magnitude.* The reasoning about errors requires that several orders of magnitude should be distinguished. There is (1) the limit within which the "typical" (above, p. 38) element is in effect (as shown at p. 115) confined, the practical range, as it may be called, of the element, usually a small multiple of the mean deviation in absolute quantity as given by the formula $|\xi| = (S|\xi|^t)^{\frac{1}{t}}$. (2) Corresponding to this limit of the element is the practical range of the compound quantity, a small multiple of its standard deviation $\sqrt{k}$ (or of its mean deviation $|x| = (S|x|^t)^{\frac{1}{t}}$). (3) It is evident, the number of the elements being large, that the last named limit belongs to a different order from the limit to which the compound observation might *possibly* extend, if every element contributed to the observation a value at the extremity of the element's practical range, on one and the same side of the centre of gravity. (4) For the element too as well as for the compound there may be a possible as well as a practical range; as is evident in the case when the element is itself composite. (5) The admission of order (4) introduces a higher order of infinity to be enjoyed by the compound observation on the immensely improbable supposition that every element contributes to the observation a value at the (same) extremity of its *possible* range. (6) There is lastly to be considered the magnitude $\Delta\xi$, or Δx (above, pp. 39, 113); which cannot be larger than the range of the least dispersed element—for example is equal to the range in the case of identical binomial elements—and may be indefinitely small. It appears to the writer that there is a propriety in treating this difference, however small, as still finite. Accordingly not much seems to be gained when, in certain versions of the Laplace-Poisson method, instead of the obvious quaesitum, what is the probability that a certain value of x meaning a certain multiple of Δx (+ a constant) should occur, there is substituted the complicating enquiry, what is the probability that x should have a value between limits $l \pm \eta$.

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2. *Degrees of approximation.* How far the series presented by the law of error are available for purposes of approximation is an investigation which bifurcates according as the representative is compared with the actual locus, (*a*) with respect to mean powers, as above proposed (Part I. § 1), or (*b*) with respect to ordinates, as usually. In both cases the general rules as to the convergence and remainder of series are to be applied to the data. It will be found that in general *ceteris paribus* the series are less available, (*a*) the higher the mean powers compared, (*b*) the larger the abscissa of the ordinates compared. A consideration of the first term in the expansion of the law of error supplies the following tests, in the case of symmetry. (*a*) $2p$, the highest power used for the purpose of comparing the actual and the representative locus, must be such that in the series

$$\frac{2p!\,k^p}{2^p p!}\left\{1+\frac{p(p-1).2^4}{4!}.\mathrm{k}_2+\ldots\right\} \qquad (\text{where } \mathrm{k}_2=k_2/(2k)^2),$$

the first term within the brackets shall be a small fraction. Whence, if k_2 is of the order $\frac{1}{m}$ in the typical case of irregular elements for which the κ coefficients do not vanish, p^2 ought to be small with respect to m. (*b*) x the abscissa of the outmost ordinate that is used for the purpose of comparison must be such that in the series $y_0+k_2\frac{1}{4}!\frac{d^4y_0}{dx^4}+\ldots$ (where y_0 is the normal error-function with k as constant) the second term shall be small with respect to the first. Whence $\mathrm{k}_2(\frac{1}{2}-2\,\mathrm{x}^2+\frac{2}{3}\mathrm{x}^4)$—where $\mathrm{x}=x/\sqrt{2k}$—ought to be small with respect to unity. When the loci are not symmetrical the conditions become more stringent.

Example. Let the locus of all the elements be the same, namely the simple one first considered by Laplace, the horizontal line extending from $-\alpha$ to $+\alpha$, with equation $\xi=\frac{1}{2\alpha}$ ($\Delta\xi$ being indefinitely small); and let the number of the elements be *three**. We have then $\kappa_0=\frac{1}{3}\alpha^2$; $\kappa_2=-\frac{2}{15}\alpha^4$; $\kappa_4=\frac{16}{63}\alpha^6$; ... and accordingly, for the aggregate of three elements $k_0=1\alpha^2$; $k_2=-\frac{2}{5}\alpha^4$; $k_4=\frac{16}{21}\alpha^6$; .... Whence for the representative locus we have

$$y=\frac{1}{\alpha\sqrt{2\pi}}\,e^{-x^2/2\alpha^2}\left(1-\frac{2}{5}\frac{1}{2^2}\left(\frac{1}{2}-2\,\frac{x^2}{2\alpha^2}+\frac{2}{3}\,\frac{x^4}{4\alpha^4}\right)+\ldots\right).$$

If (*a*) we compare the mean powers of the actual and the representative locus; the second and fourth powers are, by construction, the same for both. The sixth power of the actual locus, viz. $\alpha^6\,9\frac{16}{21}$, differs from the value given by the representative locus, not continued beyond the first term of expansion, by $\alpha^6\,\frac{16}{21}$, a considerable percentage. If we proceed to another power, the approximation breaks down, the first two terms of the series being $105\alpha^8$, $-84\alpha^8$. (*b*) The formula for the comparison of ordinates breaks down when—if not before—$\frac{2}{5}\frac{1}{2^2}\left(\frac{1}{2}-2\mathrm{x}^2+\frac{2}{3}\mathrm{x}^4\right)$ ceases to be small with respect to unity.

* The example is considered by the present writer in the *Journal of the Statistical Society* (1900, p. 73) as showing that a very small number of elements which are far from normal is sufficient to afford some approximation to the law of error. A larger number is required for a good approximation.

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These propositions may be illustrated by considering the form of the actual locus: for which the equation is* $y = \frac{1}{8\alpha^3}(3\alpha^2 - x^2)$ for values of the abscissa between $\pm\alpha$; and beyond those limits, $y = \frac{1}{16\alpha^3}(9\alpha^2 \mp 6\alpha x + x^2)$, the upper or the lower sign to be taken according as the positive or negative branch of the locus is designated. From these equations the mean sixth, eighth... powers can be found by integration. Also with the aid of proper tables† the ordinates for the actual and the representative locus can be compared at several points as in the annexed table.

Table showing ordinates (as multiples of α) corresponding to certain abscissae.

Length of abscissa (in units of α)	0	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$
First Approximation	0·399	0·3105	0·2418	0·1466
Next Approximation	0·379	0·3092‡		
Actual value	0·375	0·3125	0·25	0·1572

3. *Verification of the first method.* The fundamental formula expressing the mean of x^t, where x is the sum of numerous typical elements, this formula, proved above (Part I. § 1) by Generating Functions, may be verified by the Multinomial Law. First this law must be extended to *Mean* powers by the principle (employed above, p. 41), that if a, b, c are independently fluctuating statistical quantities, the Mean of $(a \times b \times c \ldots) =$ Mean of $a \times$ Mean of $b \times$ Mean of $c \times \ldots$. Therefore, if

$$x = \xi_1 + \xi_2 + \ldots + \xi_m, \quad x^{(t)} = t!\,\Sigma\xi_1^{(1)}\xi_2^{(1)} \ldots \xi_t^{(1)}, \quad + \frac{t!}{2!}\,\Sigma\xi_1^{(1)}\xi_2^{(1)} \ldots \xi^{(1)}{}_{t-2}\xi^{(2)}{}_{t-1}, + \ldots.$$

Consider first the case in which the ξ's resemble typical elements in that any mean power, e.g. the tth, is approximately the same for one element as for another, but differ from the typical ξ's in that negative values are not assigned to these items; they are such quantities as the deviations on one side only of the centre of gravity contemplated at p. 116, which might be designated $|\xi|$. Then $x^{(t)}$ is broken up into groups defined by the number of combinations between $|\xi|$'s. Thus in the first group the number of combinations is $\frac{m!}{t!\,m-t}$; in the second group $\frac{m!}{t-1!\,m-t+1!}$, that is smaller in the ratio $\frac{t}{m-t+1}$; and so on. The groups will form a descending series, provided that m is

* *Journ. Stat. Soc., loc. cit.*

† For the calculations on which the first two rows of the table are based there have been employed the tables given by Dr Burgess in the *Transactions of the Edinburgh Royal Society* for 1900 (vol. XXXIX.).

‡ The *following* approximation brings the value back again to the first; showing that the series is not available further.

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sufficiently large* (account being taken of the number of types, e.g. *two* in the *third* group of the series, and also of their coefficients).

An important case of this first theorem is where, for each of the positive quantities $|\xi_r|$, we put the square of a *typical* element, viz. ξ_r^2. We have thus, putting k for the mean of $\xi_1^2+\xi_2^2+\ldots+\xi_m^2$, that is the sum of the means $\xi_1^{(2)}+\xi_2^{(2)}+\ldots+\xi_m^{(2)}$,

$$k^t=(\xi_1^{(2)}+\xi_2^{(2)}+\ldots+\xi_m^{(2)})^t=(\xi_1^2+\xi_2^2+\ldots+\xi_m^2)^{(t)}$$

$$=t!\,\Sigma\xi_1^{(2)}\xi_2^{(2)}\ldots\xi_t^{(2)}+\frac{t!}{2!}\Sigma\xi_1^{(2)}\xi_2^{(2)}\ldots\xi^{(2)}{}_{t-2}(\xi^{(2)}{}_{t-1})^2+\ldots.$$

Whence $\Sigma\xi_1^{(2)}\xi_2^{(2)}\ldots\xi_t^{(2)}=\frac{1}{t!}k^t$, to a first approximation. Also

$$\frac{t!}{2!}\Sigma\xi_1^{(2)}\xi_2^{(2)}\ldots\xi^{(2)}{}_{t-2}(\xi^{(2)}{}_{t-1})^2=\frac{t!}{2!}\Sigma\xi_1^{(2)}\xi_2^{(2)}\ldots\xi^{(2)}{}_{t-2}\times\Sigma(\xi_r^{(2)})^2 \text{ approximately,}$$

= (by the former approximation)

$$\frac{t(t-1)}{2!}\times k^{t-2}\Sigma(\xi_r^{(2)})^2 \text{ approximately.}$$

Therefore $\quad \Sigma\xi_1^{(2)}\xi_2^{(2)}\ldots\xi_t^{(2)}=\frac{1}{t!}k^t-\frac{1}{t-2!}k^{t-2}\frac{\Sigma(\xi_r^{(2)})^2}{2!}$, to a second approximation.

Next, considering *typical* elements, since $\xi_1^{(1)}$, $\xi_2^{(1)}\ldots\xi_m^{(1)}$ all vanish,

$$x^{(2p)}=2p!\,\Sigma\frac{\xi_1^{(2)}}{2!}\frac{\xi_2^{(2)}}{2!}+\ldots\frac{\xi_p^{(2)}}{2!},+2p!\,\Sigma\frac{\xi_1^{(2)}}{2!}\frac{\xi_2^{(2)}}{2!}\ldots\frac{\xi^{(2)}{}_{p-2}}{2!}\frac{\xi^{(4)}{}_{p-1}}{4!}$$

$$+2p!\,\Sigma\frac{\xi_1^{(2)}}{2!}\frac{\xi_2^{(2)}}{2!}\ldots\frac{\xi^{(2)}{}_{p-3}}{2!}\frac{\xi^{(3)}{}_{p-2}}{3!}\frac{\xi^{(3)}{}_{p-1}}{3!},+\ldots.$$

Substituting for the continued products of mean-squares the approximate values given in the last paragraph, we have

$$x^{(2p)}=\frac{2p!\,k^p}{p!\,2^p}-\frac{2p!\,k^{p-2}}{p-2!\,2^{p-2}2^2}\frac{\Sigma(\xi_r^{(2)})^2}{2}+\frac{2p!\,k^{p-2}}{p-2!\,2^{p-2}}\frac{\Sigma\xi_r^{(4)}}{4!}+\frac{2p!\,k^{p-3}}{p-3!\,2^{p-3}}\frac{\Sigma\xi_r^{(3)}\xi_s^{(3)}}{3!\,3!}$$

$$=\frac{2p!\,k^p}{p!\,2^p}+\frac{2p!\,k^{p-2}}{p-2!\,2^{p-2}}\frac{1}{4!}\Sigma(\xi_r^{(4)}-3(\xi_r^{(2)})^2)+\frac{2p!\,k^{p-3}}{p-3!\,2^{p-3}}\frac{1}{2}\frac{(\Sigma\xi_r^{(3)})^2}{3!\,3!} \text{ (approximately).}$$

Replacing $\Sigma\xi_r^{(3)}$ by $k_1(=\Sigma\kappa_1)$ and $\Sigma(\xi_r^{(4)}-3(\xi_r^{(2)})^2)$ by $k_2(=\Sigma\kappa_2)$ we obtain the result already deduced (Part I. p. 43).

The expansion of odd powers may be similarly verified. The principle may also be extended to the case of two (or more) dimensions; verifying the expressions which have been found for $(X, Y)^{(t),(t')}$ (above, p. 118). Some difficulty may be caused here by the incident that *all* the terms involving mean *first* powers do not now vanish; there survive, along with terms involving only second powers, terms of the type

$$2p!\,\frac{\Xi_1^{(2)}}{2!}\frac{\Xi_2^{(2)}}{2!}\ldots\frac{\Xi^{(2)}{}_{p-1}}{2!}\Xi_p^{(1)}\Xi^{(1)}{}_{p+1}\times 2p'\frac{H_{\text{I}}^{(2)}}{2!}\frac{H_{\text{II}}^{(2)}}{2!}\ldots H_p^{(1)}H^{(1)}{}_{p+1}$$

* It must be large enough to cover any irregularities in the gradation which may be due to the absence of perfect equality between the mean powers of the different elements.

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(supposing e.g. t and t' even) where the first powers do not vanish, as they form factors $(\Xi_p, H_p)^{(1),(1)}$, $(\Xi_{p+1}, H_{p+1})^{(1),(1)}$ (in the notations of p. 116), say Λ_p and Λ_{p+1}, which, the coordinates for the same element being in general *correlated*, do not vanish. As there may occur any even number of these Λ's as factors of constituent terms in the predominant group of the multinomial expansion with which we are now concerned, it may seem at first sight that the character before ascribed to that group—a sum of products of mean squares—is no longer maintained. But it must be remembered that, as the axes X and Y are (with respect to the aggregate) *principal* axes, $(X, Y)^{(1),(1)}$, that is $(\Lambda_1 + \Lambda_2 + \ldots\ldots)$, vanishes. Accordingly $\Sigma\Lambda_1\Lambda_2$ has not the weight to which it seems entitled on account of the number of its terms. It has $\frac{m(m-1)}{2}$ terms, any one of which may in absolute quantity be of the order $\Xi^2 H^2$; yet the algebraic sum of these terms is equal to half the sum of only m squares, viz. $\Lambda_1^2 + \Lambda_2^2 + \ldots + \Lambda_m^2$ (since $(\Sigma\Lambda)^2 = 0$). If then we substitute in the last written expression for a portion of $(X, Y)^{(t),(t')}$, when $t = 2p$, $t' = 2p'$,

$$\Sigma 2p!\,\frac{\Xi_1^{(2)}}{2!}\,\frac{\Xi_2^{(2)}}{2!}\cdots\frac{\Xi^{(2)}_{p-1}}{2!} \times \Sigma 2p'!\,\frac{H_{\mathrm{I}}^{(2)}}{2!}\,\frac{H_{\mathrm{II}}^{(2)}}{2!}\cdots\frac{H^{(2)}_{p'-1}}{2!} \times \Sigma\Lambda_r\Lambda_s,$$

we see that this portion of the expansion does not rank with one which is made up of square factors only. The former is of the order $1/m$ as compared with the latter. Like reasoning is applicable to products of more than two Λ's. Thus $\Sigma\Lambda_1\Lambda_2\Lambda_3\ldots\Lambda_{2q}$ does not contribute to the terms of which it is a factor, a weight proportioned to the number of terms in the predominant group of $(\Sigma\Lambda)^q$ that is $\frac{m!}{m-q!}$, but only to the sum of the other terms in $\left(\sum_{r=1}^{r=m}\Lambda_r\right)^q$ which, m being very large with respect to q, is of an order $\frac{1}{m}$ compared with the number of terms in the seemingly predominant group of that series. So the Λ's do not affect the predominant group in the expansion of $(X, Y)^{(t),(t')}$ with which alone the first approximation is concerned; they are taken account of in the third approximation by the coefficients of the form ${}_{t,t'}K$, and in subsequent approximations.

4. *Variant of the second method.* The general solution of the leading partial differential equation in Professor Crofton's method, viz.

$$\frac{dy}{dk} = \frac{1}{2}\frac{d^2y}{dx^2},$$

may be written $\mathfrak{Q}e^{x\sqrt{2\frac{d}{dk}}}\phi(k)$*, where $\phi(k)$ is an arbitrary function involving the symbol $\sqrt{\frac{d}{dk}}$, and $\mathfrak{Q}$ denotes the *significant* part of the expression, the part not affected with that symbol. In the case before us we know by Professor Crofton's subsidiary condition

* Cp. Forsyth's *Differential Equations*, Art. 256.

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that $\phi(k)$ is of the form $\frac{1}{\sqrt{k}}f(k)$; and by theorems above given that y involves the coefficients k_1, k_2 ... in such wise that

$$\left(\frac{d}{dk_1}\right)^r y \equiv \left(-\frac{1}{3!}\left(\frac{d}{dx}\right)^3\right)^r y;$$

$$\left(\frac{d}{dk_2}\right)^r y \equiv \left(\frac{1}{4!}\left(\frac{d}{dx}\right)^4\right)^r y \ldots.$$

These conditions will be satisfied by putting for $\phi(k)$

$$Ae^{-\frac{1}{3!}k_1\left(2\frac{d}{dk}\right)^3+\frac{1}{4!}k_2\left(2\frac{d}{dk}\right)^4+\ldots}\frac{1}{\sqrt{2k}};$$

where the condition that, when k_1, k_2 ... vanish the integral of y between extreme limits should be 1, gives $A=1/\sqrt{\pi}$*. It will be observed that the operator $\left(\text{in } \frac{1}{\sqrt{2k}}\right)$ constituting $\phi(k)$ is the same operator used above in Part I. to express the law of error, say $F\left(\frac{d}{dx}\right)$, when for $\frac{d}{dx}$ is substituted $\sqrt{2}\frac{d}{dk}$. We have thus a variant form of the law of error

$$\mathfrak{D}e^{x\sqrt{2\frac{d}{dk}}}F\left(\sqrt{2\frac{d}{dk}}\right)\frac{1}{\sqrt{2\pi k}}.$$

This result may be verified by changing the order of the *two* operators on $\frac{1}{\sqrt{2\pi k}}$; substituting in F, thus transposed, for $\sqrt{2\frac{d}{dk}}$, $\frac{d}{dx}$ $\left(\text{as we are entitled to do by the equation } \frac{dy}{dk}=\frac{1}{2}\frac{d^2y}{dx^2}\right)$; and moving $\mathfrak{D}$, as after this substitution is legitimate, so as to follow F. We have then

$$F\left(\frac{d}{dx}\right)\mathfrak{D}e^{x\sqrt{2\frac{d}{dk}}}\frac{1}{\sqrt{2\pi k}}=F\left(\frac{d}{dx}\right)\frac{1}{\sqrt{2\pi k}}e^{-\frac{x^2}{2k}},$$

the original form.

5. *Test of the third method.* The ἀπορίαι of Leslie Ellis and Professor Czuber respecting Laplace's method† are not damaging to the version of it above given. They take as a test-case that in which the elements have each the locus, in our notation,

$$\eta=\frac{1}{2\gamma}e^{\mp\xi/\gamma},$$

where the upper sign is to be used for ξ positive, the lower for ξ negative. The parameter γ does not figure in the form given by the eminent authors; they treat as unity a magnitude

* Since $\mathfrak{D}e^{x\sqrt{2\frac{d}{dk}}}\frac{1}{\sqrt{2k}}=\frac{1}{\sqrt{2k\pi}}e^{-\frac{x^2}{2k}}$.

† Articles on "Method of Least Squares," *Camb. Phil. Trans.*, vol. IX. p. 605; *Theorie der Beobachtungsfehler*, pp. 267 *et seq.*

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which has been above treated as of the order $1/\sqrt{m}$ (the standard deviation of the composite locus being treated as unity).

Professor Czuber investigating, on the lines of Laplace, an expression for the probability that the aggregate of elements will lie between certain limits, questions (*loc. cit.*, p. 267) whether the results obtained by the approximations employed are not mere formal but actual approximations (*nicht blosse Näherungen der Form sondern auch Näherungen dem Werte nach sind*), unless some additional assumption is made (*loc. cit.*, p. 271). This doubt appears to be justified; but it does not touch the version above offered, in which it is proposed to supplement the reasoning as usually presented by additional—in general justifiable—assumptions: namely, (I) that the elements are such and so numerous that the coefficients $k_1, k_2, \ldots$ are each small in relation to the corresponding powers of k, the mean square of deviation for the aggregate; $k_2, k_4, \ldots$ and likewise $k_1, k_3, \ldots$ forming series descending in order of magnitude (above, p. 42); (II) that the abscissae of the ordinates for which an approximation is sought should not be large with respect to $\sqrt{k}$ (above, pp. 131—2); (III) as a consequence of (I), that the elements may be treated as virtually confined to a range which is small with respect to $\sqrt{k}$ (above, p. 115). The third assumption removes any scruple which might be felt in expanding $\chi(\alpha)$ (above, Part I. § 3) in ascending powers of α and mean powers of an element. In the case before us let us employ the rule suggested above (p. 116), and take as the limit on either side to which the range of the element is practically confined, two or three times OP where $OP = (|\xi|^{(10)})^{\frac{1}{10}}$. We have for the mean tenth power of deviations on either side $10!\,\gamma^{10}$; and, for OP, $\gamma \log_{-1} 0{\cdot}6559$, which, multiplied by 3, $> 13\gamma$. Let us be content with 10γ as the limit on either side of the centre; thus leaving out of account a proportion of area less than a twenty-thousandth (!); the operation of which cannot sensibly affect our results. Attending to the orders of magnitude involved in $\chi(\alpha)$ and $(\chi(\alpha))^m$ (*loc. cit.*) we cannot doubt the legitimacy of the step which gives

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\alpha^2+\alpha^4\frac{1}{4!}k_2+\cdots}\cos\alpha x\,d\alpha$$

for the required function. By assumption (I) we are entitled to differentiate with regard to k_2 under the sign of integration, and expand in powers of k_2, in a parenthesis outside the exponential; thus obtaining a series of definite integrals

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\alpha^2k}\cos\alpha x\,dx + \frac{1}{4!}k_2\int_{-\infty}^{\infty}\alpha^4 e^{-\alpha^2k}\cos\alpha x\,dx + \ldots = y_0 + \frac{1}{4!}k_2\frac{d^4y_0}{dx^4} + \ldots.$$

By assumption (II) such values of x are to be employed as (tested by the usual rules for series) make the above-written series convergent.

Ellis first obtains a series in ascending powers of u (his equivalent for our x) and descending powers of n (his equivalent for our m) representing the probability in the case proposed that the sum of n elements should be u (between u and $u+du$). Comparing this actual locus with the normal law of error he expresses a doubt about their coincidence, except upon the supposition that n is (not merely large, but) infinite. He remarks (comparing the two

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functions) "we retain $\frac{u^{2p}}{n^p}$ and neglect $\frac{u^{2(p-l)}}{n^p}$, although unless u be large, the former term is of the same or a lower order of magnitude than the latter." The explanation of this difficulty is that we are concerned only, in a first approximation, with the case in which u is of the order $\sqrt{n}$ [x of the order $\sqrt{m}\,\gamma$]. For u of this order it is quite proper to "retain $\frac{u^{2p}}{n^p}$ and neglect $\frac{u^{2(p-l)}}{n^p}$." The next order of magnitude (in the case of even loci) is $\left[\text{our } \frac{1}{m}, \text{ or } \frac{\gamma}{\sqrt{m}}\right]$ smaller in the proportion $\frac{1}{n}$, that is, of the order $\frac{1}{\sqrt{n}}$. When u is thus small, *both* the compared terms are to be neglected in the first approximation formed by the normal error-function. The actual and the representative function give the same value for $u\,[\equiv x] = 0$ when fractions of the order $\frac{1}{n}\left[\equiv \frac{1}{m}\right]$ are neglected. If greater accuracy is required the next approximation must be taken in; furnishing the correction $\frac{1}{2}k_2 \times y_0$ (where y_0 is the first approximation), for u zero or a quantity of the order $\frac{1}{n}$.

6. *Variants of the fourth method.* In applying the condition that the sought function must be *reproductive* (Part I. § 4) it was at first taken for granted, as a result of the fundamental theory (cp. p. 46), that in a first approximation we are concerned with only one constant (k). Other constants (k_1, k_2, ...) were introduced later. Otherwise, we might at once introduce one or more additional constants, say j, and determine ψ (*loc. cit.*) so that

$$m\psi(\alpha, j) = \psi(\rho\alpha, \sigma j).$$

A general solution of this functional equation, found by an extension of Boole's theorem referred to above (p. 55), is $A\alpha^t + B\alpha^{pt}j^{qt'} + Cj^{t'}$, where $p + q = 1$. This general solution is required for the application of the fourth method to two dimensions. But for the present purpose, if we may presume from the fundamental theorem that the constant is k_1, entering into the second approximation only in the first power, the general solution for ψ reduces to

$$A\alpha^t + B\alpha^{pt}k_1.$$

By the first approximation in the text we have $t = 2$, $\rho = \sqrt{m}$. Bringing down $B\alpha^{pt}k_1$ from the exponential we have for the first correction, if y_0 is the first approximation*,

$$(\text{Real part of } \pm 1, \text{ or } \sqrt{-1})\ Bk_1 \frac{d^{pt}y_0}{dx^{pt}};$$

assuming that pt must be an integer. We may employ the condition that k_1 enters the sought function *over* k_0^3† to determine that $pt = 3$; and accordingly $\sigma = 1/\sqrt{m}$.

Again, the investigation of the first approximation in the text may be varied by taking as subsidiary to the condition of *reproductivity* the partial differential equation which was taken as primary in Prof. Crofton's method, viz.

$$\frac{dy}{dk} = \frac{1}{2}\frac{d^2y}{dx^2} \text{ (above, p. 46, Part I. § 2).}$$

* See p. 53. † Above, p. 46 and cp. *Phil. Mag.* Feb. 1896, pp. 95, 96.

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Let the function, supposed to be reproductive, be

$$\frac{1}{c} f\left(\frac{x}{c}\right), \text{ where } c = q\sqrt{k} \text{ (above, p. 56).}$$

We have then, beginning with the case of symmetrical elements,

$$\chi(a) = \int_{-\infty}^{\infty} \frac{1}{c} f\left(\frac{x}{c}\right) \cos ax\,dx.$$

Put $\frac{x}{c} = \mathrm{x}$; and it is seen that $\chi(a)$ is of the form $\phi(ca)$, say $e^{\psi(ca)}$. By the first step taken in the text we find that $\psi(ca)$ must be of the form $\mathrm{a}(ca)^t$; and accordingly

$$y = \frac{A}{\pi} \int_0^{\infty} e^{\mathrm{a}(q\sqrt{k}\,a)^t} \cos ax\,da.$$

Now introduce the equation $\frac{dy}{dk} = \frac{1}{2}\frac{d^2y}{dx^2}$, and we have $\frac{t}{2}\mathrm{a}q^t a^t k^{\frac{t}{2}-1} \equiv -\frac{1}{2}a^2$; which can only be satisfied by making $t = 2$, $\mathrm{a}q^2 = -\frac{1}{2}$. (Substituting this value of t in the expression for y, we see that a must be negative in order to secure a finite result.) Integrate with respect to a and we obtain the normal function, A (and $\mathrm{a}q^2$) being determined by the conditions that the mean zero power $= 1$ (and the mean second power $= k$). The case of asymmetry may be analogously treated by the use of $e^{+\sqrt{-1}ax}$ for the *first* $\cos ax$, and $e^{-\sqrt{-1}ax}$ for the *second* $\cos ax$, in the preceding investigation (see the text, p. 56).

Once more, the last written partial differential might be treated as the leading one, and the condition of *reproductivity* as subsidiary. One of the general solutions of the equation

$$\frac{dy}{dk} = \frac{1}{2}\frac{d^2y}{dx^2}$$

is of the form

$$\int_0^{\infty} da \int_{-\infty}^{\infty} e^{-a^2a^2k} \cos a(x-\lambda)\,\psi(\lambda)\,d\lambda^*,$$

where $\psi(\lambda)$ is an arbitrary function. This may be regarded as the result of superposing† a statistical quantity having the frequency-locus $\psi(x)$ upon another quantity having the frequency-locus $\frac{1}{\pi}\int_0^{\infty} e^{-a^2a^2k} \cos ax\,da$, that is, the normal law of error. In order that this form should be preserved when another form is superposed, that other must be also normal (one constant only being admitted as is proper in a first approximation, above, p. 138). E.g. if $\psi(x)$ were of the form $\int_0^{\infty} e^{\mathrm{b}a^t} \cos ax$, where t is other than 2, it would not (though itself of the reproductive family), when superposed on the normal curve, produce a reproductive function of the simple species proper to the first approximation. Therefore $\psi(x)$, as well as the other component, is normal; and therefore the compound is normal.

7. *Substitute for the fifth proof.* The extension of the law of error to functions capable of being expanded in ascending powers of the elements was, in the text (above, p. 122), based

* Forsyth, *Differential Equations*, Art. 258.

† Above, p. 45.

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on a postulate which seems particularly appropriate to such a function; that the range of any element is (in effect) small. The proof may also be effected independently of that premiss, on the lines of the first method (Part I. § 1).

Let x' (as above, p. 123) be an expansible function of the elements; the first term of the expansion being, as we may put without loss of generality*,

$$\xi_1 + \xi_2 + \ldots + \xi_m (= x).$$

Consider any subsequent term of the expansion, e.g. $\mu\xi_1\xi_2\xi_3$ (when μ is small), making abstraction of other terms after the first. The mean powers of x', each divided by the corresponding factorial, are generated by a function of which the expansion in ascending powers of θ is

$$1 + \theta x'^{(1)} + \frac{1}{2!}\theta^2 x'^{(2)} + \frac{1}{3!}\theta^3 x'^{(3)} + \ldots,$$

where
$$x'^{(1)} = x^{(1)} + \mu\Sigma\xi_1^{(1)}\xi_2^{(1)}\xi_3^{(1)} = 0;$$
$$x'^{(2)} = x^{(2)} + \mu^2(\Sigma\xi_1\xi_2\xi_3)^{(2)}$$
$$= k + \mu^2\Sigma\xi_1^{(2)}\xi_2^{(2)}\xi_3^{(2)} = k + \mu^2\frac{1}{3!}k^3$$

(terms which vanish being omitted).

Likewise
$$x'^{(3)} = k_1 + \ldots,$$
$$x'^{(4)} = 3k^2 + k_2 + 24\mu\Sigma\xi_1^2\xi_2^2\xi_3^2 + \ldots = 3k^2 + k_2 + 4\mu k^3$$

(terms of an order lower than μ, as well as those which vanish, being omitted). And so on. Consider, as the generating function of mean powers,

$$\text{Exp. log}\left[1 + \theta^2\frac{1}{2!}\left(k + \frac{1}{3!}\mu^2k^3\right) + \theta^3\frac{1}{3!}(k_1) + \theta^4\frac{1}{4!}(k_2 + 4\mu k^3) + \ldots\right].$$

The character of the law of error will be preserved if the coefficients of θ^2, θ^3, $\theta^4, \ldots$, say $k_1', k_2', \ldots$ continue, when referred to the corresponding powers of k_0', to form a descending series. This may be verified by expanding the above-written logarithm in powers of θ. It will be found that no power of θ after θ^4 is affected with a coefficient of so high an order as μ if μ is of the order $\frac{1}{m}$†. The generating function up to the third degree of approximation is

$$e^{+\frac{1}{2!}k\theta^2 + \frac{1}{3!}k_1\theta^3 + \frac{1}{4!}\theta^4(k_2 + 4\mu k^3)};$$

as may be verified by comparing the value of $\frac{1}{2p!}x'^{(2p)}$ as given by the coefficient of θ^{2p} in the expansion of this function with the value obtained from the equation

$$x'^{(2p)} = (x + \mu\Sigma\xi_a\xi_b\xi_c)^{(2p)} = x^{(2p)} + 2p\mu\, 2p - 1\,!\, c\Sigma\xi_{\frac{1}{2!}}^{(2)}\xi_{\frac{2}{2!}}^{(2)}\xi_{\frac{3}{2!}}^{(2)} \ldots \xi_{\frac{p-2}{2!}}^{(2)}\xi_a^2\xi_b^2\xi_c^2,$$

* If the first term is
$\alpha_1\xi_1 + \alpha_2\xi_2 +$ &c., put $\xi_1 = \xi_1'/\alpha_1$, $\xi_2 = \xi_2'/\alpha_2 \ldots$.
(Cp. above, p. 125).

† No exception to this statement is made by a term of the form $\theta^6 c\mu k^3\Sigma\xi_r^{(4)}$ (c a small numerical constant) since $\Sigma\xi_r^{(4)}$ is of the order $1/m = \mu$.

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where c is the number of different arrangements that may be made with the same $(p+1)$ elements so as to form the same product of squares, viz. $(p+1)p(p-1)/3!$,

$$= x^{(2p)} + 2p!\,\mu k^{p+1}/2^{p-2}\,3!\,p-2!,$$

as it ought.

Analogously, verification may be obtained with respect to any other terms in the expansion of x', e.g. $\lambda\Sigma\xi_r^3$ or $\nu\Sigma\xi_1\xi_2\xi_3\xi_4$; where λ should be of the order $1/\sqrt{m}$, ν of the order $1/m^{\frac{3}{2}}$; and it is to be observed that $x'^{(1)}$, in the first instance, does not vanish, but $= \lambda k$.

Corresponding verifications are to be obtained in the cognate case of elements not perfectly independent*; when in that expression for the mean value of $e^{\theta\Sigma\xi}$ from which the generating function is derived (above, p. 41) the means of products such as $\xi_1\xi_2$, $\xi_1\xi_2^2$, $\xi_1\xi_2\xi_3$, $\xi_1^2\xi_2^2$, &c. (in the notation above used $(\xi_1, \xi_2)^{(1),(1)}$, $(\xi_1, \xi_2)^{(1),(2)}$, $(\xi_1, \xi_2, \xi_3)^{(1),(1),(1)}$, $(\xi_1, \xi_2)^{(2),(2)}$) differ by small quantities from what they would be if the elements were perfectly independent, viz. respectively,

$$0,\ 0,\ 0,\ \xi_1^{(2)}\xi_2^{(2)}.$$

In both cases the propositions admit of generalization.

* See p. 126 above on the analogy between curvilinear functions of independent elements and a linear function of interdependent elements.

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VI. *Memoir on the Orthogonal and other Special Systems of Invariants.*

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PART I.

Professor Elliott has constructed a complete syzygetic theory of the absolute orthogonal concomitants of binary quantics; his method is not symbolic, but it is quite convenient for obtaining the fundamental covariants of a quantic whose order is small or of a system of such quantics. In general the process leads to a Diophantine Equation which is difficult to handle for an order which is not small.

I propose, in the present research, to obtain results of a general character by the employment of a symbolical calculus involving imaginary umbræ. I obtain an inferior limit to the maximum degree of an irreducible covariant of given order, appertaining to a quantic of given order, and in certain cases I shew that there are also superior limits, but I have not succeeded in establishing this for the general case. For the first three degrees I obtain the actual numbers of irreducible absolute concomitants of the quantic of general degrees, and also some results in respect of the fourth degree. These results enable me to obtain the fundamental syzygies of the third degree.

§ 1.

Let the binary quantic be

$$a_x^n = b_x^n = c_x^n = \dots,$$

and the transformed quantic, due to the orthogonal substitution,

$$A_X^n = B_X^n = C_X^n = \dots;$$

then it is easy to shew that

$$x_1 \pm ix_2 = e^{\pm i\theta}(X_1 \pm iX_2),$$

$$a_1 \pm ia_2 = e^{\pm i\theta}(A_1 \pm iA_2).$$

I write

$$x_1 + ix_2 = \xi_1, \quad x_1 - ix_2 = \xi_2,$$

and I take new complex umbræ, linear combinations of the old umbræ, viz.: I write

$$a_1 + ia_2 = \alpha_1, \quad a_1 - ia_2 = \alpha_2,$$

$$b_1 + ib_2 = \beta_1, \quad b_1 - ib_2 = \beta_2.$$

$$\vdots \qquad\qquad \vdots$$

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The general expression of a non-absolute orthogonal concomitant of degree θ and order ϵ is

$$\alpha_1^{s_1}\alpha_2^{n-s_1}\beta_1^{s_2}\beta_2^{n-s_2}\ldots\xi_1^{\epsilon-p}\xi_2^{p},$$

where there are θ different symbols α, β, ...

For the form to denote an absolute concomitant the second of the exponents of symbols with suffix unity must be equal to the sum of exponents of symbols with suffix two.

Hence $$s_1+s_2+\ldots+s_\theta+\epsilon-p=\theta n-s_1-s_2\ldots-s_\theta+p,$$

or $$\Sigma s=\tfrac{1}{2}(\theta n-\epsilon)+p.$$

Since $$\xi_1\xi_2=x_1^2+x_2^2, \text{ is a covariant,}$$

and we are concerned with irreducible forms, we may put $p=\epsilon$ and take the form to be

$$\alpha_1^{s_1}\alpha_2^{n-s_1}\beta_1^{s_2}\beta_2^{n-s_2}\ldots\xi_2^{\epsilon},$$

where $$\Sigma s=\tfrac{1}{2}(\theta n+\epsilon).$$

The weight w may be taken to be

$$\tfrac{1}{2}(\theta n-\epsilon),$$

so that $$\theta n-\epsilon=2w.$$

The form $$\alpha_1^{s_1}\alpha_2^{n-s_1}\beta_1^{s_2}\beta_2^{n-s_2}\ldots\xi_2^{\epsilon},$$

for different values of s_1, s_2, ..., satisfying the condition

$$\Sigma s=\tfrac{1}{2}(\theta n+\epsilon),$$

denotes the asyzygetic invariantive forms of the degree θ and order ϵ. When $\epsilon>0$ we have the associated form

$$\alpha_1^{n-s_1}\alpha_2^{s_1}\beta_1^{n-s_2}\beta_2^{s_2}\ldots\xi_1^{\epsilon},$$

obtained by interchange of suffixes; hence in this case the invariantive form connotes two concomitants.

When $\epsilon=0$ there is a one-to-one correspondence between the invariantive forms and the invariants.

For example the linear quantic has the invariantive form of degree 2

$$\alpha_1\alpha_2^0\beta_1^0\beta_2,$$

which is unchanged by interchange of suffixes, and therefore yields one and not two invariants; in fact

$$\alpha_1\beta_2=(a_1+ia_2)(b_1-ib_2)=a_b-i(ab),$$

wherein (ab) is a vanishing form.

The covariantive form may be written

$$P+iQ,$$

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and denotes the two covariants

$$P,\ Q;$$

the reducibility of P and Q depends upon the reducibility of $P + iQ$ and conversely.

The symbolic identities in the umbræ, a, b, c, ... can be very simply obtained from the complex umbræ α, β, γ, ... Ex. gr. from the identity

$$\alpha_1\alpha_2 \,.\, \beta_1\beta_2 = \alpha_1\beta_2 \,.\, \alpha_2\beta_1,$$

we derive

$$a_a b_b = \{a_b - i\,(ab)\}\,\{a_b + i\,(ab)\} = a_b^2 + (ab)^2;$$

and from

$$\alpha_1\alpha_2 \,.\, \beta_1\beta_2 \,.\, \gamma_1\gamma_2 = \alpha_1\beta_2 \,.\, \beta_1\gamma_2 \,.\, \gamma_1\alpha_2,$$

we derive

$$a_a b_b c_c = a_b b_c c_a - a_b\,(bc)\,(ca) - b_c\,(ca)\,(ab) - c_a\,(ab)\,(bc),$$

$$a_b b_c\,(ca) + b_c c_a\,(ab) + c_a a_b\,(bc) = (ab)\,(bc)\,(ca),$$

by equation of real and imaginary parts.

It will be convenient at this point to have before us the irreducible invariantive forms for quantics of the first six orders for comparison with Elliott's list and for easy verification of general results to be presently established.

For brevity I will denote the form

$$\alpha_1^{s_1}\alpha_2^{n-s_1}\beta_1^{s_2}\beta_2^{n-s_2}\ldots\xi_2^{\epsilon},$$

by

$$(s_1 s_2 \ldots s_\theta;\ \epsilon).$$

Linear Quantic.

The irreducible forms are

$$(10;\ 0),\ (1;\ 1),\ \xi_1\xi_2;$$

where observe that the covariantive forms other than $\xi_1\xi_2$ connote two covariants; in fact

$$(1;\ 1) = \alpha_1\xi_2 = a_x - i\,(ax),$$

and we have $1 + 2 + 1 = 4$ invariants, viz.

$$a_b,\quad a_x,\quad (ax),\quad x_1^2 + x_2^2.$$

The table is

Degree \ Order	0	1	2
0			1
1		2	
2	1		

Quadratic.

The irreducible forms are

$$(1;\ 0),\quad (20;\ 0),\quad (2;\ 2),\quad \xi_1\xi_2;$$

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yielding five forms which in the umbræ $a, b, c, \ldots$ are

$$a_a, \quad a_b^2, \quad a_x^2, \quad a_x(xa), \quad x_x.$$

I give these in the most suitable forms, not restricting myself to those which are immediately obtained from the complex umbræ.

The table is

Degree \ Order	0	1	2
0			1
1	1		2
2	1		

Cubic.

The irreducible forms are

$$(30;\,0),\quad (21;\,0),\quad (3111;\,0),\quad (2220;\,0),$$
$$(2;\,1),\quad (311;\,1),\quad (31;\,2),\quad (3;\,3),\quad \xi_1\xi_2,$$

viz. 4 invariants and 9 covariants.

The table is

Degree \ Order	0	1	2	3
0			1	
1		2		2
2	2		2	
3		2		
4	2			

Quartic.

The irreducible forms are

$$(2;\,0),\quad (40;\,0),\quad (31;\,0),\quad (411;\,0),\quad (330;\,0),$$
$$(3;\,2),\quad (41;\,2),\quad (4;\,4),\quad \xi_1\xi_2,$$

viz. 5 invariants and 7 covariants.

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The table is

Degree \ Order	0	1	2	3	4
0			1		
1	1		2		2
2	2		2		
3	2				

Quintic.

The irreducible forms are

(50; 0), (41; 0), (32; 0),
(5311; 0), (4420; 0), (5221; 0), (4330; 0), (4222; 0), (3331; 0),
(552111; 0), (444300; 0), (522222; 0), (333330; 0),
(55511111; 0), (44444000; 0),
(3; 1), (521; 1), (440; 1), (422; 1),
(55111; 1), (52222; 1),
(51; 2), (42; 2), (5222; 2), $\xi_1\xi_2$,
(4; 3), (522; 3),
(52; 4),
(5; 5),

viz. 15 invariants and 27 covariants.

The table is

Degree \ Order	0	1	2	3	4	5
0			1			
1		2		2		2
2	3		4		2	
3		6		2		
4	6		2			
5		4				
6	4					
7						
8	2					

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Sextic.

The irreducible forms are

(3; 0), (60; 0), (51; 0), (42; 0),
(621; 0), (540; 0), (522; 0), (441; 0),
(6222; 0), (4440; 0), (6411; 0), (5520; 0),
(66111; 0), (55500; 0),
(4; 2), (61; 2), (52; 2), (622; 2), (550; 2), $\xi_1\xi_2$,
(5; 4), (62; 4), (6; 6),

viz. 14 invariants and 17 covariants.

The table is

Degree \ Order	0	1	2	3	4	5	6
0			1				
1	1		2		2		2
2	3		4		2		
3	4		4				
4	4						
5	2						

§ 2.

It has been shewn that the asyzygetic forms of given degree and order ϵ for a quantic of order n are given by

$$\alpha_1^{s_1}\alpha_2^{n-s_1}\beta_1^{s_2}\beta_2^{n-s_2}\ldots\xi_2^{\epsilon},$$

where the number of symbols α, β, ... is equal to the degree θ, and

$$s_1+s_2+\ldots+s_\theta=\tfrac{1}{2}(\theta n+\epsilon).$$

Hence the number of these asyzygetic forms is equal to the number of partitions of

$$\tfrac{1}{2}(\theta n+\epsilon)$$

into θ, or fewer, parts, none of which exceed n; the number is, in fact, the coefficient of

$$a^\theta x^{\frac{1}{2}(\theta n+\epsilon)},$$

in the ascending expansion of

$$\frac{1}{(1-a)(1-ax)\ldots(1-ax^n)};$$

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or, denoting $1-x^s$ by (s) for brevity, this is the coefficient of $x^{\frac{1}{2}(\theta n+\epsilon)}$ in

$$\frac{(n+1)(n+2)\dots(n+\theta)}{(1)(2)\dots(\theta)}.$$

The form $$\alpha_1^{s_1}\alpha_2^{n-s_1}\beta_1^{s_2}\beta_2^{n-s_2}\dots\xi_2^{\epsilon}$$

is reducible only if it be factorizable into other forms. This fact enables us to specify the conditions which lead to irreducibility and thus to specify those partitions of

$$\tfrac{1}{2}(\theta n+\epsilon),$$

into θ or fewer parts, which give irreducible forms. If the form can be exhibited in two ways as the product of irreducible forms of lower degree and order, we have evidence of one or two syzygies according as $\epsilon =$ or > 0. These observations will be verified as the work proceeds. The first of them enables us to find forthwith an inferior limit to the maximum order of covariants of given degree to a quantic of given order. A preliminary observation is that, if the form

$$\alpha_1^{s_1}\alpha_2^{n-s_1}\beta_1^{s_2}\beta_2^{n-s_2}\dots\xi_2^{\epsilon},$$

of degree θ, has a factor

$$\alpha_1^{t_1}\alpha_2^{n-t_1}\beta_1^{t_2}\beta_2^{n-t_2}\dots\xi_2^{\epsilon_1},$$

of degree $\theta_1\,(<\theta)$ and order $\epsilon_1\,(\leqslant\epsilon)$, the partitions of $\frac{1}{2}(\theta n+\epsilon)$ into θ, or fewer, parts must include some partition of

$$\tfrac{1}{2}(\theta_1 n+\epsilon_1)$$

into θ_1, or fewer, parts.

Inferior Limit to the maximum degree of an invariant to a quantic of even order $2n$.

Consider the invariant

$$\alpha_1^{2n}\alpha_2^{0}\beta_1^{2n}\beta_2^{0}\dots\lambda_1\lambda_2^{2n-1}\mu_1\mu_2^{2n-1}\dots,$$

of degree $2n-1$, to the quantic of order $2n$; there being $n-1$ symbolic factors with exponent $2n$ and n with the exponent unity.

Denote it briefly by

$$(2n)^{n-1}\,1^n.$$

This is a partition of

$$\tfrac{1}{2}(2n-1)\,2n$$

into $2n-1$ parts none greater than $2n$.

Assuming this partition to contain a partition of tn into t parts where $t<2n-1$ we must have

$$t_1\,.\,2n+(t-t_1)\,1=tn,$$

where $t_1\leqslant t$.

Hence $$t=\frac{2n-1}{n-1}\,t_1;$$

but this condition cannot be satisfied, for on the one hand $t<2n-1$, and on the other t_1 must contain $n-1$ as a factor, circumstances which are incompatible.

Hence the form is irreducible and so also is its conjugate obtained by interchange of suffixes.

THEOREM. *Two irreducible orthogonal invariants, of degree* $2n-1$, *appertain to the binary quantic of order* $2n$.

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Inferior Limit to the maximum degree of an invariant to a quantic of uneven order $2n+1$.

Consider the invariant

$$\alpha_1^{2n+1}\alpha_2^0\beta_1^{2n+1}\beta_2^0 \ldots \lambda_1\lambda_2^{2n}\mu_1\mu_2^{2n}\ldots,$$

of degree $4n$, to the quantic of order $2n+1$; there being $2n-1$ symbolic factors with exponent $2n+1$ and $2n+1$ with exponent unity.

Denote it briefly by

$$(2n+1)^{2n-1}\,1^{2n+1}.$$

This is a partition of

$$\tfrac{1}{2}\,.\,4n\,(2n+1)$$

into $4n$ parts none greater than $2n+1$.

Assuming this partition to contain a partition of $t(2n+1)$ into $2t$ parts, where $t<2n$, we must have

$$t_1(2n+1)+(2t-t_1)\,1=t\,(2n+1),$$

or

$$t=\frac{2n}{2n-1}\,t_1,$$

which, as in the case of even order, is absurd. Hence the form and its conjugate are irreducible.

THEOREM. *Two irreducible orthogonal invariants of degree* $4n$ *appertain to the binary quantic of order* $2n+1$.

Inferior Limit to the maximum degree of an irreducible covariant of given order 2ϵ *to the quantic of even order* $2n$, ϵ *being* >0.

Consider the form

$$\alpha_1^{2n}\alpha_2^0\beta_1^{n-1}\beta_2^{n+1}\gamma_1^{n-1}\gamma_2^{n+1}\ldots\xi_2^{2\epsilon};$$

there being $n-\epsilon$ symbolic factors with exponent $n-1$. It is of degree $n-\epsilon+1$ and order 2ϵ.

Denote it by

$$2n\,.\,n-1^{n-\epsilon}\xi_2^{2\epsilon},\quad\text{or}\quad(2n\,.\,n-1^{n-\epsilon};\ 2\epsilon);$$

and suppose a factor of the form to be

$$(s_1s_2\ldots s_\theta;\ 2\epsilon'),$$

where

$$\theta<n-\epsilon+1,\ \text{and}\ \epsilon'\leqslant\epsilon.$$

We have

$$s_1+s_2+\ldots+s_\theta=n\theta+\epsilon';$$

since the numbers $s_1, s_2, \ldots s_\theta$ must be drawn from the numbers

$$2n,\quad n-1,\quad n-1,\ldots n-1,$$

this necessitates either

$$\text{(i)}\quad 2n+(\theta-1)(n-1)=n\theta+\epsilon',$$

or

$$\text{(ii)}\quad \theta\,(n-1)=n\theta+\epsilon';$$

(i) leads to

$$\theta=n-\epsilon'+1,$$

or

$$n-\epsilon'+1<n-\epsilon+1,$$

$$\epsilon<\epsilon',$$

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which is inconsistent with the assumption

$$\epsilon' \leqslant \epsilon;$$

(ii) leads to $$\theta + \epsilon' = 0,$$

or $$\theta = \epsilon' = 0.$$

Hence the form is irreducible.

THEOREM. *Two irreducible orthogonal covariants of degree $n - \epsilon + 1$ and even order 2ϵ appertain to the binary quantic of even order $2n$, ϵ being > 0.*

Inferior Limit to the maximum degree of an irreducible covariant of given order ϵ to the quantic of uneven order $2n+1$, ϵ being > 0.

Consider the form

$$\alpha_1^{2n+1}\alpha_2^0\beta_1^n\beta_2^{n+1}\gamma_1^n\gamma_2^{n+1}\ldots\xi_2^\epsilon;$$

there being $2n+1-\epsilon$ symbolic factors with exponent n. This form is of degree $2n+2-\epsilon$ and order ϵ; denote it by

$$2n+1\,.\,n^{2n+1-\epsilon}\,\xi_2^\epsilon, \text{ or } (2n+1\,.\,n^{2n+1-\epsilon};\ \epsilon).$$

Suppose a factor of the form to be

$$(s_1 s_2 \ldots s_\theta;\ \epsilon'),$$

of degree θ and order ϵ', where

$$\theta < 2n+2-\epsilon, \quad \epsilon' \leqslant \epsilon;$$

then $$\Sigma s = \theta n + \tfrac{1}{2}(\theta + \epsilon').$$

Since the numbers $s_1, s_2, \ldots s_\theta$ must be drawn from the numbers

$$2n+1,\ n,\ n, \ldots n,$$

we have either

$$\text{(i)} \quad 2n+1+(\theta-1)\,n = \theta n + \tfrac{1}{2}(\theta+\epsilon'),$$

or $$\text{(ii)} \quad \theta n = \theta n + \tfrac{1}{2}(\theta+\epsilon');$$

(i) leads to $$\theta = 2n+2-\epsilon',$$

$$\therefore\ 2n+2-\epsilon' < 2n+2-\epsilon,$$

or $$\epsilon < \epsilon',$$

which is inconsistent with

$$\epsilon' \leqslant \epsilon;$$

(ii) leads to $$\theta = \epsilon' = 0.$$

Hence the form is irreducible.

THEOREM. *Two irreducible orthogonal covariants of degree $2n+2-\epsilon$ and order ϵ appertain to the binary quantic of uneven order $2n+1$, ϵ being > 0.*

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§ 3. Irreducible Invariants and Covariants.

Degree in coefficients 1; *order of quantic* $2n+1$.

The form of symbolic product is

$$\alpha_1^s \alpha_2^{2n+1-s} \xi_2^{2\epsilon+1},$$

where $$s = n + \epsilon + 1;$$

leading to $$\alpha_1^{n+\epsilon+1} \alpha_2^{n-\epsilon} \xi_2^{2\epsilon+1}.$$

There is thus one irreducible symbolic product yielding two irreducible covariants in respect of each uneven order. The order of the covariant evidently cannot exceed the order of the quantic.

Degree in coefficients 1; *order of quantic* $2n$.

The form of symbolic product is

$$\alpha_1^s \alpha_2^{2n-s} \xi_2^{2\epsilon},$$

where $$s = n + \epsilon;$$

leading to $$\alpha_1^{n+\epsilon} \alpha_2^{n-\epsilon} \xi_2^{2\epsilon},$$

giving one irreducible invariant and two irreducible covariants in respect of each even order. The order of the covariant cannot exceed that of the quantic.

Degree in coefficients 2; *order of quantic* $2n+1$.

The form of symbolic product is

$$\alpha_1^{s_1} \alpha_2^{2n+1-s_1} \beta_1^{s_2} \beta_2^{2n+1-s_2} \xi_2^{2\epsilon},$$

where $$s_1 + s_2 = 2n + 1 + \epsilon.$$

The number of asyzygetic products is equal to the number of binary partitions of

$$2n + 1 + \epsilon,$$

zero being admissible as a part, and no part exceeding $2n+1$.

The product may contain a factor

$$\alpha_1^{n+\epsilon'+1} \alpha_2^{n-\epsilon'} \xi_2^{2\epsilon'+1},$$

and then $$s_1 = n + \epsilon' + 1, \quad s_2 = n + \epsilon - \epsilon'.$$

Hence, for irreducibility, it suffices to have no part of the binary partition equal to

$$n + \epsilon' + 1,$$

for all values of ϵ' from 0 to $\epsilon - 1$; that is, no part must be taken from the series

$$n+1, \quad n+2, \ldots n+\epsilon.$$

The binary partitions satisfying these conditions are

$$2n+1, \epsilon;\ 2n, \epsilon+1;\ \ldots\ n+\epsilon+1, n;$$

$n - \epsilon + 1$ in number.

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Hence there are $$n-\epsilon+1$$
irreducible symbolic products and for
$$\epsilon=0,\ 1,\ 2,\ 3,\ \ldots$$
the number of irreducible covariants is
$$n+1,\quad 2n,\quad 2(n-1),\quad 2(n-2),\ \ldots$$
respectively.

Generally for the order 2ϵ, $(\epsilon>0)$, there are
$$2(n-\epsilon+1)$$
irreducible covariants.

The theorem given above in regard to the inferior limit to the maximum order shews that, for degree 2, the order of an irreducible covariant is at least equal to n; hence the results are consistent and in this case the inferior limit is also the superior limit.

Degree in coefficients 2; *order of quantic* $2n$.

The form of symbolic product is
$$\alpha_1^{s_1}\alpha_2^{2n-s_1}\beta_1^{s_2}\beta_2^{2n-s_2}\xi_2^{2\epsilon},$$
where $$s_1+s_2=2n+\epsilon.$$

The number of asyzygetic products is equal to the number of binary partitions of
$$2n+\epsilon,$$
zero being admissible as a part, but no part exceeding $2n$.

The product may contain a factor
$$\alpha_1^{n+\epsilon'}\alpha_2^{n-\epsilon'}\xi_2^{2\epsilon'},$$
and then $$s_1=n+\epsilon',\quad s_2=n+\epsilon-\epsilon'.$$

Hence, for irreducibility, it suffices to have no part of the binary partition equal to
$$n+\epsilon',$$
for all values of ϵ' from 0 to ϵ; that is, no part must be drawn from the series
$$n,\ n+1,\ \ldots\ n+\epsilon.$$

The binary partitions satisfying these conditions are
$$2n,\ \epsilon;\quad 2n-1,\ \epsilon+1;\ \ldots\ n+\epsilon+1,\ n-1;$$
$n-\epsilon$ in number.

Hence there are $$n-\epsilon$$
irreducible symbolic products and for
$$\epsilon=0,\ 1,\ 2,\ 3,\ \ldots$$
the number of irreducible covariants is
$$n,\ 2(n-1),\ 2(n-2),\ 2(n-3),\ \ldots$$
respectively.

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Generally for the order 2ϵ, $(\epsilon > 0)$, there are

$$2(n-\epsilon)$$

irreducible covariants.

No irreducible covariants exist when

$$\epsilon \geqslant n.$$

Moreover we have established above that 2 is an inferior limit to the maximum degree of a covariant of order $2n-2$ to a quantic of order $2n$, or, the same thing, that if the degree be 2 the order of an irreducible covariant is at least equal to $2n-2$. Hence the results are consistent and the inferior limit to the maximum order of an irreducible covariant of degree 2 to a quantic of order $2n$ is also a superior limit.

Degree in coefficients 3; *order of quantic* $2n+1$.

The form of symbolic product is

$$\alpha_1^{s_1}\alpha_2^{2n+1-s_1}\beta_1^{s_2}\beta_2^{2n+1-s_2}\gamma_1^{s_3}\gamma_2^{2n+1-s_3}\xi_2^{2\epsilon+1},$$

where $$s_1+s_2+s_3 = 3n+\epsilon+2.$$

We know that there is one irreducible product of this degree when ϵ is equal to $n-1$ and it will be shewn subsequently that ϵ cannot exceed $n-1$.

Put therefore $$n = \nu+\epsilon,$$

so that 2ν is the excess of the order of the quantic over the order of the covariant.

The symbolic product is then

$$\alpha_1^{s_1}\alpha_2^{2\nu+2\epsilon+1-s_1}\beta_1^{s_2}\beta_2^{2\nu+2\epsilon+1-s_2}\gamma_1^{s_3}\gamma_2^{2\nu+2\epsilon+1-s_3}\xi_2^{2\epsilon+1},$$

where $$s_1+s_2+s_3 = 3\nu+4\epsilon+2.$$

We are concerned with the partitions of

$$3\nu+4\epsilon+2$$

into 3 parts, zero parts being admissible but no part being greater than

$$2\nu+2\epsilon+1.$$

If the product be reducible there must be a factor

$$\alpha_1^{s}\alpha_2^{2\nu+2\epsilon+1-s}\xi_2^{2\epsilon'+1},$$

where $$s = \nu+\epsilon+\epsilon'+1,$$

and $$\epsilon' \leqslant \epsilon.$$

Hence the condition of irreducibility is that no part of the ternary partition of

$$3\nu+4\epsilon+2$$

must be drawn from the series

$$\nu+\epsilon+1,\quad \nu+\epsilon+2,\ \ldots\ \nu+2\epsilon+1.$$

I shall shew that the number of these partitions is a function of ν only.

The parts which may compose

$$3\nu+4\epsilon+2$$

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are separable into two series:

first series $\qquad 0,\ 1,\ 2,\ \ldots\ \nu+\epsilon;$

second series $\qquad \nu+2\epsilon+2,\ \ \nu+2\epsilon+3,\ \ldots\ 2\nu+2\epsilon+1.$

Since the sum of three parts of the first series is at most equal to

$$3\nu+3\epsilon,$$

and that of three parts of the second series at least equal to

$$3\nu+6\epsilon+6,$$

it is seen that the partitions must consist either of one part from the first series and two from the second, or of two parts from the first series and one from the second.

Taking the first alternative the parts may be $\nu-s-t-2$ from the first, $\nu+2\epsilon+2+s$, $\nu+2\epsilon+2+t$ from the second; where s, t assume all values from 0 to $\nu-2$ subject to the conditions

$$s+t\leqslant\nu-2,\quad s\leqslant t.$$

We have therefore a partition of the required form corresponding to every partition of

$$\nu-2$$

and lower numbers into 2 or fewer parts.

This number is the coefficient of

$$a^2x^{\nu-2}$$

in the expansion of the algebraic fraction

$$\frac{1}{(1-x)(1-a)(1-ax)(1-ax^2)}.$$

On the second alternative the parts may be $\nu+\epsilon-s$, $\nu+\epsilon-t$ from the first series, $\nu+2\epsilon+s+t+2$ from the second, where s, t assume all integer values subject to the conditions

$$s+t\leqslant\nu-1,\quad s\leqslant t.$$

We have, therefore, a partition of the required form corresponding to every partition of

$$\nu-1$$

and lower numbers into 2 or fewer parts.

This number is

$$\text{Co } a^2x^{\nu-1} \text{ in } \frac{1}{(1-x)(1-a)(1-ax)(1-ax^2)};$$

so that taking both alternatives the number is

$$\text{Co } a^2x^{\nu-1} \text{ in } \frac{1+x}{(1-x)(1-a)(1-ax)(1-ax^2)}$$

$$=\text{Co}_{\nu-1}\frac{1+x}{(1-x)^2(1-x^2)}=\text{Co}_{\nu-1}\frac{1}{(1-x)^3}$$

$$=\binom{\nu+1}{2}=\binom{n-\epsilon+1}{2}.$$

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Hence the number of irreducible symbolic products is

$$\binom{\nu+1}{2};$$

and the number of irreducible covariants of degree 3 and order $2\epsilon+1$ to the quantic of order $2n+1$ is

$$2\binom{\nu+1}{2} \text{ or } (n-\epsilon)(n-\epsilon+1).$$

As a verification it is not difficult to prove that the coefficient of

$$a^3x^{3n+\epsilon+2}$$

in the expansion of the fraction

$$\frac{(1-ax^{n+1})(1-ax^{n+2})\dots(1-ax^{n-\epsilon+1})}{(1-a)(1-ax)\dots(1-ax^{2n+1})},$$

is equal to $\frac{1}{2}(n-\epsilon)(n-\epsilon+1)$, and this supplies us with a proof that the order of an irreducible covariant of degree 3 cannot exceed

$$2n-2.$$

Degree in coefficients 3; *order of quantic* $2n$.

The form of symbolic product is

$$\alpha_1^{s_1}\alpha_2^{2n-s_1}\beta_1^{s_2}\beta_2^{2n-s_2}\gamma_1^{s_3}\gamma_2^{2n-s_3}\xi_2^{2n},$$

where $$s_1+s_2+s_3=3n+\epsilon.$$

For irreducibility the ternary partition of

$$3n+\epsilon$$

must have no part drawn from the series

$$n,\ n+1,\ \dots\ n+\epsilon.$$

Putting $$n=\nu+\epsilon,$$

consider the ternary partitions of

$$3\nu+4\epsilon$$

no part $>2\nu+2\epsilon$ and no part drawn from the series

$$\nu+\epsilon,\quad \nu+\epsilon+1,\ \dots\ \nu+2\epsilon.$$

The possible parts are in two series:

first series $$0,\ 1,\ 2,\ \dots\ \nu+\epsilon-1;$$

second series $$\nu+2\epsilon+1,\quad \nu+2\epsilon+2,\ \dots\ 2\nu+2\epsilon;$$

and, clearly, we must take one part from the first series and two from the second or two from the first and one from the second.

On the first alternative the parts may be $\nu-s-t-2$ from the first and $\nu+2\epsilon+1+s$, $\nu+2\epsilon+1+t$ from the second; and on the second alternative, $\nu+\epsilon-s-1$, $\nu+\epsilon-t-1$ from the first and $\nu+2\epsilon+s+t+2$ from the second.

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In both cases a partition corresponds to every partition of

$$\nu - 2$$

and lower numbers into 2 or fewer parts.

Hence the enumeration is given by

$$2\,\mathrm{Co}\, a^2 x^{\nu-2} \text{ in } \frac{1}{(1-x)(1-a)(1-ax)(1-ax^2)}$$

$$= 2\,\mathrm{Co}_{\nu-2} \frac{1}{(1-x)^2(1-x^2)} = 2\,\mathrm{Co}_{\nu-2} \frac{1+x^2+2x}{(1-x^2)^3}.$$

If ν be even this is

$$2\,\mathrm{Co}_{\frac{1}{2}(\nu-2)} \frac{1+x}{(1-x)^3}$$

$$= 2\binom{\frac{1}{2}(\nu+2)}{2} + 2\binom{\frac{1}{2}\nu}{2}$$

$$= \tfrac{1}{2}\nu^2.$$

If ν be uneven it is

$$2\,\mathrm{Co}_{\frac{1}{2}(\nu-1)} \frac{2x}{(1-x)^3}$$

$$= 4\binom{\frac{1}{2}(\nu+1)}{2}$$

$$= \tfrac{1}{2}(\nu^2-1).$$

Hence the number of irreducible symbolic products is $\frac{1}{2}\nu^2$ or $\frac{1}{2}(\nu^2-1)$ according as ν is even or uneven. Thence the number of irreducible invariants of degree 3 to the quantic of order $2n$ is

$$\tfrac{1}{2}n^2 \text{ or } \tfrac{1}{2}(n^2-1),$$

according as n is even or uneven; and the number of irreducible covariants, of degree 3 and order $2\epsilon\,(\epsilon > 0)$, is equal to

$$(n-\epsilon)^2 \text{ or } (n-\epsilon)^2 - 1,$$

according as $n-\epsilon$ is even or uneven.

This result is clearly consistent with the previously established theorem which states that for the degree 3 the inferior limit to the order of an irreducible covariant is $n-2$. The facts that have been discovered in regard to the quantic of general order are complete as far as the third degree in the coefficients inclusive and we may now enumerate the syzygies of the second and third degrees in the coefficients.

The syzygies, between symbolic products, are of two kinds, (i) those which involve the covariant $\xi_1\xi_2$; (ii) those which are free from this covariant. To obtain a syzygy of the first kind it is merely necessary to multiply a covariantive symbolic product by its conjugate to obtain a form which is divisible by some power of $\xi_1\xi_2$. I propose to avoid these by restricting myself to symbolic products of the form

$$\alpha_1^{s_1}\alpha_2^{n-s_1} \ldots \xi_2^{\epsilon};$$

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and I proceed to the study of the syzygies of the second kind appertaining to such forms. The syzygies of the first kind might be made the basis of the study of the whole question. By the method of the present paper they are not necessary.

The table of fundamental symbolic products is, for a quantic of order $2n+1$,

Degree \ Order	0	1	2	3	4	5	6	7	8	9
0			1							
1		1		1		1		1		1
2	$n+1$		n		$n-1$		$n-2$		$n-3$	
3		$\binom{n+1}{2}$		$\binom{n}{2}$		$\binom{n-1}{2}$		$\binom{n-2}{2}$		$\binom{n-3}{2}$

The corresponding table of invariants is obtained by multiplying all numbers except those in the first row and first column by two.

Of any degree-order we have the known theorem which states that the number of syzygies is equal to the excess over the asyzygetic forms of the sum of the number of compounds and the number of irreducible forms.

There are clearly no syzygies of degree 1.

Syzygies of Degree 2.

Order of Quantic $2n+1$.

The number of asyzygetic symbolic products of type

$$\alpha_1^{s_1}\alpha_2^{2n+1-s_1}\beta_1^{s_2}\beta_2^{2n+1-s_2}\xi_2^{2\epsilon}$$

is equal to the number of binary partitions of

$$2n+\epsilon+1,$$

no part being greater than $2n+1$.

This number, when ϵ is of the form $2\epsilon_1$ or $2\epsilon_1+1$, is equal to

$$n-\epsilon_1+1.$$

Moreover there is one irreducible symbolic product of degree one, for each uneven order, and, compounded of these, we have for the degree 2, and order 2ϵ, ϵ_1 or ϵ_1+1 compounds; also of degree 2 and order 2ϵ we have $n-\epsilon+1$ irreducible symbolic products. Hence

$$n-\epsilon_1+1+\text{number of syzygies}=\epsilon_1+n-\epsilon+1,$$

$$\text{or } =\epsilon_1+1+n-\epsilon+1,$$

according as ϵ is of the form $2\epsilon_1$ or $2\epsilon_1+1$.

In either case we find the number of syzygies equal to zero.

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SYZYGIES OF DEGREE 2.

Order of Quantic $2n$.

For a quantic of even order $2n$ the table of fundamental symbolic products is

Degree \ Order	0	1	2	3	4	5	6	7	8
0			1						
1	1		1		1		1		1
2	n		$n-1$		$n-2$		$n-3$		$n-4$
3	$\frac{1}{2}n^2$		$\frac{1}{2}(n-1)^2$		$\frac{1}{2}(n-2)^2$		$\frac{1}{2}(n-3)^2$		$\frac{1}{2}(n-4)^2$

where, in the fourth row, we take

$$\tfrac{1}{2}\{(n-\epsilon)^2-1\}$$

if $n-\epsilon$ be uneven.

Here, for degree 2, the number of asyzygetic symbolic products of type

$$\alpha_1^{s_1}\alpha_2^{2n-s_1}\beta_1^{s_2}\beta_2^{2n-s_2}\xi_2^{2\epsilon}$$

is equal to the number of binary partitions of

$$2n+\epsilon,$$

no part being greater than $2n$. This number is equal to $n-\epsilon_1+1$ or $n-\epsilon_1$ according as ϵ is equal to $2\epsilon_1$ or $2\epsilon_1+1$. Moreover there is one irreducible product of degree 1 for each even order, and, compounded of these, we have for the degree 2 and order 2ϵ, ϵ_1+1 compounds; also of degree 2 and order 2ϵ there are

$$n-\epsilon$$

irreducible symbolic products.

Hence, for $\epsilon=2\epsilon_1$,

$$\text{number of syzygies}=(\epsilon_1+1)+(n-2\epsilon_1)-(n-\epsilon_1+1)=0,$$

and for $\epsilon=2\epsilon_1+1$,

$$\text{number of syzygies}=(\epsilon_1+1)+(n-2\epsilon_1-1)-(n-\epsilon_1)=0.$$

In both cases the number of syzygies is zero.

Hence for no quantic is there a syzygy of degree 2.

SYZYGIES OF DEGREE 3.

Order of Quantic $2n+1$.

I proceed to prove that the number of fundamental syzygies of order $2\epsilon+1$ is the nearest integer to

$$\tfrac{1}{4}(\epsilon+1)^2, \text{ where } n\geqslant\epsilon.$$

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The number of asyzygetic symbolic products of degree and order 3, $2\epsilon+1$ is, by previous work,

$$\mathrm{Co}_{3n+\epsilon+2}\frac{(2n+2)(2n+3)(2n+4)}{(1)(2)(3)}.$$

Compounds are obtained by multiplying products of degree 1 and order $2t+1$ by products of degree 2 and order $2\epsilon-2t$ for all values of t from 0 to ϵ; the number of these is

$$(n-\epsilon+1)+(n-\epsilon)+\ldots+(n+1)=n(\epsilon+1)-\binom{\epsilon}{2}+1.$$

Compounds are also obtained by taking the products of degree 1, three together; the number of these by a well-known theorem is equal to

$$\mathrm{Co}_{\epsilon-1}\frac{1}{(1)(2)(3)}.$$

Moreover the number of irreducible products is

$$\tfrac{1}{2}(n-\epsilon)(n-\epsilon+1).$$

Hence the number of syzygies is

$$n(\epsilon+1)-\binom{\epsilon}{2}+1+\tfrac{1}{2}(n-\epsilon)(n-\epsilon+1)+\mathrm{Co}_{\epsilon-1}\frac{1}{(1)(2)(3)}-\mathrm{Co}_{3n+\epsilon+2}\frac{(2n+2)(2n+3)(2n+4)}{(1)(2)(3)}.$$

It is somewhat remarkable that the number, denoted by this expression, is independent of n and probably there is some easy way of establishing this *a priori*. I found this to be the case by actually working out the value of the number. The result is easily verified, so that I do not intend to reproduce the investigation here.

Assuming the value to be independent of n it follows that n may be assigned any suitable integer value for the purpose of arriving at the number of the syzygies.

I find it most convenient to put n equal to $\epsilon+1$, viz., to assign to n the least value which yields an irreducible product of degree 3 and order 2ϵ.

The expression to be evaluated then becomes

$$(\epsilon+1)^2-\tfrac{1}{2}\epsilon(\epsilon-1)+1+1+\mathrm{Co}_{\epsilon-1}\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon+5}\frac{(2\epsilon+4)(2\epsilon+5)(2\epsilon+6)}{(1)(2)(3)}$$

$$=\tfrac{1}{2}\epsilon^2+\tfrac{5}{2}\epsilon+3+\mathrm{Co}_{\epsilon-1}\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon+5}\frac{1}{(1)(2)(3)}+\mathrm{Co}_{2\epsilon+1}\frac{1}{(1)^2(2)}.$$

When $\epsilon=2\epsilon_1$ the last three terms may be written

$$\mathrm{Co}_{2\epsilon_1}\frac{x(1+x)(1+x^2)}{(2)^2(6)}-\mathrm{Co}_{8\epsilon_1+5}\frac{x(1+x)(1+x^2)}{(2)^2(6)}+\mathrm{Co}_{4\epsilon_1+2}\frac{x(1+x)^2}{(2)^2},$$

or

$$\mathrm{Co}_{\epsilon_1}\frac{x+x^2}{(1)^2(3)}-\mathrm{Co}_{4\epsilon_1+3}\frac{x+x^2}{(1)^2(3)}+2\mathrm{Co}_{2\epsilon_1+1}\frac{x}{(1)^2},$$

or

$$\mathrm{Co}_{\epsilon_1}\frac{x+x^2}{(1)^2(3)}-\mathrm{Co}_{2\epsilon_1+1}\frac{(1+x)^2}{(1)^2(3)}+2\binom{2\epsilon_1+2}{2},$$

or

$$\mathrm{Co}_{\epsilon_1}\frac{x+x^2}{(1)^2(3)}-\mathrm{Co}_{\epsilon_1+1}\frac{5x+11x^2+11x^3+5x^4}{(1)^2(3)}+2\binom{2\epsilon_1+2}{2},$$

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or
$$-\mathrm{Co}_{\epsilon_1}\frac{5(1+2x+2x^2+x^3)}{(1)^2(3)}+2\binom{2\epsilon_1+2}{2},$$

or
$$-5\mathrm{Co}_{\epsilon_1}\frac{1}{(1)^3}-5\mathrm{Co}_{\epsilon_1-1}\frac{1}{(1)^3}+2\binom{2\epsilon_1+2}{2},$$

or
$$-5\binom{\epsilon_1+2}{2}-5\binom{\epsilon_1+1}{2}+2\binom{2\epsilon_1+2}{2}.$$

Hence, for $\epsilon=2\epsilon_1$, the whole expression is
$$2\epsilon_1^2+5\epsilon_1+3-\tfrac{5}{2}(\epsilon_1^2+3\epsilon_1+2)-\tfrac{5}{2}(\epsilon_1^2+\epsilon_1)+4\epsilon_1^2+6\epsilon_1+2,$$
which is $\epsilon_1^2+\epsilon_1$ or $\frac{1}{4}(\epsilon+1)^2-\frac{1}{4}$.

On the other hand, when
$$\epsilon=2\epsilon_1+1,$$
the three terms may be written
$$\mathrm{Co}_{2\epsilon_1}\frac{1}{(1)(2)(3)}-\mathrm{Co}_{8\epsilon_1+9}\frac{1}{(1)(2)(3)}+\mathrm{Co}_{4\epsilon_1+3}\frac{1}{(1)^2(2)}.$$

Now
$$\mathrm{Co}_{2\epsilon_1}\frac{1}{(1)(2)(3)}=\mathrm{Co}_{\epsilon_1}\frac{1+x^2}{(1)^2(3)},$$
$$\mathrm{Co}_{8\epsilon_1+9}\frac{1}{(1)(2)(3)}=\mathrm{Co}_{4\epsilon_1+5}\frac{x+x^2}{(1)^2(3)}=\mathrm{Co}_{4\epsilon_1+4}\frac{(1+x)^3(1+x^2)}{(2)^2(6)}$$
$$=\mathrm{Co}_{2\epsilon_1+2}\frac{(1+x)^3}{(1)^2(3)}=\mathrm{Co}_{2\epsilon_1+2}\frac{(1+x)^5(1+x^2)}{(2)^2(6)}$$
$$=\mathrm{Co}_{\epsilon_1+1}\frac{1+10x+10x^2+10x^3+x^4}{(1)^2(3)}.$$

Therefore
$$\mathrm{Co}_{2\epsilon_1}\frac{1}{(1)(2)(3)}-\mathrm{Co}_{8\epsilon_1+9}\frac{1}{(1)(2)(3)}$$
$$=-\mathrm{Co}_{\epsilon_1+1}\frac{1+9x+10x^2+9x^3+x^4}{(1)(2)(3)}$$
$$=-\mathrm{Co}_{\epsilon_1+1}\frac{1}{(1)^3}-8\mathrm{Co}_{\epsilon_1}\frac{1}{(1)^3}-\mathrm{Co}_{\epsilon_1-1}\frac{1}{(1)^3}$$
$$=-\binom{\epsilon_1+3}{2}-8\binom{\epsilon_1+2}{2}-\binom{\epsilon_1+1}{2}.$$

Hence for $\epsilon=2\epsilon_1+1$ the whole expression is
$$\tfrac{1}{2}(2\epsilon_1+1)^2+\tfrac{5}{2}(2\epsilon_1+1)+3-\tfrac{1}{2}(\epsilon_1^2+5\epsilon_1+6)-4(\epsilon_1^2+3\epsilon_1+2)-\tfrac{1}{2}(\epsilon_1^2+\epsilon_1)$$
$$+2\mathrm{Co}_{2\epsilon_1+1}\frac{1}{(1)^3},$$
which is
$$-3\epsilon_1^2-8\epsilon_1-5+4\epsilon_1^2+10\epsilon_1+6=\epsilon_1^2+2\epsilon_1+1=(\epsilon_1+1)^2,$$
and this is
$$\tfrac{1}{4}(\epsilon+1)^2.$$

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Hence the number of syzygies of degree 3 and order 2ϵ to the quantic or order $2n+1$ is in every case the nearest integer to

$$\tfrac{1}{4}(\epsilon+1)^2,$$

where, from previous work, ϵ does not exceed n.

As an illustration we have one syzygy deg-order 3, 3 for the cubic, viz.:—

$$\alpha_1^2\alpha_2\xi_2 \,.\, \beta_1^2\gamma_1\gamma_2^2\xi_2^2 = \alpha_1^2\alpha_2\gamma_1\gamma_2^2 \,.\, \beta_1^2\xi_2^3,$$

one syzygy deg-order 3, 3 for the quintic, viz.:—

$$\alpha_1^3\alpha_2^2\xi_2 \,.\, \beta_1^4\beta_2\gamma_1^2\gamma_2^3\xi_2^2 = \alpha_1^3\alpha_2^2\gamma_1^2\gamma_2^3\beta_1^4\beta_2\xi_2^3,$$

and generally one syzygy deg-order 3, 3 for the $2n+1$-ic, viz.:—

$$\alpha_1^{n+1}\alpha_2^n\xi_2 \,.\, \beta_1^{n+2}\beta_2^{n-1}\gamma_1^n\gamma_2^{n+1}\xi_2^2 = \alpha_1^{n+1}\alpha_2^n\gamma_1^n\gamma_2^{n+1} \,.\, \beta_1^{n+2}\beta_2^{n-1}\xi_2^3.$$

Similarly for the quintic we have two syzygies of deg-order 3, 5, viz.:—

$$\alpha_1^5\beta_1\beta_2^4\xi_2^2 \,.\, \gamma_1^4\gamma_2\xi_2^3 = \alpha_1^5\xi_2^5 \,.\, \gamma_1^4\gamma_2\beta_1\beta_2^4,$$

$$\alpha_1^3\alpha_2^2\xi_2 \,.\, \beta_1^5\gamma_1^2\gamma_2^3\xi^4 = \beta_1^5\xi_2^5 \,.\, \alpha_1^3\alpha_2^2\gamma_1^2\gamma_2^3,$$

and generally, for the $2n+1$-ic, $n \geqslant 2$,

$$\alpha_1^{n+3}\alpha_2^{n-2}\beta_1^{n-1}\beta_2^{n+2}\xi_2^2 \,.\, \gamma_1^{n+2}\gamma_2^{n-1}\xi_2^3 = \alpha_1^{n+3}\alpha_2^{n-2}\xi_2^5 \,.\, \gamma_1^{n+2}\gamma_2^{n-1}\beta_1^{n-1}\beta_2^{n+2},$$

$$\alpha_1^{n+1}\alpha_2^n\xi_2 \,.\, \beta_1^{n+3}\beta_2^{n-2}\gamma_1^n\gamma_2^{n+1}\xi_2^4 = \beta_1^{n+3}\beta_2^{n-2}\xi_2^5 \,.\, \alpha_1^{n+1}\alpha_2^n\gamma_1^n\gamma_2^{n+1};$$

the syzygies for the $2n+3$-ic being obtained from these by multiplication throughout by

$$\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2.$$

Syzygies of Degree 3.

Order of Quantic 2n.

It will be shewn that the number of fundamental syzygies of order 2ϵ is the nearest integer to

$$\tfrac{1}{4}\epsilon^2.$$

The number of asyzygetic symbolic products of degree and order 3, 2ϵ is, by previous work,

$$\mathrm{Co}_{2n+\epsilon}\,\frac{(2n+1)(2n+2)(2n+3)}{(1)(2)(3)}.$$

Compounds are obtained by multiplying products of degree 1 and order $2t$ by products of degree 2 and order $2\epsilon-2t$ where t has all values from 0 to ϵ. The number of these is

$$(n-\epsilon)+(n-\epsilon+1)+\ldots+n \text{ or } n(\epsilon+1)-\binom{\epsilon+1}{2}.$$

Compounds are also obtained by taking the products of degree 1 three together. The number of these is

$$\mathrm{Co}_\epsilon\,\frac{1}{(1)(2)(3)}.$$

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Moreover the number of irreducible products of degree 3 and order 2ϵ is

$$\tfrac{1}{2}(n-\epsilon)^2 \text{ or } \tfrac{1}{2}(n-\epsilon)^2-\tfrac{1}{2}$$

according as $n-\epsilon$ is even or uneven. Hence when $n-\epsilon$ is even the number of syzygies is

$$n(\epsilon+1)-\tfrac{1}{2}\epsilon(\epsilon+1)+\tfrac{1}{2}(n-\epsilon)^2+\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{2n+\epsilon}\frac{(2n+1)(2n+2)(2n+3)}{(1)(2)(3)}.$$

It may be verified that this number is independent of n; we may therefore give n any value such that $n-\epsilon$ is even; it is convenient to put $n=\epsilon$ and then the number has the expression

$$\tfrac{1}{2}\epsilon(\epsilon+1)+\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon}\frac{1}{(1)(2)(3)}+\mathrm{Co}_{2\epsilon-1}\frac{1}{(1)^2(2)},$$

which is

$$\tfrac{1}{2}\epsilon(\epsilon+1)+\epsilon(\epsilon+1)+\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon}\frac{1}{(1)(2)(3)}.$$

Now
$$\mathrm{Co}_{4\epsilon}\frac{1}{(1)(2)(3)}=\mathrm{Co}_{4\epsilon}\frac{(1+x)(1+x^2)}{(2)^2(6)}=\mathrm{Co}_{2\epsilon}\frac{1+x^2}{(1)^2(3)}$$
$$=\mathrm{Co}_{2\epsilon}\frac{(1+x^2)(1+x)^2(1+x^2)}{(2)^2(6)}$$
$$=\mathrm{Co}_\epsilon\frac{(1+x)(1+x+2x^2)}{(1)^2(3)}.$$

Hence
$$\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon}\frac{1}{(1)(2)(3)}=\mathrm{Co}_\epsilon\frac{1-(1+x)^2(1+x+2x^2)}{(1)(2)(3)}$$
$$=-\mathrm{Co}_\epsilon\frac{3x(1+x+x^2)+2x^2(1+x+x^2)}{(1)(2)(3)}$$
$$=-3\mathrm{Co}_{\epsilon-1}\frac{1}{(1)^2(2)}-2\mathrm{Co}_{\epsilon-2}\frac{1}{(1)^2(2)}.$$

If $\epsilon=2\epsilon_1$ this is

$$-3\mathrm{Co}_{2\epsilon_1}\frac{x(1+x)^2}{(2)^3}-2\mathrm{Co}_{2\epsilon_1-2}\frac{(1+x)^2}{(2)^3},$$

or
$$-3\mathrm{Co}_{\epsilon_1}\frac{2x}{(1)^3}-2\mathrm{Co}_{\epsilon_1-1}\frac{1+x}{(1)^3},$$

or
$$-6\binom{\epsilon_1+1}{2}-2\binom{\epsilon_1+1}{2}-2\binom{\epsilon_1}{2} \text{ or } -5\epsilon_1^2-3\epsilon_1;$$

giving a number of syzygies equal to

$$\tfrac{3}{2}2\epsilon_1(2\epsilon_1+1)-5\epsilon_1^2-3\epsilon_1,$$

which is
$$\epsilon_1^2 \text{ or } \tfrac{1}{4}\epsilon^2.$$

If $\epsilon=2\epsilon_1+1$,

$$\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon}\frac{1}{(1)(2)(3)}$$
$$=-3\mathrm{Co}_{2\epsilon_1}\frac{(1+x)^2}{(2)^3}-2\mathrm{Co}_{2\epsilon_1}\frac{x(1+x)^2}{(2)^3}$$
$$=-3\mathrm{Co}_{\epsilon_1}\frac{1+x}{(1)^3}-4\mathrm{Co}_{\epsilon_1-1}\frac{1}{(1)^3};$$

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which is $$-3\binom{\epsilon_1+2}{2}-3\binom{\epsilon_1+1}{2}-4\binom{\epsilon_1+1}{2},$$

or $$-5\epsilon_1^2-8\epsilon_1-3;$$

giving a number of syzygies equal to

$$\tfrac{3}{2}(2\epsilon_1+1)(2\epsilon_1+2)-5\epsilon_1^2-8\epsilon_1-3,$$

or $$\epsilon_1^2+\epsilon_1,$$

or $$\tfrac{1}{4}\epsilon^2-\tfrac{1}{4}.$$

Hence when $n-\epsilon$ is even, the number of syzygies is the nearest integer to $\frac{1}{4}\epsilon^2$.

Next consider the case when $n-\epsilon$ is uneven; the number of syzygies is then

$$n(\epsilon+1)-\tfrac{1}{2}\epsilon(\epsilon+1)+\tfrac{1}{2}(n-\epsilon)^2-\tfrac{1}{2}+\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{3n+\epsilon}\frac{(2n+1)(2n+2)(2n+3)}{(1)(2)(3)}.$$

Putting $n=\epsilon+1$, we find

$$(\epsilon+1)^2-\tfrac{1}{2}\epsilon(\epsilon+1)+\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon+3}\frac{1}{(1)(2)(3)}+\mathrm{Co}_{3\epsilon}\frac{1}{(1)^2(2)};$$

or, after some simplification,

$$\tfrac{1}{2}(\epsilon+1)(3\epsilon+n)+\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon+3}\frac{1}{(1)(2)(3)}.$$

Now $$\mathrm{Co}_{4\epsilon+3}\frac{1}{(1)(2)(3)}=\mathrm{Co}_{4\epsilon+4}\frac{x(1+x)(1+x^2)}{(2)^2(6)}$$

$$=\mathrm{Co}_{2\epsilon+2}\frac{x+x^2}{(1)^2(3)}=\mathrm{Co}_{2\epsilon+2}\frac{x(1+x)^2(1+x^2)}{(2)^2(6)}$$

$$=\mathrm{Co}_{\epsilon+1}\frac{3x+2x^2+3x^3}{(1)^2(3)}=\mathrm{Co}_\epsilon\frac{3+2x+3x^2}{(1)^2(3)}.$$

Hence $$\mathrm{Co}_\epsilon\frac{1}{(1)(2)(3)}-\mathrm{Co}_{4\epsilon+3}\frac{1}{(1)(2)(3)}$$

$$=\mathrm{Co}_\epsilon\frac{1-(1+x)(3+2x+3x^2)}{(1)(2)(3)}$$

$$=-\mathrm{Co}_\epsilon\frac{2(1+x+x^2)+3x(1+x+x^2)}{(1)(2)(3)}$$

$$=-2\mathrm{Co}_\epsilon\frac{1}{(1)^2(2)}-3\mathrm{Co}_{\epsilon-1}\frac{1}{(1)^2(2)}.$$

If $\epsilon=2\epsilon_1$ this is

$$-2\mathrm{Co}_{2\epsilon_1}\frac{(1+x)^2}{(2)^3}-3\mathrm{Co}_{2\epsilon_1}\frac{x(1+x)^2}{(2)^3},$$

or $$-2\mathrm{Co}_{\epsilon_1}\frac{1+x}{(1)^3}-6\mathrm{Co}_{\epsilon_1-1}\frac{1}{(1)^3},$$

or $$-2\binom{\epsilon_1+2}{2}-2\binom{\epsilon_1+1}{2}-6\binom{\epsilon_1+1}{2},$$

or $$-5\epsilon_1^2-7\epsilon_1-2 \quad\text{or}\quad -\tfrac{5}{4}\epsilon^2-\tfrac{7}{2}\epsilon-2.$$

Hence the number of syzygies is

$$\tfrac{1}{2}(\epsilon+1)(3\epsilon+4)-\tfrac{5}{4}\epsilon^2-\tfrac{7}{2}\epsilon-2,$$

or $$\tfrac{1}{4}\epsilon^2.$$

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If $\epsilon = 2\epsilon_1 + 1$,

$$\mathrm{Co}_\epsilon \frac{1}{(1)(2)(3)} - \mathrm{Co}_{4\epsilon+3} \frac{1}{(1)(2)(3)}$$

$$= -2\mathrm{Co}_{2\epsilon_1+2} \frac{x(1+x)^2}{(2)^3} - 3\mathrm{Co}_{2\epsilon_1} \frac{(1+x)^2}{(2)^3}$$

$$= -4\mathrm{Co}_{\epsilon_1} \frac{1}{(1)^3} - 3\mathrm{Co}_{\epsilon_1} \frac{1+x}{(1)^3},$$

or

$$-7\binom{\epsilon_1+2}{2} - 3\binom{\epsilon_1+1}{2}, \quad \text{or} \quad -5\epsilon_1^2 - 12\epsilon_1 - 7.$$

Hence the number of syzygies is

$$\tfrac{1}{2}(\epsilon+1)(3\epsilon+4) - \tfrac{5}{4}(\epsilon-1)^2 - 6(\epsilon-1) - 7,$$

or

$$\tfrac{1}{4}\epsilon^2 - \tfrac{1}{4}.$$

Hence when $n - \epsilon$ is uneven the number of syzygies is always the nearest integer to $\frac{1}{4}\epsilon^2$.

To resume these results we have the theorem:—

The number of syzygies between symbolic products of order 3 and order $2\epsilon + 1$ or 2ϵ to the quantic of order $2n + 1$ or $2n$, the products being free from ξ_1, is the nearest integer to

$$\tfrac{1}{4}(\epsilon+1)^2 \text{ or } \tfrac{1}{4}\epsilon^2 \text{ respectively.}$$

Doubling these numbers we obtain the number of syzygies, free from $\xi_1\xi_2$, amongst the covariants.

Illustrations are: one syzygy, degree 3 order 4 for the quartic, viz.:—

$$\alpha_1^3\alpha_2\xi_2^2 \,.\, \beta_1^4\gamma_1\gamma_2^3\xi_2^2 = \alpha_1^3\alpha_2\gamma_1\gamma_2^3 \,.\, \beta_1^4\xi_2^4;$$

one syzygy, degree 3 order 4 for the sextic, viz.:—

$$\alpha_1^4\alpha_2^2\xi_2^2 \,.\, \beta_1^5\beta_2\gamma_1^2\gamma_2^4\xi_2^2 = \alpha_1^4\alpha_2^2\gamma_1^2\gamma_2^4 \,.\, \beta_1^5\beta_2\xi_2^4;$$

two syzygies, degree 3 order 6 for the sextic, viz.:—

$$\gamma_1^4\gamma_2^2\xi_2^2 \,.\, \alpha_1^6\beta_1^2\beta_2^4\xi_2^4 = \beta_1^2\beta_2^4\gamma_1^4\gamma_2^2 \,.\, \alpha_1^6\xi_2^6,$$

$$\beta_1^6\gamma_1\gamma_2^5\xi_2^2 \,.\, \alpha_1^5\alpha_2\xi_2^4 = \alpha_1^5\alpha_2\gamma_1\gamma_2^5 \,.\, \beta_1^6\xi_2^6.$$

The corresponding syzygies for the quantic of order $2n$ are:—

Degree 3 *order* 4,

$$\alpha_1^{n+1}\alpha_2^{n-1}\xi_2^2 \,.\, \beta_1^{n+2}\beta_2^{n-2}\gamma_1^{n-1}\gamma_2^{n+1}\xi_2^2 = \alpha_1^{n+1}\alpha_2^{n-1}\gamma_1^{n-1}\gamma_2^{n+1} \,.\, \beta_1^{n+2}\beta_2^{n-2}\xi_2^4.$$

Degree 3 *order* 6,

$$\gamma_1^{n+1}\gamma_2^{n-1}\xi_2^2 \,.\, \alpha_1^{n+3}\alpha_2^{n-3}\beta_1^{n-1}\beta_2^{n+1}\xi_2^4 = \beta_1^{n-1}\beta_2^{n+1}\gamma_1^{n+1}\gamma_2^{n-1} \,.\, \alpha_1^{n+3}\alpha_2^{n-3}\xi_2^6;$$

$$\beta_1^{n+3}\beta_2^{n-3}\gamma_1^{n-2}\gamma_2^{n+2}\xi_2^2 \,.\, \alpha_1^{n+2}\alpha_2^{n-2}\xi_2^4 = \alpha_1^{n+3}\alpha_2^{n-3}\gamma_1^{n-2}\gamma_2^{n+2} \,.\, \beta_1^{n+3}\beta_2^{n-3}\xi_2^6.$$

So far the results are complete in regard to the third degree in the coefficients for all quantics.

I have obtained incomplete results in regard to the fourth degree in the coefficients which I reserve for a future communication.

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VII. *A comparison of the results from the Falmouth Declination and Horizontal Force Magnetographs on quiet days in years of Sun-spot maximum and minimum.* (From the National Physical Laboratory.) By C. Chree, Sc.D., F.R.S.

[*Received and read* February 12, 1906.]

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§ 1. In a recent paper, described for shortness as (B)*, I have discussed the mean results from the Falmouth magnetic records for 1891 to 1902 on the "quiet" days selected annually by the Astronomer Royal. A previous paper described as (A)† contained a similar discussion of the Kew results for 1890 to 1900, and in addition investigated the differences at Kew between a sun-spot maximum period 1892 to 1895 and a sun-spot minimum period consisting of 1890, 1899, and 1900. The present paper aims at investigating the difference between a sun-spot maximum and a sun-spot minimum period at Falmouth. For the former period I have taken the same years as for Kew, viz. 1892 to 1895, but for the sun-spot minimum period the four consecutive years 1899 to 1902.

It was shown in (B) that in the average year there is a remarkable agreement between a number of phenomena at Kew and Falmouth, especially the times of occurrence of maxima and minima. In (A) it was shown that a considerable difference exists between sun-spot maximum and minimum years at Kew, not merely in the amplitude but in other features of the diurnal inequalities, and it was obviously desirable to make sure that this was no purely local phenomenon. It is not of course safe to assume that the difference between two sub-groups of years from a single sun-spot cycle is an absolutely exact and unchanging measure of sun-spot influence. One 11-year period differs from another in its average sun-spot frequency. There is

* *Phil. Trans. Roy. Soc.*, Vol. 204, A, p. 373. † *Ibid.* Vol. 202, A, p. 335.

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also some direct evidence* that the relation between sun-spot frequency and magnetic diurnal variation varies somewhat from one epoch to another. Still it would appear that no great change in this relation has taken place since regular magnetic observations were commenced, and the sun-spot cycle now dealt with is a fairly average one, thus the results to which the investigation leads may be accepted with considerable confidence as fairly representative. Even if less representative than I believe them to be, they will serve a useful purpose in bringing out the notable differences that exist in magnetic phenomena within a short series of years.

§ 2. In (A), § 2 gave particulars of Prof. Wolfer's sun-spot frequency from 1890 to 1900. The following Table I supplies the additional details necessary for dealing with the period 1891 to 1902. Wolfer's great Table† of revised frequencies ended with 1901. The data in Table I for 1902 are from his quarterly lists in the *Met. Zeitschrift* which are liable to revision. It is unusual, however, for revision to introduce more than minute changes in Prof. Wolfer's figures, and as 1902 was a year of very small sun-spot frequency, revision could hardly modify the mean results to an appreciable extent. The differences in the last line of Table I are employed in calculating the constants of Wolfer's sun-spot magnetic relation formula by the method of groups of years described in (A) § 52.

TABLE I.

Sun-Spot Frequency (after Wolfer).

	Jan.	Feb.	March	April	May	June	July	August	Sept.	Oct.	Nov.	Dec.	Mean for whole Year
1901	0·2	2·4	4·5	0·0	10·2	5·8	0·7	1·0	0·6	3·7	3·8	0·0	2·7
1902	6·7	0·0	14·8	0·0	3·5	1·1	0·9	1·8	8·0	12·7	6·3	0·8	4·7
Means for 1891 to 1902	36·64	39·25	33·73	38·02	40·35	42·11	40·27	41·12	40·63	38·71	32·16	36·28	38·27
1892 „ 1895	72·65	75·10	57·22	79·05	83·25	83·72	79·85	92·45	66·07	73·40	61·07	75·77	74·97
1899 „ 1902	8·95	6·30	11·50	7·55	9·15	9·87	5·85	2·50	6·32	10·57	5·60	2·90	7·25
Difference S max. − S min.	63·70	68·80	45·72	71·50	74·10	73·85	74·00	89·95	59·75	62·83	55·47	72·87	67·72

As has been remarked by Mr Ellis‡, when we consider the whole series of years for which Wolf's and Wolfer's figures exist, the mean value is so nearly alike for each season of the year as to discourage the idea of an appreciable annual variation in sun-spot frequency. But if we confine ourselves to 11, 22, or even 33 years, very considerable differences do exist between the means for different months of the year. In the present case it happens that certain months

* *Phil. Trans. Roy. Soc.*, Vol. 203, A, pp. 154, 158, &c.

† *Met. Zeitschrift*, 1902, p. 193.

‡ *Monthly Notices, Royal Astronomical Society*, Vol. 60, p. 153.

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show relatively low frequencies in years of sun-spot maximum and relatively high frequencies in years of sun-spot minimum, whilst other months have high relative frequencies in years of maximum and low frequencies in years of minimum. Thus the differences in the last line of Table I do vary somewhat conspicuously, the value for August being nearly double that for March. The uncertainties thus introduced are discussed in a recent paper*.

§ 3. The present paper makes use to a minor extent of data for areas of faculae, umbrae and whole sun-spots as recorded photographically at Greenwich, Mauritius, and Dehra Dûn, and communicated by the Astronomer Royal to the Royal Astronomical Society. The figures employed here are taken from the Society's *Monthly Notices*, Vol. LXIII, Table II, p. 465. This Table gives two sets of data for the mean daily areas of the year, described here for brevity as "Projected" and "Corrected," or "P" and "C." The former are the areas as seen and measured on photographs, expressed as millionths of the sun's apparent disc; the latter are the areas as corrected for foreshortening, expressed as millionths of the sun's visible hemisphere. For the present purpose the size of the unit employed in any measurement of solar activity is immaterial, but it is desirable to compare the modes of variation of the several quantities throughout the 12 years. This object is secured in Table II by expressing the faculae, umbrae or whole spots, as the case may be, in terms of their mean value throughout the 12-year period taken as unity. The absolute values of the several 12-year means, in the units employed by the Astronomer Royal, are given at the foot of the Table. Comparative data are also given for Wolfer's frequencies.

Table II shows that whilst the mean absolute P and C values differ by from 10 to 40 per cent. according to the element, their variation proceeds in each case in a very similar way throughout the twelve years. We thus see at once that so far as concerns the existence of a general connection between magnetic elements and the Greenwich faculae, umbrae, or whole spots, the conclusion come to is sure to be pretty much the same whether use is made of the projected or corrected areas. It is also evident that the mode of variation in the areas of whole spots and umbrae is so closely alike that it is sure to be pretty nearly indifferent whether we employ whole spots or umbrae in Wolf's formula in place of his sun-spot frequency.

It will be noticed that the means of the corrected areas for whole spots and umbrae are as nearly as possible in the ratio of 6 to 1. If this somewhat curious fact be borne in mind, the parallelism in the variation of whole spots and umbrae is readily recognised even in the Astronomer Royal's absolute values.

The mode of variation of the faculae is conspicuously different from that of whole spots or umbrae in the years of sun-spot maximum, there being an enormous excess in the figures for 1892 as compared to subsequent years, especially 1894, which is not seen in the case of the other elements.

* *Phil. Trans. Roy. Soc.*, Vol. 203, A, p. 151 (especially § 18, p. 166).

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TABLE II.

Mean annual values of Faculae, Umbrae, and Sun-spots, expressed as decimals of their arithmetic mean from 1891—1902.

Year	GREENWICH						Wolfer's Sun-spots
	Faculae		Umbrae		Whole Sun-spots		
	Projected	Corrected	Projected	Corrected	Projected	Corrected	
1891	1·120	1·100	0·860	0·861	0·918	0·947	0·930
1892	2·735	2·546	1·827	1·857	1·966	2·020	1·907
1893	1·937	1·872	2·343	2·336	2·443	2·436	2·218
1894	1·411	1·462	2·271	2·306	2·129	2·133	2·038
1895	1·744	1·774	1·698	1·687	1·639	1·621	1·672
1896	1·053	1·098	0·910	0·898	0·918	0·903	1·092
1897	0·827	0·895	0·874	0·879	0·856	0·855	0·685
1898	0·650	0·694	0·666	0·639	0·655	0·624	0·698
1899	0·252	0·262	0·193	0·180	0·196	0·185	0·316
1900	0·127	0·140	0·158	0·170	0·125	0·125	0·248
1901	0·019	0·023	0·100	0·086	0·051	0·048	0·071
1902	0·127	0·134	0·100	0·100	0·106	0·103	0·123
Mean absolute value	1180·9	1284·1	139·58	100·15	811·75	601·00	38·27

DIURNAL INEQUALITIES. RANGES, &c.

§ 4. Table III gives for both Declination and Horizontal Force—referred to generally as D and H for brevity—the ranges and the sum of the 24 hourly differences from the mean of the day in the diurnal inequalities for each month of the year and the year as a whole. "Range" here means the difference between the greatest and least mean hourly values. There is a considerable "non-cyclic" or aperiodic element in the annually published Falmouth Tables, especially in H (see (B), p. 376), which has been eliminated in forming the diurnal inequalities.

The quiet days numbered five a month, so that an hourly mean for a particular month from a four-year period is based on 20 curve measurements. Individual measurements aim at $0'\cdot1$ in D and 1γ in H, so that the degree of accuracy in the hourly values should be of

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the order 0′·01 in D, and 0·1 γ in H. For convenience in eliminating the non-cyclic element the hourly values were really taken out to 0·01 γ, but the last figure has been discarded in Table III in all the H columns, as well as in the D columns devoted to the sum of the 24 differences. The discarded figures were used, however, in calculating the results dealt with later in Table XV.

TABLE III.

Month	Declination (unit 1′)				Horizontal Force (unit 0·1 γ)			
	Ranges		Sum 24 differences		Ranges		Sum 24 differences	
	S max.	*S* min.	*S* max.	*S* min.	*S* max.	*S* min.	*S* max.	*S* min.
January	5·06	3·37	31·2	15·9	231	128	1270	561
February	6·71	3·77	43·0	18·9	296	116	1620	702
March.....	10·74	7·26	51·7	33·0	368	207	2289	1087
April	12·27	9·20	56·8	39·9	491	316	3017	1752
May	11·99	8·32	59·8	36·3	509	274	3340	1755
June.....	12·42	8·59	65·2	43·2	513	328	3279	1989
July	12·18	8·80	60·2	41·0	505	312	3286	1806
August	12·79	8·93	62·7	42·5	495	309	3146	1750
September ...	11·48	7·90	57·1	37·2	423	284	2453	1635
October.....	8·70	6·47	46·3	29·4	345	248	2195	1540
November ...	6·01	3·85	31·3	16·8	278	155	1738	881
December ...	4·26	2·24	28·0	12·7	191	71	1071	416
Year.....	9·47	6·25	49·6	30·2	373	219	2258	1236

The range as defined here is necessarily an underestimate unless both maximum and minimum should happen to coincide with exact hours G.M.T. The sum of the 24 differences is less dependent on the choice of the origin of time, and is probably the better measure of the activity of the forces to which the diurnal inequality is due.

It will be noticed that whilst December is unquestionably the month when the diurnal inequality is least, the difference between December and January is less marked in the case of the 24 differences than in the case of the ranges. It is also less marked in the sun-spot maximum than in the sun-spot minimum period.

§ 5. The *relative* variability of the diurnal inequality throughout the year is, as at Kew, decidedly less in years of many than in years of few sun-spots. This will be readily seen on

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comparing the two curves of Fig. 1. The full line curve answers to the sun-spot maximum, the broken line curve to the sun-spot minimum group of years. The ordinates represent

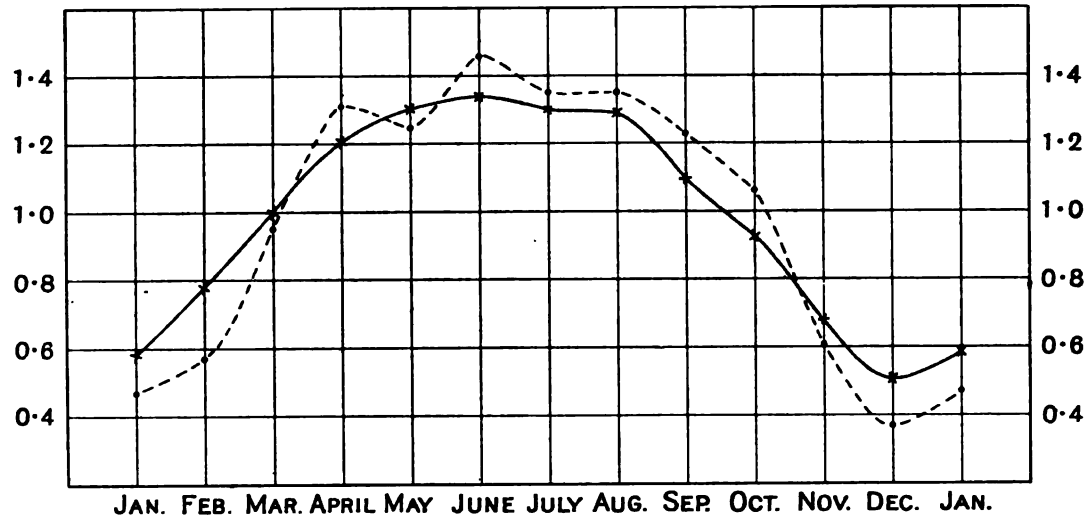

FIG. 1.

the ratios borne by the sum of the 24 hourly differences for individual months, as given in Table III, to their arithmetic mean. The ratios shown are the means of those calculated for D and H.

§ 6. A complete record of the mean hourly values in the diurnal inequalities for each month of the year in sun-spot maximum and sun-spot minimum years would occupy too much space. A good general idea of the phenomena can be derived from the accompanying vector diagrams for December, March, June, and the year as a whole (Figs. 2 to 5). The dotted curves are for the sun-spot maximum, the full line for the sun-spot minimum period.

As already mentioned, December represents the minimum of diurnal movement. March is fairly representative of the equinoctial months (March, April, September, October), and June represents the four summer months (May to August). NS and EW represent *astronomical* north-south and east-west directions. Their intersection serves as origin, and the line MM' represents the mean position of the magnetic meridian during the 12-year period, viz. 18° 47′·4 west of north. The numerals indicate the hours of the day, counted from midnight G.M.T. The line drawn from the origin to any numeral represents in direction and magnitude the value at the hour specified of the horizontal component of the disturbing force to which the diurnal inequality may be ascribed. The scales on which the curves are drawn are indicated on the diagrams

$$(10\,\gamma \equiv \cdot 0001 \text{ C.G.S.}).$$

The sun-spot maximum and sun-spot minimum curves are drawn from a common origin except in the case of December.

The difference between the size of the vectors in years of S max. and S min. is too obvious to require detailed comment.

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DECEMBER.

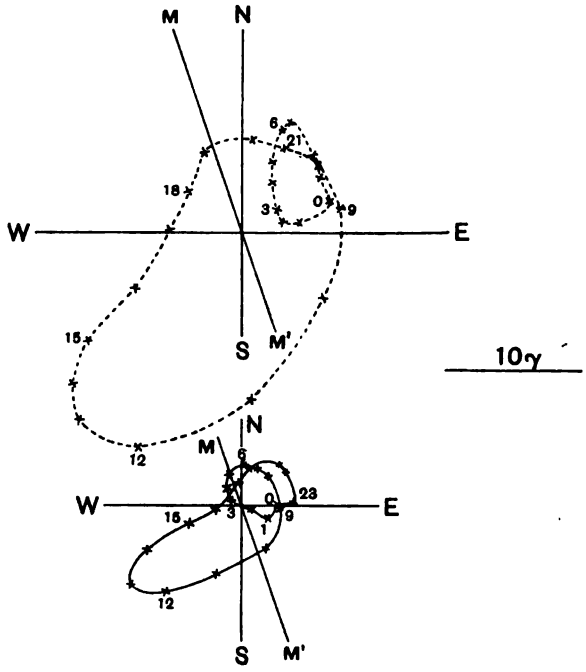

FIG. 2.

MARCH.

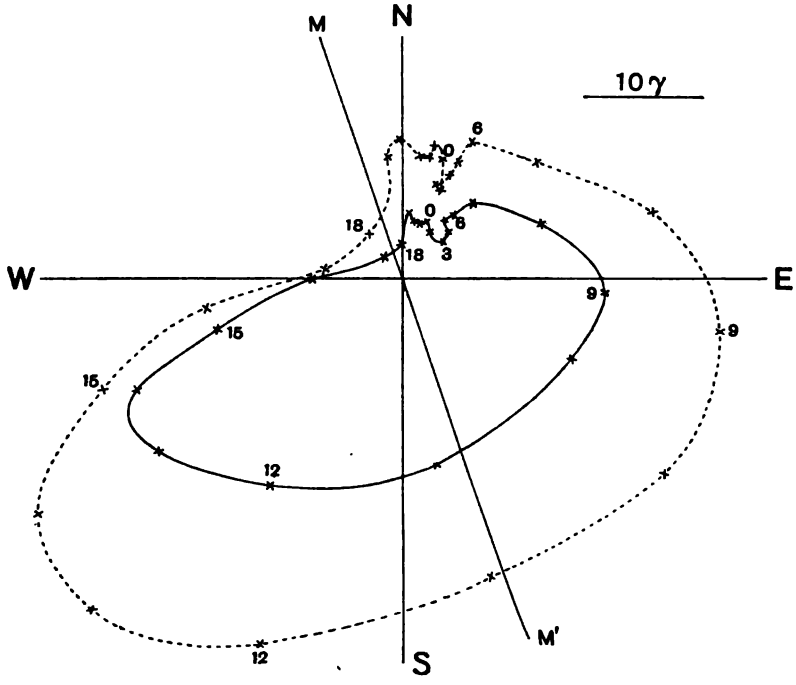

FIG. 3.

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JUNE.

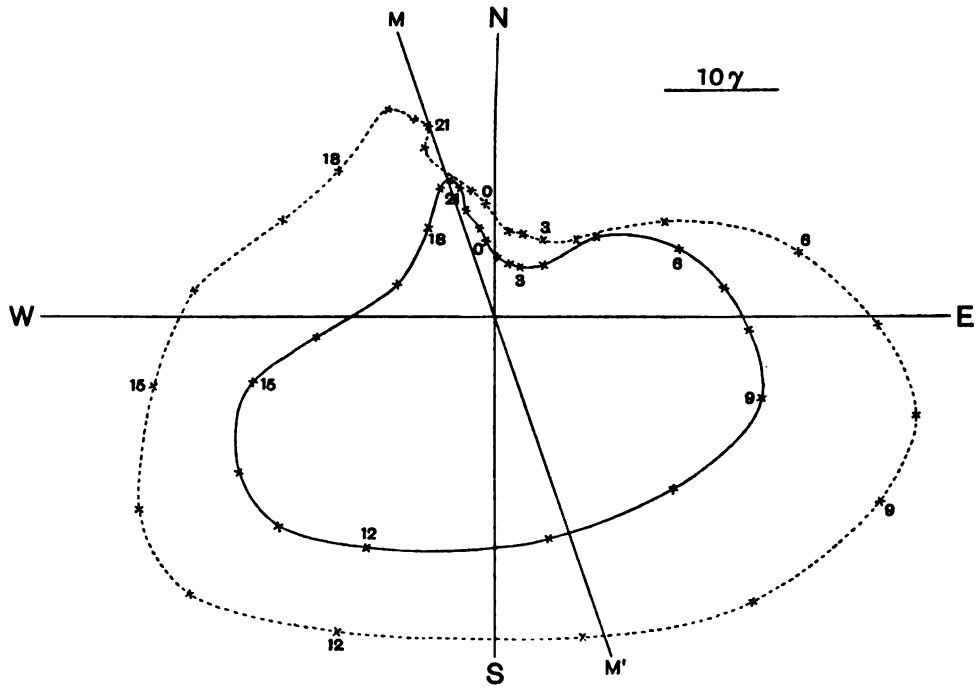

FIG. 4.

YEAR.

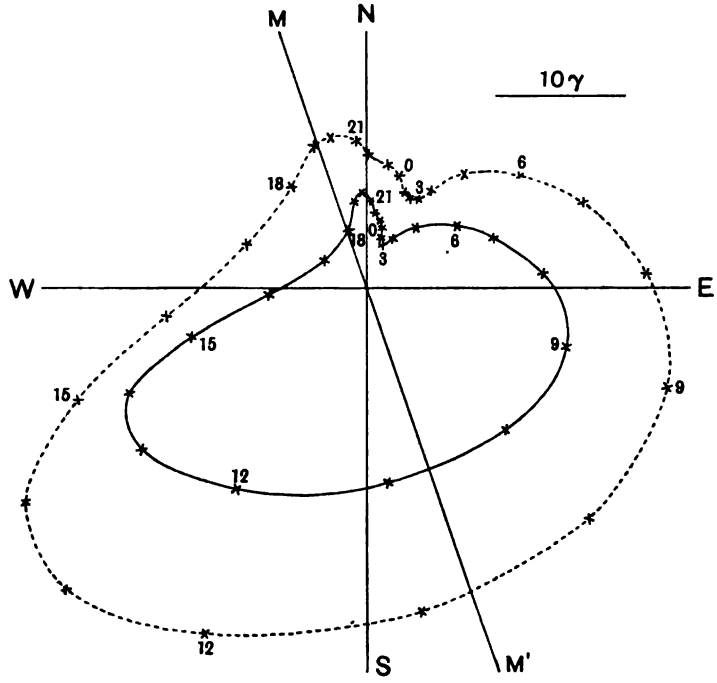

FIG. 5.

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§ 7. The mean declination for the *S* max. period really exceeded that for the *S* min. period by 36′·5, but so small an angle could not have been shown clearly in the diagrams. It is obvious to the eye that the vector for the *S* min. curve is in advance of that for the *S* max. curve at noon and in the early afternoon, but this is not generally the case throughout the whole 24 hours. Owing however to the slowness of the angular change throughout the night, vector diagrams would require to be drawn on an excessively large scale to facilitate a study of this phenomenon. I have accordingly calculated the exact hourly values of the vectorial angles in both the *S* max. and *S* min. periods. The approach of the *S* min. period to an ideal definite condition, viz. a total absence of sun-spots, would naturally lead one to pay it chief attention; but owing to the smallness of the diurnal changes, the calculated vectorial angles for this group of years are exposed to considerably larger probable errors than those for the *S* max. group of years. The latter have thus been selected for exhibition in Table IV. The angles, though recorded only to whole degrees, were calculated to the nearest 0°·1. They are

TABLE IV.

Vectorial Angles for S max. period.

Forenoon

Hour.....	1	2	3	4	5	6	7	8	9	10	11	12
January...	48°	38°	27°	20°	15°	4°	16°	37°	77°	134°	172°	202°
February..	52	58	62	51	43	40	35	45	84	132	173	203
March	23	19	20	23	23	27	49	75	99	126	164	201
April	6	15	26	32	35	42	62	78	99	125	157	194
May	4	18	32	47	57	71	86	103	118	142	172	203
June	10	18	31	47	60	77	91	103	116	137	164	206
July	2	18	23	32	55	74	84	96	114	137	166	198
August ...	8	15	25	35	50	65	84	101	120	147	180	212
September	18	17	29	33	44	59	79	98	121	150	183	221
October ...	26	22	18	15	17	24	37	72	101	129	165	203
November	25	12	12	5	10	15	24	51	97	131	165	197
December	77	76	55	29	22	20	23	41	77	129	176	205
Year...	21	25	29	33	41	52	68	86	108	136	170	205

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TABLE IV *continued.*

Afternoon

Hour.....	1	2	3	4	5	6	7	8	9	10	11	12
January...	221	240	233	228	229	274	334	378	399	409	410	413
February..	221	233	239	244	255	305	321	344	364	385	397	402
March	223	237	249	261	276	323	352	358	368	373	372	378
April	215	231	246	263	289	327	340	339	343	341	351	361
May	225	239	255	280	308	328	340	340	345	342	348	356
June	227	241	258	275	294	313	332	337	341	337	350	355
July	219	236	254	270	300	324	337	340	342	340	345	354
August ...	232	244	255	275	314	346	347	348	351	354	357	363
September	239	248	252	253	293	349	352	358	359	361	365	374
October ...	223	233	242	240	263	315	340	349	369	376	384	386
November	213	227	235	247	286	310	328	353	375	390	401	401
December	220	228	234	241	271	309	336	366	387	408	413	430
Year...	224	237	248	261	290	323	340	347	355	360	369	377

measured from an initial line drawn towards astronomical north, the positive direction of rotation being supposed clock-wise (N., E., S., W.). To avoid the appearance of discontinuity, angles in excess of 360° usually appear in the late afternoon hours, but the meaning attaching to, say, 400° is of course the same as that attaching to 40°.

The Table will, it is hoped, afford a useful check on speculations as to the immediate cause of the diurnal inequality. In any comparisons it is important to remember that the hours being G.M.T. differ considerably—especially in months when the equation of time is large and positive—from local solar time*.

The variation in the vectorial angle with the season will be seen to be comparatively small from 10 a.m. to 4 p.m.; it is large from 10 p.m. to 2 a.m., and also at 6, 7 and 8 a.m. and at 5 and 6 p.m.

§ 8. To give full details of the differences between the *S* max. and *S* min. vectorial angles for every hour of every month would have entailed another long Table; accordingly in Table V I give only mean results for the four months comprising Winter, Equinox and Summer, and results for the year. The last, it should be noticed, are derived directly from the mean annual diurnal inequality, and so, unlike those for the three seasons, are not arithmetic means of differences for individual months. The sign has been taken as plus

* Falmouth local time is 20·3 minutes later than Greenwich.

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TABLE V.

Excess of Vectorial Angle in S min. as compared to S max. period.

Forenoon

Hour ...	1	2	3	4	5	6	7	8	9	10	11	12
Winter	+21°	+6°	−65°?	−40°?	−27°	−12°	−2°	+1°	+9°	+13°	+15°	+14°
Equinox	− 4	+2	0	+ 2	+ 2	+ 3	−1	−2	−2	− 2	+ 4	+ 5
Summer	− 4	−3	− 4	+ 1	+ 2	+ 3	0	−3	−4	− 3	+ 2	+10
Year	− 5	−5	− 9	− 5	− 1	+ 3	+2	−1	−1	0	+ 4	+ 8

Afternoon

Hour ...	1	2	3	4	5	6	7	8	9	10	11	12
Winter	+14°	+11°	+9°	+8°	+24°	+59°	+58°	+30°	+22°	+11°	+16°	+15°
Equinox	+ 8	+ 8	+7	+12	+40	+22	+12	+ 9	+ 7	+ 6	+ 8	− 1
Summer	+ 9	+ 6	+1	− 5	− 1	+11	+ 6	+ 3	+ 3	+ 3	0	− 2
Year	+10	+ 8	+6	+ 5	+14	+21	+16	+10	+ 9	+ 6	+ 5	− 2

when the vectorial angle is larger for the *S* min. than the *S* max. group of years. A query is attached to the 3 and 4 a.m. data in Winter. At these hours the February *S* min. vectorial angle presented outstanding values, and owing to the minute size of the vector itself much might have been due to an element of chance. It will at once be seen that the differences between the *S* max. and *S* min. vectorial angles are systematic and have a well marked diurnal variation.

On the average of the 24 hours the *S* min. angle is the larger, by 8°·75 in Winter, 6°·0 in Equinox, 1°·3 in Summer, and by 4°·1 in the mean diurnal inequality for the year. The excess of the *S* min. angle is most marked about 6 p.m. and 1 p.m.; from midnight to 4 or 5 a.m. the *S* max. angle is usually in advance.

§ 9. The times during which the *S* min. and *S* max. angles are respectively the larger are shown graphically in Fig. 6 for the twelve months individually and for the year as a whole. The line is existent or non-existent according as the *S* min. or the *S* max. vectorial angle is the larger. The transition time is calculated in each case by supposing the change from plus to minus or from minus to plus to take place uniformly throughout the hour during which it occurs. In one month, August, the *S* min. angle was the larger throughout the whole 24 hours, but at one hour the excess was only 1° so the phenomenon is probably in part accidental.

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On the average of the 12 months the *S* min. angle was the larger during 16, and the *S* max. during 8 hours of the day. This is practically the same proportion as in the mean diurnal inequality for the year.

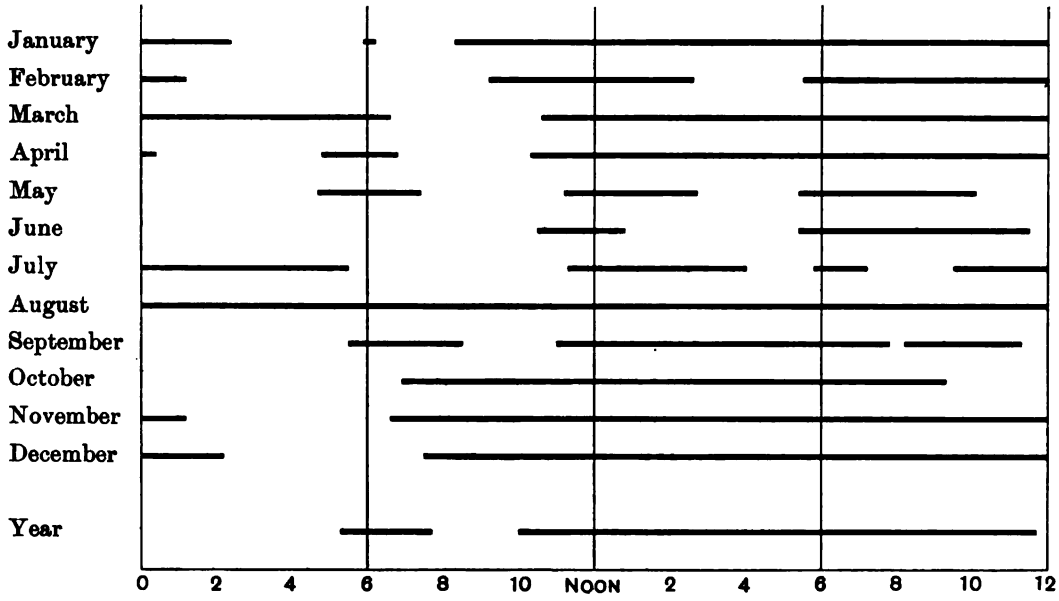

FIG. 6.

§ 10. From some points of view the rates of change of the vectorial angle are of more interest than its absolute values. In the case of the *S* max. group of years the hourly increments in individual months are immediately derivable from Table IV. As a comparison with *S* min. years appeared however desirable, I show side by side in Table VI, for the two groups of years, mean hourly increments of the vectorial angles for the three seasons and for the year.

The values for the seasons are arithmetic means of the values for the individual months; those for the year are derived from the mean diurnal inequality for the year.

The *S* min. increment from 3 to 4 a.m. in winter is queried for reasons already explained.

As one would have anticipated from the remarks in § 7, the hourly increments are markedly less regular in the *S* min. than the *S* max. period; but in addition to irregularities ascribable to observational uncertainties, there is in the *S* min. period an unmistakable tendency to a greater difference between the fastest and slowest (or most retrogressive) hourly increments than in the *S* max. period. The phenomenon, I may add, is even more distinctly shown when one considers the individual months. If the difference between the algebraically greatest and least hourly increments be defined as the *range*, then the mean value of the ratios borne by the *S* min. to the *S* max. ranges from the 12 months of the year exceeded 1·8, the *S* min. range being the smaller in only one month, June.

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TABLE VI.

Hourly increments in Vectorial Angles.

Forenoon

Hour		0—1	1—2	2—3	3—4	4—5	5—6	6—7	7—8	8—9	9—10	10—11	11—noon
Winter	*S* max.	−1°	−4°	−7°	−13°	−4°	−3°	+5°	+19°	+40°	+48°	+40°	+30°
	S min.	+6	−19	−81 ?	+15	+9	+13	+15	+22	+48	+52	+41	+28
Equinox	*S* max.	+3	0	+5	+3	+4	+8	+19	+24	+24	+27	+35	+37
	S min.	0	+7	+2	+5	+4	+9	+15	+22	+24	+28	+41	+39
Summer	*S* max.	+9	+11	+11	+13	+15	+16	+15	+15	+16	+24	+30	+34
	S min.	+7	+13	+10	+17	+16	+17	+12	+11	+16	+25	+35	+42
Year....	*S* max.	+4	+4	+4	+4	+8	+11	+16	+18	+22	+28	+34	+35
	S min.	+1	+4	0	+8	+12	+15	+15	+15	+22	+29	+38	+39

Afternoon

Hour		Noon—1	1—2	2—3	3—4	4—5	5—6	6—7	7—8	8—9	9—10	10—11	11—12
Winter	*S* max.	+17°	+13°	+3°	+5°	+20°	+39°	+30°	+31°	+21°	+17°	+7°	+6°
	S min.	+18	+11	+1	+4	+36	+75	+29	+3	+13	+5	+13	+4
Equinox	*S* max.	+20	+12	+10	+7	+26	+48	+18	+5	+9	+3	+5	+7
	S min.	+23	+12	+9	+12	+54	+30	+7	+3	+6	+2	+7	−2
Summer	*S* max.	+21	+14	+15	+17	+29	+24	+11	+2	+3	−2	+7	+7
	S min.	+20	+11	+11	+13	+33	+36	+5	−1	+4	−1	+4	+4
Year....	*S* max.	+19	+13	+11	+13	+29	+33	+17	+7	+8	+5	+9	+8
	S min.	+21	+11	+9	+12	+38	+40	+12	+1	+7	+2	+8	+1

DIURNAL INEQUALITY. FOURIER COEFFICIENTS.

§ 11. The diurnal inequality in D or H can be expressed with high accuracy by four terms of a Fourier series whose periods are respectively 24, 12, 8 and 6 hours. We may suppose the result expressed in the form

$$c_1 \sin(t+\alpha_1) + c_2 \sin(2t+\alpha_2) + c_3 \sin(3t+\alpha_3) + c_4 \sin(4t+\alpha_4),$$

the c's and α's representing the amplitudes and phase angles, and t being time counted from midnight (G.M.T.), an hour being taken as equivalent to 15°. The suffix $_1$ answers to the longest period, 24 hours. The values of the c's and α's for each month of the year are given in Tables VII and VIII for the S max. and S min. periods already indicated. The calculations were carried further than appears in the Tables, the phase angles being worked

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TABLE VII.

Declination. Fourier Coefficients, Amplitudes and Phase Angles.

Month	c_1		c_2		c_3		c_4		α_1		α_2		α_3		α_4	
	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.
January...	1′·75	0′·76	1′·12	0′·82	0′·56	0′·40	0′·36	0′·21	234°	243°	16°	24°	225°	232°	37°	45°
February..	2·54	1·04	1·28	0·85	0·81	0·46	0·40	0·22	227	245	23	33	214	228	21	21
March	2·63	1·70	2·49	1·58	1·41	1·10	0·56	0·53	216	222	25	33	209	211	34	33
April	3·22	2·05	2·71	2·18	1·58	1·21	0·43	0·40	204	210	27	27	207	210	38	37
May	3·51	2·03	2·85	2·08	1·13	0·90	0·13	0·20	206	203	39	39	230	230	66	51
June	3·95	2·53	2·92	2·15	0·96	0·81	0·03	0·12	201	204	34	35	231	223	102	22
July	3·62	2·42	2·82	2·07	1·05	0·92	0·01	0·23	203	203	34	35	219	215	261 ?	− 8
August....	3·50	2·29	3·05	2·16	1·41	1·07	0·32	0·17	217	222	46	50	230	234	44	34
September	3·11	1·91	2·66	1·82	1·37	1·05	0·39	0·35	224	226	44	45	230	231	55	55
October ...	2·51	1·47	1·98	1·46	1·19	0·94	0·53	0·48	221	225	22	30	213	220	30	41
November	1·62	0·74	1·37	0·91	0·66	0·57	0·44	0·31	223	248	17	33	226	241	38	59
December	1·55	0·64	1·06	0·64	0·43	0·30	0·25	0·12	234	266	17	31	230	235	30	44

out to minutes of arc, and use is made of the more complete values later in the paper. If Falmouth solar time were used the phase angles would require the corrections given in (A) Table XIX, with the constant addition of $+4° 45'·6$ in the case of α_1, $+9° 31'$ in the case of α_2, and so on, Falmouth local time being 19 minutes 3 seconds after Kew (20 minutes 18 seconds after Greenwich).

All the c's, both for D and H, show a well marked winter minimum, nearly always in December. In c_3 and c_4 there is also a well marked summer minimum, in some cases the principal minimum of the year.

The excess of the sun-spot maximum over the sun-spot minimum value is always large in c_1 and c_2, especially the former. It is less marked though generally apparent in c_3. In c_4 it is not clearly shown except in winter.

In the case of D the phase angles α_1, α_2, α_3 are almost invariably smaller in S max. than in S min. years. This indicates a later hour for the daily maximum in the former than in the latter epoch. The difference is most distinct in winter. In the case of α_4 in D, and all the phase angles in H, the difference between S max. and S min. years is not

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sufficiently prominent to make its presence certain in individual months, when data from only one sun-spot cycle are available.

TABLE VIII.

Horizontal Force. Amplitudes and Phase Angles (Unit for c's 1 γ).

Month	c_1		c_2		c_3		c_4		a_1		a_2		a_3		a_4	
	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.
January...	6·8	2·4	5·1	3·0	2·4	1·8	1·9	1·2	71°	72°	264°	269°	137°	155°	− 23°	− 3°
February..	9·0	3·4	6·7	2·8	2·8	1·3	2·1	1·0	95	75	252	251	125	145	− 25	− 9
March	13·6	6·1	8·0	4·3	4·2	2·9	1·6	1·5	98	103	281	284	141	138	− 25	− 20
April	18·3	10·3	10·0	6·7	4·4	3·7	1·6	2·1	105	108	278	291	124	126	− 25	− 33
May.....	21·1	11·4	9·6	4·0	2·6	1·4	1·0	0·8	125	124	298	305	155	194	+ 70	+ 33
June	20·7	12·7	8·4	5·4	2·4	1·9	0·2	0·8	128	119	303	288	187	173	+ 85	− 8
July	20·5	11·6	9·4	4·7	2·0	2·8	0·6	1·1	120	121	295	302	139	171	+ 17	− 30
August....	19·5	11·3	9·1	4·4	3·6	3·3	1·4	1·5	118	126	307	334	175	200	+ 4	+ 16
September	16·0	10·4	6·3	4·9	4·7	3·2	2·3	1·7	113	117	313	320	188	180	+ 21	+ 19
October ...	13·0	9·1	7·3	5·5	3·4	3·0	1·9	1·5	92	95	280	284	151	146	− 4	− 3
November	9·3	4·7	7·3	4·2	2·7	1·9	1·5	0·9	90	105	270	284	131	154	− 14	+ 3
December	5·0	1·2	5·9	2·5	1·4	0·7	1·0	0·3	87	97	253	271	132	165	− 17	− 21

§ 12. Table IX gives the values of the c coefficients in the mean diurnal inequalities for winter, equinox and summer in D and H, and also in the mean diurnal inequality for the whole year in these elements and in W and N (the westerly and northerly components of the horizontal force). The differences between S max. and S min. years are similar to those at Kew described in (A). The general nature of the difference is perhaps more readily recognised from Table X, showing the ratios borne by c_2, c_3 and c_4 to c_1 in the case of the mean diurnal inequality for the year. The shorter period terms diminish in importance as compared to the 24-hour term as sun-spots increase. This is shown more prominently in D and W than in H and N. Similar phenomena are shown in Table XXXVII of (A) at Kew.

§ 13. Table XI records the phase angles for the seasons and the year corresponding to the amplitudes appearing in Table IX; it also gives the algebraical excess of the angles for the S min. years over those for the S max. years. The tendency for this difference to be

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positive, *i.e.* for the daily maxima to occur earlier in *S* min. than in *S* max. years, is much more marked in winter than in summer.

TABLE IX.

Amplitudes (Unit for c's in H, W, and N, 1 γ).

		D				*H*				*W*	*N*
		Winter	Equinox	Summer	Year	Winter	Equinox	Summer	Year	Year	Year
c_1	Max. ...	1′·86	2′·84	3′·62	2′·74	7·4	15·1	20·4	13·9	13·6	15·1
	Min.	0·78	1·77	2·30	1·57	2·8	8·9	11·7	7·6	7·7	8·4
c_2	Max. ...	1·21	2·43	2·90	2·16	6·2	7·7	9·1	7·3	10·6	8·8
	Min.	0·80	1·75	2·10	1·55	3·1	5·2	4·4	4·1	7·7	5·2
c_3	Max. ...	0·61	1·37	1·13	1·03	2·3	3·8	2·6	2·8	5·7	2·7
	Min.	0·43	1·06	0·92	0·80	1·4	3·0	2·3	2·1	4·4	1·9
c_4	Max. ...	0·36	0·47	0·11	0·31	1·6	1·8	0·7	1·3	1·9	0·9
	Min.	0·21	0·43	0·16	0·27	0·9	1·6	1·0	1·1	1·6	0·8

TABLE X.

Ratios of Amplitudes in mean diurnal inequality for the year.

	D		*H*		*W*		*N*	
	S max.	*S* min.	*S* max.	*S* min.	*S* max.	*S* min.	*S* max.	*S* min.
c_2/c_1 ...	·79	·99	·53	·54	·78	1·00	·58	·62
c_3/c_1 ...	·38	·51	·20	·28	·42	·58	·18	·22
c_4/c_1 ...	·11	·17	·09	·15	·14	·21	·06	·10

Comparing Table XI with Table XXXIX of (A) it will be seen that the differences shown in the case of α_1 for the mean diurnal inequality for the year are closely alike at Falmouth and Kew in all the elements. In the case of α_2 and α_3 the Kew differences are decidedly the larger; whilst in the case of α_4 the results are somewhat indecisive at both stations.

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TABLE XI.

Phase Angles.

		D				H				W	N
		Winter	Equinox	Summer	Year	Winter	Equinox	Summer	Year	Year	Year
a_1	Max. ...	229° 14′	216° 6′	206° 37′	214° 52′	86° 36′	102° 42′	122° 53′	109° 34′	196° 20′	91° 51′
	Min.	249 23	220 0	207 54	218 49	88 40	106 33	122 36	112 29	200 56	94 23
Min. – Max.		20 9	3 54	1 17	3 57	2 4	3 51	– 0 17	2 55	4 36	2 32
a_2	Max. ...	18 8	29 52	38 27	31 31	260 0	285 57	300 35	284 52	19 12	260 43
	Min.	30 15	33 41	40 5	35 59	270 39	294 27	306 7	292 52	26 22	262 40
Min. – Max.		12 7	3 49	1 38	4 28	10 39	8 30	5 32	8 0	7 10	1 57
a_3	Max. ...	222 23	214 27	227 28	220 46	131 0	151 31	165 59	150 19	212 4	110 56
	Min.	234 28	217 24	226 3	223 42	153 52	146 37	185 2	161 6	215 48	119 24
Min. – Max.		12 5	2 57	– 1 25	2 56	22 52	– 4 54	19 3	10 47	3 44	8 28
a_4	Max. ...	31 31	38 16	53 8	37 25	– 20 36	– 5 33	+30 26	– 5 51	28 56	– 30 14
	Min.	43 27	40 24	23 50	37 54	– 5 6	–10 48	+ 2 10	– 5 42	29 5	– 28 59
Min. – Max.		11 56	2 8	– 29 18	0 29	15 30	– 5 15	–28 16	0 9	0 9	1 15

Taking means for the year from *D*, *H*, *N* and *W*, and converting differences of angle into time, we find for the retardation in *minutes* in the *S* max. as compared with the *S* min. period the following values:

	24-hour wave	12-hour wave	8-hour wave	6-hour wave
At Falmouth	14·0	10·8	8·5	0·5
At Kew.....	15·5	18·5	14·0	– 0·1

The retardation though apparently less at Falmouth than at Kew is well marked in the 24, 12 and 8-hour waves. In the 6-hour wave it is doubtful whether the phase angle is affected in the mean diurnal inequality for the year; there is a decided retardation during sun-spot maximum in winter, but an acceleration apparently in summer.

§ 14. Supposing the diurnal inequality in years of sun-spot maximum to be composed of a fundamental part representing the inequality in the total absence of sun spots, and of a second part associated with sun spots, it is important to know whether the two parts are or are not essentially different in type. A conspicuous difference in type would naturally suggest a difference in source, while close agreement in type would suggest a mere intensification of action in years of sun-spot maximum.

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Subtracting the hourly values for the sun-spot minimum period from the corresponding values for the sun-spot maximum period, we obtain a diurnal inequality which should be wholly free from what was called above the fundamental part of the real diurnal inequality. Fourier coefficients were calculated for this difference inequality for the three seasons and the year, and a summary is given in Table XII.

TABLE XII.

Difference Inequality (Diurnal) S max. – S min. (Units for c's 1′ in Declination, 1 γ in Horizontal Force.)

	Declination				Horizontal Force			
	Winter	Equinox	Summer	Year	Winter	Equinox	Summer	Year
c_1	1·28	1·08	1·33	1·18	4·6	6·3	9·0	6·3
c_2	0·45	0·70	0·80	0·63	3·2	2·6	4·7	3·3
c_3	0·21	0·31	0·22	0·24	1·2	0·8	0·8	0·8
c_4	0·16	0·04	0·09	0·05	0·8	0·2	0·5	0·1
α_1	223°	210°	204°	210°	85°	97°	122°	106°
α_2	– 4	20	34	20	250	269	295	275
α_3	197	204	233	211	103	169	103	121
α_4	17	15	164	35	– 38	36	141	– 7

§ 15. In the *S* min. period the mean sun-spot frequency was only 7·25. Thus if the diurnal inequality really consists of a fundamental part and a sun-spot contribution, the phase angles for the *S* min. years in Table XI cannot differ very much from those of the fundamental part. Consequently if the sources of the fundamental part and the sun-spot contribution were wholly different we should expect notable differences between the angles in Table XII and the corresponding angles for the *S* min. years in Table XI.

In two cases—the summer values of α_4 in *D* and *H*—there is a conspicuous difference, but in all other cases the differences are hardly such as to suggest a radically different source.

Even the *S* min. angles in Table XI are slightly modified by sun-spot frequency. In the case of α_1 and α_2, by assuming the difference in phase angle proportional to difference in sun-spot frequency, we can calculate angles for the fundamental part alone. The assumption is probably not exactly true, but the results can hardly be much in error. The phase angles thus calculated exceed those for the pure sun-spot element, whose approximate values are given in Table XII, by the following amounts:

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	a_1				a_2			
	Winter	Equinox	Summer	Year	Winter	Equinox	Summer	Year
D	28°·2	12°·9	3°·7	9°·6	35°·0	14°·0	6°·1	16°·0
H	3·5	9·9	0·5	6·8	21·6	26·8	11·2	18·8
Mean..	15·9	11·4	2·1	8·2	28·3	20·4	8·6	17·4

The mean differences represent a retardation in the sun-spot contribution as compared to the fundamental part, which in the case of the mean diurnal inequality for the year represents fully half an hour in both the 24 and 12 hour waves.

The summer values of α_4 in D and H in Table XII already alluded to differ from the corresponding S min. angles by about 140°, so that there is here an approach to a reversal of phase.

The phenomenon appearing remarkable, I calculated the values of α_4 at Kew corresponding to those in Table XII. They exceeded the Kew S min. angles by 226° in D and 144° in H. The amplitudes corresponding to the phase angles α_4 in Table XII are so small that accident may play a considerable part in the result; but the coincidence certainly points to something more than mere accident.

Annual Variation, Fourier Coefficients.

§ 16. In (A) and (B) the annual variations of the range and the sum of the 24 differences from the mean, in the diurnal inequality, and likewise of the amplitudes of the 24-hour and other contributory waves, were shown by means of Fourier series with annual and semi-annual terms.

In (B), § 16, attention was called to the remarkable resemblance between results for the average year at Kew and Falmouth. It was thus important to ascertain whether the results varied at all with sun-spot frequency. Accordingly Fourier series of the type

$$M + P_1 \sin(t + \theta_1) + P_2 \sin(2t + \theta_2)$$

were calculated for the S max. and S min. periods separately, and the results appear in Table XIII.

M denotes the mean value of the element throughout the year; P_1 and P_2 are the amplitudes, θ_1 and θ_2 the phase angles of the annual and semi-annual terms.

The time t is counted from the beginning of the year, a month representing 30° in the annual term.

R denotes the range, and Σ the sum of the 24 differences in the diurnal inequality.

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TABLE XIII.

Annual Variation.

		Declination (Unit 1′)								Horizontal Force (Unit 1γ)							
		M	P_1	θ_1	P_2	θ_2	P_1/M	P_2/M	P_2/P_1	M	P_1	θ_1	P_2	θ_2	P_1/M	P_2/M	P_2/P_1
R	Max.	9·55	4·10	277°	1·40	286°	·43	·15	·34	38·7	15·7	277°	2·7	268°	·40	·07	·17
	Min.	6·56	3·19	275	1·14	277	·49	·17	·36	22·9	11·4	266	3·1	263	·50	·14	·28
Σ	Max.	49·4	17·1	277	4·8	304	·35	·10	·28	239	110	276	17	243	·46	·07	·15
	Min.	30·6	14·7	273	4·2	291	·48	·14	·28	132	72	265	20	245	·54	·15	·28
c_1	Max.	2·79	1·09	279	0·17	328	·39	·06	·15	14·4	7·7	275	1·2	254	·54	·08	·15
	Min.	1·63	0·91	274	0·15	311	·56	·09	·17	7·9	5·3	264	1·2	231	·67	·16	·24
c_2	Max.	2·19	1·00	273	0·28	280	·45	·13	·28	7·8	1·8	290	0·6	223	·23	·07	·31
	Min.	1·56	0·78	275	0·20	267	·50	·13	·25	4·4	1·0	265	1·0	242	·22	·22	·97
c_3	Max.	1·05	0·31	280	0·38	283	·30	·36	—	3·1	0·3	278	1·2	284	·08	·41	—
	Min.	0·81	0·29	273	0·27	275	·36	·34	—	2·3	0·6	249	0·9	292	·27	·37	—
c_4	Max.	0·32	0·15	94	0·18	279	·46	·57	—	1·4	0·5	104	0·6	301	·33	·45	—
	Min.	0·28	0·03	71	0·15	268	·11	·55	—	1·2	0·1	278	0·5	293	·11	·43	—

In both D and H as we pass from sun-spot minimum to maximum P_1/M and P_2/M diminish alike for the ranges, the sum of the differences, and c_1. Thus, relatively considered, these elements are less variable throughout the year when sun-spots are many than when they are few. In H the semi-annual term is of reduced importance relatively to the annual term as sun-spots increase; but in D the dependence of P_2/P_1 on sun-spot frequency seems small. In the case of c_3 and c_4 the absolute size of the element was so small that the values of P_2/P_1 seemed too uncertain to record.

Before discussing the phase angles in Table XIII as a whole it is convenient to consider an exceptional case presented by the annual term in c_4 for H. In the sun-spot maximum period the angle is 104°, which is comparatively near the value 119° for the 12-year period at Falmouth, and the corresponding value 126° for the 11-year period at Kew. But in the sun-spot minimum period the angle becomes 278°, or the phase is very nearly reversed. Suspecting some error, I calculated the corresponding angle for the sun-spot minimum period at Kew and obtained 290°, thus showing a close approach to the phenomena at Falmouth. In the case of a forced vibration, a change of this kind might arise from a change in the natural period either of the vibrator or of the disturbing force, that which was the shorter period of the two becoming the longer.

Omitting c_4 in H, Table XIII shows in ten cases out of eleven a larger phase angle in the annual term in years of sun-spot maximum than in years of sun-spot minimum, and the same phenomenon is exhibited in the semi-annual term by all the six D quantities. This

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means an earlier occurrence of maximum when sun-spots are numerous than when they are few. As a check on the accuracy and generality of this result—corresponding data not having been given in (A)—I calculated the amplitudes and phase angles at Kew for the ranges, the sums of the differences, and c_1, corresponding to those in Table XIII.

In ten cases out of twelve—both exceptions occurring in H in the semi-annual term—there appeared as at Falmouth a larger phase angle in the maximum than in the minimum sun-spot period.

§ 17. As the difference between phase angles is less suggestive than that in the times of occurrence of the maximum, I give the latter in Table XIV, giving only the means found in the case of Kew. Except in the case of the semi-annual term in H—where there are considerable differences between the ranges, the 24 differences and c_1—corresponding Kew and Falmouth means approach closely to one another. The phase differences in D from the ranges, 24 differences and c_1 are, it may be added, in closer agreement at Kew than at Falmouth. Thus the acceleration of the epoch of maximum in the years of sun-spot maximum seems fairly established. The fact that an acceleration of the annual maximum should be combined with a retardation of the diurnal maximum seems a little curious.

TABLE XIV.

Annual Variation. Difference of times of occurrence of Maxima in years of Sun-spot Maximum (1892 *to* 1895) *and Sun-spot Minimum* (1899 *to* 1902), *in* days (+ *denotes* earlier *occurrence in years of Sun-spot Maximum*).

	Annual term		Semi-annual term	
	D	*H*	*D*	*H*
Ranges	+ 1·6	+ 11·0	+ 4·4	+ 2·4
Sum of 24 Differences	+ 4·4	+ 11·4	+ 6·6	− 0·8
c_1	+ 4·4	+ 11·0	+ 8·2	+ 11·3
c_2	− 1·5	+ 25·6	+ 6·7	− 9·9
c_3	+ 6·8	+ 29·4	+ 4·2	− 3·9
c_4	+ 23·0	?	+ 5·7	+ 4·1
Means from first three of above quantities	+ 3·5	+ 11·15	+ 6·4	+ 4·3
Corresponding means at Kew	+ 3·4	+ 10·3	+ 7·0	− 3·1

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APPLICATION OF WOLF'S FORMULA.

§ 18. The nature of Wolf's formula

$$R = a + bS \quad \text{.....(1)},$$

where S is sun-spot frequency, a and b constants, and R the value of a magnetic element, and the methods of determining a and b have been discussed in (A) and a subsequent paper*.

Table XV gives the values of a, b and b/a in the case of the ranges and 24 differences for D and H for the several months of the year, also the arithmetic means of a and b from

TABLE XV.

Values of Constants in Sun-spot Formula, calculated from Groups of Years (Units 1′ for D, and 1 γ for H).

	Declination						Horizontal Force					
	Ranges			Sum of 24 differences			Ranges			Sum of 24 differences		
	a	$b \times 10^4$	$(b/a) \times 10^4$	a	$b \times 10^3$	$(b/a) \times 10^4$	a	$b \times 10^3$	$(b/a) \times 10^4$	a	$b \times 10^2$	$(b/a) \times 10^4$
January...	3·19	265	83	13·76	240	174	12·1	161	133	44·1	111	252
February..	3·32	427	129	15·63	351	224	8·8	261	296	53·8	133	248
March	6·19	761	123	27·07	409	151	14·8	351	237	67·3	263	390
April	8·70	429	49	37·73	236	63	29·2	244	84	160·6	177	110
May.....	8·25	495	60	35·71	316	89	26·2	317	121	165·8	214	129
June	7·89	519	66	39·05	297	76	29·2	251	86	175·1	175	100
July.....	8·32	457	55	38·97	259	66	28·2	261	93	167·6	200	119
August....	8·74	429	49	43·34	224	52	31·0	206	66	173·4	155	90
September	7·15	599	84	33·50	333	99	26·4	234	88	149·9	137	91
October ...	6·24	355	57	27·47	269	98	24·2	154	64	146·0	104	72
November	3·66	389	106	16·52	263	159	14·6	222	153	77·6	155	199
December	2·25	277	123	10·75	211	196	5·9	165	279	34·0	90	264
Mean of monthly values for Year	6·16	450	73	28·29	284	100	20·9	236	113	117·9	159	135
Winter	3·10	340	109	14·17	266	188	10·3	202	196	52·4	122	233
Equinox...	7·07	536	76	31·44	312	99	23·7	246	104	130·9	170	130
Summer...	8·30	475	57	39·27	274	70	28·7	259	90	170·5	186	109

* *Phil. Trans. Roy. Soc.*, Vol. 203, A, p. 151.

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the months forming the three seasons and the year, with the seasonal and yearly values calculated for b/a from these arithmetic means. The results were obtained by grouping the years (cf. (A) § 52) and are immediately comparable with the Kew results in (A) Table XLI.

It will be seen that b and b/a in both D and H tend to be larger at Falmouth than at Kew—*i.e.* sun-spot influence is greater at the former station than the latter—but the mode of variation throughout the year is remarkably similar at the two places. In the case of the H ranges the mean values of b and b/a for the year at Falmouth are in excess of those at Kew by only 18 and 13 per cent. respectively, and yet taking individual months of the year the Falmouth value of b is the greater in 11 months and the Falmouth value of b/a in 10 months. Again in the case of the 24 differences in D the mean values of b and b/a at Falmouth exceed those at Kew by only 8 and 18 per cent. respectively, but there is only one month in the year in which the Kew values of b and b/a are the larger. This is far from what one would expect if chance were the dominant factor in individual months.

The mean values of b/a for the year and the equinox are closely alike in all cases, as at Kew; also the winter value of b/a is notably the largest and the summer value the least.

It will be seen that the value of b/a for the 24 differences is in excess of that for the ranges in every instance in D and in all but two instances in H. The ratios of the seasonal values of b/a for the 24 differences to those for the ranges vary in H only between 1·19 and 1·25; but in D the ratio is 1·23 in summer as against 1·72 in winter. The corresponding ratios for the year are 1·19 for H and 1·37 for D.

§ 19. Table XVI gives the values obtained by the method of groups for a, b and b/a in the case of c_1, c_2, c_3 and c_4 for the mean diurnal inequalities for the year in D, H, N and W, and for the three seasons as well in D and H. The data correspond to those for Kew in (A) Table XLII, and closely resemble them. The sun-spot influence as measured by the size of b/a diminishes when we pass from the 24- to the 12-hour term, and from the 12- to the 8-hour term. The same tendency is generally visible in passing from the 8- to the 6-hour term, but to this the winter values appear an exception especially in D.

§ 20. The values of a and b assigned to the year in Table XV are arithmetic means from the monthly values. Table XVII gives values of a, b and b/a obtained by the method of groups from the mean diurnal inequality for the year; they answer to the Kew values in Table XLIII of (A).

Taking b/a as the best measure of sun-spot influence, it is according to Table XVII fairly alike in D and W, and again in H and N; but it is considerably greater in the second pair of elements than in the first.

The values of b/a for the 24 differences bear to the corresponding values for the ranges the following ratios:

D	*W*	*H*	*N*	mean
1·24	1·51	1·21	1·11	1·27

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TABLE XVI.

Values of Constants in Sun-spot Formula, calculated from groups of years. Fourier Coefficients in Expressions for Diurnal Inequality.

Element	Season	c_1			c_2			c_3			c_4		
		a	$b \times 10^4$	$(b/a) \times 10^4$	a	$b \times 10^4$	$(b/a) \times 10^4$	a	$b \times 10^4$	$(b/a) \times 10^4$	a	$b \times 10^4$	$(b/a) \times 10^4$
D Unit = 1′	Winter.	0·69	165	239	0·76	62	82	0·41	28	68	0·19	23	123
	Equinox	1·61	179	111	1·60	114	71	0·94	52	55	0·42	6	15
	Summer	2·22	170	76	2·01	102	51	0·89	28	31	0·16	−7	−42
	Year ...	1·46	174	119	1·46	91	62	0·76	35	45	0·26	7	26
H Unit = 1γ	Winter.	2·11	705	334	2·98	481	162	1·08	140	129	0·85	111	130
	Equinox	7·84	1036	132	4·64	410	88	2·84	132	46	1·67	26	16
	Summer	11·14	1115	100	3·99	598	150	2·06	33	16	0·71	−36	−51
	Year ...	6·89	922	134	3·74	477	128	2·03	100	49	1·17	21	18
N Unit = 1γ	Year ...	7·64	990	130	4·81	530	110	1·71	121	70	0·81	13	17
W Unit = 1γ	Year ...	7·16	865	121	7·26	427	59	4·27	179	42	1·62	39	24

TABLE XVII.

Values of Constants in Sun-spot Formula, from mean Diurnal Inequalities for the year (years grouped).

Element	Ranges			24 Differences		
	a	$b \times 10^4$	$(b/a) \times 10^4$	a	$b \times 10^3$	$(b/a) \times 10^4$
D.....	5·72	476	83	27·8	287	103
W.....	31·5	2318	74	132·0	1484	112
H.....	20·2	2274	113	110·0	1509	137
N... ..	20·6	2377	115	130·3	1673	128

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The corresponding ratios for Kew—obtainable from Table XLIII of (A)—are somewhat larger except in W, their mean being 1·31.

The considerable excess in these ratios over unity shows that the difference between years of many and years of few sun spots may be materially under-estimated when attention is directed solely to the ranges, at least when these are based on hourly readings.

Comparing Table XVII with Table XLIII of (A), we find for the mean values of the ratios of b/a at Falmouth to b/a at Kew, from ranges and 24 differences combined,

D	W	H	N
1·23	1·33	1·07	1·08

This points again to the sun-spot relation being more important at Falmouth than at Kew.

§ 21. In order to ascertain whether the connection between terrestrial magnetism and solar activity became closer when Wolfer's sun-spot frequency was replaced by the area of faculae, umbrae or whole spots, I have calculated values for the a and b of (1)—applied to the ranges from the mean diurnal inequalities of D and H in individual years—making S represent in turn the Greenwich solar data described in § 3 and Wolfer's frequencies. The method of least squares was used. From the values of a and b thus found ranges were then calculated for individual years.

Tables XVIII and XIX show the excess of the observed ranges in D and H respectively over those thus calculated, with corresponding results at Kew for Wolfer's frequencies. The tables also give the arithmetic mean of the differences irrespective of sign between observed and calculated values, and the "probable error" of a single calculated value. At the foot are shown the values of a and b in the formulae reached, and the value of B, where B is what b transforms into when the mean value of the variable S during the twelve years is taken as unity. The values obtained for a and b in the case of Wolfer's frequencies answer to those for Kew in (A) Table XLIV.

It is immaterial to the fit of a formula of the type (1)—*i.e.* to our estimate of the closeness of the connection between magnetic ranges and the particular measure of solar activity—what unit is actually employed. If, for instance, we multiplied all Wolfer's frequencies by any constant n, the only result would be that every b would have to be divided by n. The amount of agreement between the several *formulae* is most clearly seen by comparing the values of B.

It was certain *a priori* for the reasons stated in § 3 that results obtained by the intermediary of the Greenwich projected and corrected areas would be closely similar. I thus thought it unnecessary to take more than one series of projected areas, selecting the umbrae. The agreement between the results from the P and C umbrae will be seen to be remarkably close. The ranges calculated for the same year differ in no instance by more than 0′·04 in D, or 0·2γ in H. In the case of H the formula from the corrected umbrae agrees slightly better with observation than that from the projected umbrae, but in the case of D they agree equally well.

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TABLE XVIII.

Observed less calculated Ranges in mean Diurnal Inequality of Declination for the year.

Year	Greenwich Solar Data				Wolfer's Sun-spots	
	Faculae	Umbrae		Sun-spots		
	Corrected	Projected	Corrected	Corrected	Falmouth	Kew
1891	+ 0′·26	+ 0′·63	+ 0′·63	+ 0′·50	+ 0′·53	+ 0′·40
1892	− 0·69	+ 0·42	+ 0·38	+ 0·13	+ 0·15	+ 0·24
1893	+ 1·08	+ 0·32	+ 0·36	+ 0·20	+ 0·34	+ 0·32
1894	+ 0·95	− 0·33	− 0·37	− 0·10	− 0·12	− 0·16
1895	− 0·41	− 0·30	− 0·27	− 0·17	− 0·36	− 0·28
1896	− 0·13	+ 0·16	+ 0·18	+ 0·17	− 0·14	− 0·14
1897	− 0·89	− 0·86	− 0·87	− 0·83	− 0·51	− 0·53
1898	− 0·47	− 0·42	− 0·38	− 0·36	− 0·43	− 0·41
1899	+ 0·12	+ 0·24	+ 0·25	+ 0·25	+ 0·15	+ 0·07
1900	+ 0·17	+ 0·16	+ 0·13	+ 0·20	+ 0·13	+ 0·01
1901	+ 0·16	+ 0·05	+ 0·06	+ 0·12	+ 0·24	
1902	− 0·16	− 0·09	− 0·10	− 0·10	+ 0·01	
Mean difference observed ~ calculated	0′·457	0′·332	0′·332	0′·261	0′·259	
Probable Error ...	0·402	0·281	0·281	0·234	0·216	
Value of a	6′·068	6′·051	6′·066	6′·062	5′·901	
" " b	·001215	·01130	·01559	·002604	·0451	
" " $10^3 \times B$ ...	1560	1577	1561	1565	1727	

As was anticipated in § 3, the ranges calculated from the faculae differ considerably from those calculated from the other Greenwich data, especially near sun-spot maximum; their accordance with observation is also conspicuously inferior.

In the case of D the ranges calculated from Greenwich whole spot areas accord better with observation than those calculated from the umbrae, but not quite so well as the ranges

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TABLE XIX.

Observed less calculated Ranges in mean Diurnal Inequality of Horizontal Force for the year (unit = 1 γ).

Year	Greenwich Solar Data				Wolfer's Sun-spots	
	Faculae	Umbrae		Sun-spots		
	Corrected	Projected	Corrected	Corrected	Falmouth	Kew
1891	+3·4	+5·4	+5·3	+4·6	+4·8	+3·0
1892	−4·8	+0·6	+0·4	−0·8	−0·7	−1·4
1893	+2·8	−1·3	−1·2	−1·8	−1·1	+0·9
1894	+8·1	+1·2	+1·1	+2·6	+2·5	+0·8
1895	−1·1	−0·7	−0·6	0·0	−1·0	−0·4
1896	−1·2	+0·4	+0·4	+0·4	−1·2	−1·4
1897	−4·2	−4·0	−4·0	−3·8	−2·2	−1·1
1898	−2·1	−1·7	−1·6	−1·5	−1·8	−1·9
1899	+0·3	+1·1	+1·2	+1·1	+0·6	+0·3
1900	−0·6	−0·5	−0·6	−0·3	−0·7	+0·6
1901	+1·7	+1·4	+1·5	+1·7	+2·3	
1902	−2·5	−1·9	−1·9	−2·1	−1·5	
Mean difference observed ~ calculated	2·73	1·68	1·65	1·725	1·70	
Probable Error ...	2·43	1·57	1·54	1·54	1·44	
Value of a	21·11	20·77	20·84	20·94	20·09	
„ „ b	·00614	·0589	·0814	·01339	·2326	
„ „ $10^2 \times B$ ...	788	823	815	805	890	

calculated from Wolfer's frequencies. In the case of H there is little to choose between the results from umbrae, whole spot areas, and Wolfer's frequencies; the last however still give the smallest probable error.

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§ 22. Excluding the faculae, the several calculated ranges at Falmouth for the same year agree on the whole decidedly better amongst themselves than they do with the observed ranges. The mean numerical difference, for instance, between the ranges calculated from the whole spot areas and from Wolfer's frequencies is only 0′·12 in D and 0·64γ in H, the latter quantity being less than 40 per cent. of the mean difference between either set of calculated ranges and the observed ranges.

Whenever the calculated ranges, excluding those from faculae, differ substantially from the observed—as in 1891, 1897 and 1898—all the calculated values are too large, or else all are too small. Further when the values calculated from Wolfer's frequencies for Falmouth differ notably from observation, so also do those at Kew. There are two years when the calculated range for H is in excess at Falmouth whilst the observed is in excess at Kew, but in D the differences between observed and calculated values have invariably the same sign at the two stations. It would thus appear that the differences between the observed and calculated values, though comparatively small, cannot be ascribed in any large measure to accident, or to observational or local peculiarities. If the variation of magnetic range is due to direct solar action, then its intensity at a single station cannot be *exactly* expressed as a linear function of any one of the measures of solar disturbance considered above.

§ 23. In comparing the goodness of fit of Wolf's formula at different stations one has to take into account the extent of the variation of the element dealt with and the size of its mean value. During the 12 years considered the difference between the greatest and least of the mean annual ranges in D amounted to 3′·95, the mean value being 7′·63. The corresponding quantities in H were respectively 21·0γ and 29·0γ. Thus the mean differences in Tables XVIII and XIX between observed values and those calculated from Wolfer's sun-spot frequencies represent 3·4 per cent. of the mean range in D, and 5·9 per cent. in H; whilst the probable error represents 5·5 per cent. of the excursion of the range in D, and 6·9 per cent. in H. The above agreement is about the same for D, but not quite so close for H, as the agreements found in a previous paper* at Kew, Pawlowsk and Katharinenburg.

§ 24. Looking at the values of a and B in Tables XVIII and XIX one cannot fail to notice the remarkable accordance between the results from the several Greenwich series of data. The agreement in the values of the constants derived from the faculae and sun-spot areas, for instance, is much closer than that between the constants derived from the spot areas and Wolfer's frequencies, notwithstanding the much closer agreement between the ranges calculated from the two latter sets of data.

* *Phil. Trans. Roy. Soc.*, Vol. 203, A, p. 151.

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VIII. *The influence of very strong electromagnetic fields on the spark spectra of* (1) *vanadium and* (2) *platinum and iridium.* By J. E. Purvis, M.A., St John's College, Cambridge.

[*Received* 27 February 1906.]

In developing Zeeman's important discovery various experimenters* have measured the separation of many spectral lines of elements when vibrating in very strong electromagnetic fields, and they have shown, amongst other facts, that the lines of various elements which correspond to the laws of series are affected similarly, and that on the scale of vibration numbers the components of corresponding lines have the same distances in the same strength of field, and that the separations of the same type agree to the smallest detail for different elements†.

The following paper is to describe the general effect upon a large number of the spectral lines of vanadium and of the stronger lines of platinum and iridium, and also to give the value of $d\lambda/\lambda^2$ for the constituents of the separated lines.

The 21-foot concave grating spectroscope and the apparatus used in these experiments have already been described in a preliminary paper in the *Proc. Camb. Phil. Soc.*, Vol. XIII. Pt. II. p. 82. It will be sufficient to state now that the strength of the field used in these experiments was 39,980 C.G.S. units. The maximum time of exposure was 30 minutes, and only those lines are noted which appear on the photographic plate after this period of exposure. The polarised condition of the constituents was determined by a calcite prism placed between the source of light and the quartz lens used to focus the spark on the slit of the spectroscope. The spectrum of the second order was used for wave lengths from about λ 3480 and extending into the ultra-violet, and that of the first order for lines whose wave lengths are below λ 3480. The column λ gives the wave length of the undisturbed line; $d\lambda$ gives the distances of the constituents from the undisturbed line, a + sign meaning lines of greater wave length, and a − sign, lines of shorter wave length; $d\lambda/\lambda^2$ is multiplied by 10^6; the letter *s* signifies that the line vibrates so that its electric vector is perpendicular to the lines of force, and *p* that it is parallel to the lines of force. The remark "the middle constituent removed by calcite" means that the constituents of the divided line were so close, or so intense, or so broad, that they could only be seen separated when those vibrating parallel to the lines of force were blotted out by introducing a calcite prism between the source of light and the quartz condensing lens. The constituents of the divided lines are usually diffuse, and they vary in their relative diffuseness; some constituents are much more diffuse than others. This varying diffuseness renders the measurements very difficult; but for all lines whose constituents are more than three, there were eight or nine separate measurements; whilst for those lines which give triplets there were at least three separate measurements.

* Preston: *Royal Dublin Soc. Trans.* Vol. VI. p. 385, 1898; and Vol. VII. p. 7, 1899. *Phil. Mag.* Vol. XLV. p. 325, 1898; and Vol. XLVII. p. 165, 1899. *Royal Soc. Proc.* Vol. LXIII. p. 26, 1898. Reese: *Astrophys. Journ.* Vol. XII. p. 120, 1900. Kent: *Astrophys. Journ.* Vol. XIII. p. 289, 1901.

† Runge and Paschen, *Astrophys. Journ.* Vol. XV. pp. 235, 333; and Vol. XVI. p. 123.

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VANADIUM.

The lines were identified by comparison with the numbers and descriptions of Exner and Haschek* for wave lengths above 4670·45. They describe over 2000 lines, and about 1000 of these are sufficiently strong to leave their images on the photographic plate in the time exposure of these observations.

The two following lines are divided into *six* constituents. The general appearance is that of two strong inside constituents which are more diffuse on the inner edges than on the outer edges, and two weak constituents, one on each of the outer sides of these. On analysis by the calcite the two stronger inner constituents further subdivide.

λ	dλ	dλ/λ²		λ	dλ	dλ/λ²	
	+ 0·610	+ 3·60 *s*			+ 0·616	+ 3·65 *s*	
	+ 0·214	+ 1·25 *s*			+ 0·211	+ 1·25 *s*	
	+ 0·175	+ 1·03 *p*			+ 0·174	+ 1·03 *p*	
4123·70	0	0		4109·98	0	0	
	− 0·175	− 1·03 *p*			− 0·174	− 1·03 *p*	
	− 0·214	− 1·25 *s*			− 0·211	− 1·25 *s*	
	− 0·636	− 3·74 *s*			− 0·626	− 3·70 *s*	

The following lines are divided into *five* constituents; and each constituent is measured from the centre of the middle one.

λ	dλ	dλ/λ²		λ	dλ	dλ/λ²	
	+ 0·550	+ 2·84 *s*	The middle constituent is the strongest, and the two outside ones are very weak.		+ 0·277	+ 3·21 *s*	There are three very narrow and sharp constituents, and two broader and more diffuse ones, one on each side of these.
	+ 0·302	+ 1·56 *p*			+ 0·141	+ 1·63 *p*	
4400·80	0	0 *s*		2934·48	0	0 *p*	
	− 0·308	− 1·59 *p*			− 0·136	− 1·58 *p*	
	− 0·564	− 2·91 *s*			− 0·273	− 3·17 *s*	
	+ 0·845	+ 4·99 *s*	The middle constituent is much stronger than the others. The most refrangible one, marked ?, widens into the constituents of the triplet of λ 4115·38, and could not be isolated. The least refrangible constituent is very weak.		+ 0·266	+ 3·16 *s*	do.
	+ 0·370	+ 2·18 *s*			+ 0·126	+ 1·50 *p*	
4116·70	0	0 *p*		2903·20	0	0 *p*	
	− 0·431	− 2·51 *s*			− 0·135	− 1·60 *p*	
	?	? *s*			− 0·269	− 3·19 *s*	

The following lines are divided into *four* constituents. The separation of the two middle constituents of λλ 4438·08 and 3692·38 is not very wide, and it is not easy to say if there is a real separation or if it is an ordinary reversal.

λ	dλ	dλ/λ²		λ	dλ	dλ/λ²	
	+ 0·617	+ 3·12 *s*	The two middle constituents are much sharper and stronger than the two outside ones.		+ 0·454	+ 2·30 *s*	The two middle constituents are much sharper and stronger than the two outside ones.
	+ 0·224	+ 1·13 *p*			+ 0·078	+ 0·39 *p*	
4444·42	0	0		4438·08	0	0	
	− 0·224	− 1·13 *p*			− 0·078	− 0·39 *p*	
	− 0·614	− 3·13 *s*			− 0·463	− 2·35 *s*	
	+ 0·468	+ 2·37 *s*	do.		+ 0·533	+ 2·71 *s*	do.
	+ 0·124	+ 0·62 *p*			+ 0·118	+ 0·59 *p*	
4441·90	0	0		4436·34	0	0	
	− 0·124	− 0·62 *p*			− 0·118	− 0·59 *p*	
	− 0·483	− 2·44 *s*			− 0·541	− 2·74 *s*	

* *Sitz. d. k. Akad. Wien*, 1898, Vol. CVII. IIA Abth. p. 184.

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λ	dλ	dλ/λ²	
	+0·657	+3·33 *s*	
	+0·130	+0·66 *p*	
4421·82	0	0	
	−0·130	−0·66 *p*	
	−0·633	−3·24 *s*	
	+0·820	+4·20 *s*	
	+0·311	+1·59 *p*	
4416·63	0	0	
	−0·311	−1·59 *p*	
	−0·831	−4·26 *s*	
4406·96	....	.....	The constituents widen into those of 4407·89 and it was impossible to isolate them.

λ	dλ	dλ/λ²	
	+0·421	+3·08 *s*	The inside constituents are much stronger than the outside ones.
	+0·167	+1·22 *p*	
3696·00	0	0	
	−0·167	−1·22 *p*	
	−0·415	−3·05 *s*	
	+0·316	+2·32 *s*	There is no difference in the intensities of the four constituents.
	+0·093	+0·68 *p*	
3692·38	0	0	
	−0·093	−0·68 *p*	
	−0·324	−2·38 *s*	
	+0·321	+3·56 *s*	
	+0·135	+1·49 *p*	
3013·12	0	0	
	−0·135	−1·49 *p*	
	−0·328	−3·62 *s*	

Lines divided into four constituents by means of calcite. These lines have the general appearance of doublets, with the inner edge of each constituent sharp and strong, whilst the outer edges are weak and diffuse. On analysis by the calcite they are divided again, so that the lines are quadruplets with two inner strong and sharp constituents impinging upon two outer weaker and more diffuse constituents.

λ	dλ	dλ/λ²	
	?	? *s*	The outer constituents are too diffuse and weak to measure; only the two inside stronger members were measured.
	+0·205	+1·06 *p*	
4395·49	0	0	
	−0·205	−1·06 *p*	
	?	? *s*	
	?	? *s*	do.
	+0·295	+1·66 *p*	
4210·02	0	0	
	−0·295	−1·66 *p*	
	?	? *s*	
	+0·323	+1·99 *s*	
	+0·174	+1·07 *p*	
4036·95	0	0	
	−0·174	−1·07 *p*	
	−0·323	−1·99 *s*	
	+0·284	+1·79 *s*	
	+0·166	+1·05 *p*	
3977·88	0	0	
	−0·166	−1·05 *p*	
	−0·284	−1·79 *s*	
	+0·262	+1·73 *s*	
	+0·178	+1·17 *p*	
3890·35	0	0	
	−0·178	−1·17 *p*	
	−0·262	−1·73 *s*	

λ	dλ	dλ/λ²	
	+0·250	+1·71 *s*	
	+0·166	+1·14 *p*	
3813·63	0	0	
	−0·166	−1·14 *p*	
	−0·250	−1·71 *s*	
	+0·159	+1·46 *s*	
	+0·114	+1·04 *p*	
3298·89	0	0	
	−0·114	−1·04 *p*	
	−0·159	−1·46 *s*	
	+0·150	+1·38 *s*	
	+0·119	+1·10 *p*	
3290·40	0	0	
	−0·119	−1·10 *p*	
	−0·150	−1·38 *s*	
	+0·146	+1·34 *s*	
	+0·138	+1·27 *p*	
3289·52	0	0	
	−0·138	−1·27 *p*	
	−0·146	−1·34 *s*	
	?	? *s*	The outer constituents are too diffuse and weak to measure.
	+0·116	+1·08 *p*	
3263·45	0	0	
	−0·116	−1·08 *p*	
	?	? *s*	

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λ	dλ	dλ/λ²	
	+ 0·158	+ 1·49 *s*	
	+ 0·094	+ 0·88 *p*	
3249·71	0	0	
	− 0·094	− 0·88 *p*	
	− 0·158	− 1·49 *s*	
	+ 0·134	+ 1·29 *s*	
	+ 0·103	+ 0·99 *p*	
3207·52	0	0	
	− 0·103	− 0·99 *p*	
	− 0·134	− 1·29 *s*	
	+ 0·149	+ 1·45 *s*	
	+ 0·108	+ 1·05 *p*	
3202·50	0	0	
	− 0·108	− 1·05 *p*	
	− 0·149	− 1·45 *s*	
	?	? *s*	The outer constituents are too weak and too diffuse to measure. The less refrangible constituent of the two outside ones is stronger than the more refrangible one.
	+ 0·141	+ 1·38 *p*	
3193·29	0	0	
	− 0·141	− 1·38 *p*	
	?	? *s*	
	?	? *s*	The least refrangible constituent is stronger than the most refrangible one. The outside constituents are too weak and too diffuse to measure.
	+ 0·062	+ 0·62 *p*	
3146·95	0	0	
	− 0·062	− 0·62 *p*	
	?	? *s*	
	+ 0·143	+ 1·46 *s*	
	+ 0·128	+ 1·30 *p*	
3130·40	0	0	
	− 0·128	− 1·30 *p*	
	− 0·143	− 1·46 *s*	

λ	dλ	dλ/λ²	
	+ 0·142	+ 1·45 *s*	
	+ 0·110	+ 1·13 *p*	
3126·31	0	0	
	− 0·110	− 1·13 *p*	
	− 0·142	− 1·45 *s*	
	+ 0·188	+ 1·93 *s*	
	+ 0·073	+ 0·74 *p*	
3121·27	0	0	
	− 0·073	− 0·74 *p*	
	− 0·188	− 1·93 *s*	
	?	? *s*	The outer constituents are too weak and too diffuse to measure.
	+ 0·093	+ 0·98 *p*	
3062·80	0	0	
	− 0·093	− 0·98 *p*	
	?	? *s*	
	+ 0·312	+ 3·52 *s*	Of the two middle stronger constituents the less refrangible one is slightly stronger than the more refrangible one.
	+ 0·139	+ 1·57 *p*	
2975·70	0	0	
	− 0·139	− 1·57 *p*	
	− 0·312	− 3·52 *s*	
	+ 0·128	+ 1·54 *s*	
	+ 0·103	+ 1·23 *p*	
2882·60	0	0	
	− 0·103	− 1·23 *p*	
	− 0·128	− 1·54 *s*	
	+ 0·144	+ 1·73 *s*	
	+ 0·057	+ 0·68 *p*	
2880·40	0	0	
	− 0·057	− 0·68 *p*	
	− 0·144	− 1·73 *s*	

Lines whose general appearance is that of doublets with the inner edge of each constituent sharp and strong, whilst the outer edges are weak and diffuse. They are probably quadruplets like the last but they are too weak to measure when analysed by the calcite in the time exposures of these experiments. The separations have been measured between the centres of the separated lines.

	+ 0·139	+ 1·30	
4606·84	0	0	
	− 0·139	− 1·30	
	+ 0·189	+ 1·30	
3807·69	0	0	
	− 0·189	− 1·30	
	+ 0·261	+ 2·14	
3480·01	0	0	
	− 0·261	− 2·14	

	+ 0·101	+ 0·92	It is very near to line 3315·00 which is slightly widened, and the closeness of the two lines makes the analysis very difficult.
3315·35	0	0	
	− 0·101	− 0·92	
	+ 0·117	+ 1·24	
3076·12	0	0	
	− 0·117	− 1·24	
	+ 0·116	+ 1·26	
3023·99	0	0	
	− 0·116	− 1·26	

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λ	dλ	dλ/λ²		λ	dλ	dλ/λ²	
	+0·084	+0·93	The division is a very narrow one, and the two constituents are very diffuse.		+0·127	+1·73	
2996·05	0	0		2708·00	0	0	
	−0·084	−0·93			−0·127	−1·73	
	+0·134	+1·61			+0·093	+1·26	
2890·69	0	0		2706·87	0	0	
	−0·134	−1·61			−0·093	−1·26	

The following lines have almost the appearance of doublets; the difference is that in these cases the *outside* edges of the constituents are sharper and stronger than the *inside* edges, which are diffuse and blur into each other. On analysis by the calcite, they appear to be triplets, the two outer members of each being stronger and narrower than the inner one and it appears as if the latter impinged upon and overlapped the former. The distance between the two outside strong constituents has been measured because the middle constituent is weak and diffuse and it is as broad as the distance between the two outside constituents. It will be noticed that the constituents of these triplets do not vibrate in the normal directions.

	+0·265	+1·65 *p*			+0·142	+1·40 *p*
4003·12	0	0 *s*		3198·09	0	0 *s*
	−0·265	−1·65 *p*			−0·142	−1·40 *p*
	+0·220	+1·51 *p*			+0·179	+1·83 *p*
3808·70	0	0 *s*		3133·48	0	0 *s*
	−0·220	−1·51 *p*			−0·179	−1·83 *p*

The following lines are divided into triplets. The middle member of each is usually the strongest; and variations from this are noted. The constituents of weak lines are usually separated further apart than those of strong lines. The two outside constituents vibrate perpendicular, and the inner one parallel, to the lines of force, except where specially mentioned. The measurements were taken from the centre of the middle constituent.

	+0·402	+1·68		4600·40	.....	.....	Too weak and diffuse to measure.
4881·75	0	0					
	−0·402	−1·68			+0·322	+1·52	
				4594·31	0	0	
	+0·344	+1·45			−0·328	−1·55	
4875·66	0	0					
	−0·378	−1·59		4591·41	.....	.....	do.
	+0·278	+1·31			+0·312	+1·48	
4864·94	0	0		4586·55	0	0	
	−0·270	−1·27			−0·308	−1·46	
	+0·151	+0·64	The middle constituent removed by calcite.		+0·292	+1·39	
4851·69	0	0		4580·60	0	0	
	−0·151	−0·64			−0·274	−1·30	
4670·65	.....	.....	Too weak and diffuse to measure.	4578·90	.....	.....	do.
	+0·265	+1·24	The middle constituent is slightly nearer to the more refrangible constituent than the less refrangible one. The middle constituent removed by calcite, as the middle and the more refrangible constituents were too close to measure.		+0·190	+0·90	The middle constituent removed by calcite.
4619·93	0	0		4577·36	0	0	
	−0·265	−1·24			−0·190	−0·90	
					+0·251	+1·20	
				4572·00	0	0	do.
					−0·251	−1·20	

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λ	dλ	dλ/λ²	
4564·80	+ 0·293 0 − 0·293	+ 1·40 0 − 1·40	The middle constituent removed by calcite.
4560·90	+ 0·255 0 − 0·255	+ 1·22 0 − 1·22	do.
4553·22	.....	.....	Too weak and diffuse to measure.
4549·88	.....	.....	
4545·60	+ 0·242 0 − 0·258	+ 1·17 0 − 1·24	
4528·69	+ 0·316 0 − 0·316	+ 1·54 0 − 1·54	The middle constituent removed by calcite.
4524·41	.....	.....	Too weak and diffuse to measure.
4489·11	+ 0·228 0 − 0·250	+ 1·13 0 − 1·24	
4474·93	+ 0·310 0 − 0·310	+ 1·54 0 − 1·54	The middle constituent removed by calcite.
4469·92	+ 0·296 0 − 0·274	+ 1·48 0 − 1·37	
4464·49	.....	.....	Too weak and diffuse to measure.
4462·60	+ 0·286 0 − 0·292	+ 1·43 0 − 1·46	
4460·52	+ 0·286 0 − 0·334	+ 1·43 0 − 1·67	This widens into 4459·98, so that it was not easy to isolate the constituents, and the measurement of the most refrangible constituent is probably too high.
4459·98	.....	.....	Widens into 4460·52, and impossible to isolate its constituents.
4452·23	+ 0·308 0 − 0·292	+ 1·55 0 − 1·47	
4428·71	+ 0·518 0 − 0·492	+ 2·64 0 − 2·50	The most refrangible constituent is stronger and more diffuse than the least refrangible. The middle constituent also seems to shew signs of a division, but it is not very clear, and may be an ordinary reversal.
4426·23	+ 0·534 0 − 0·523	+ 2·72 0 − 2·66	

λ	dλ	dλ/λ²	
4390·23	+ 0·265 0 − 0·265	+ 1·37 0 − 1·37	The middle constituent removed by calcite.
4384·88	+ 0·317 0 − 0·317	+ 1·65 0 − 1·65	
4379·40	+ 0·321 0 − 0·321	+ 1·67 0 − 1·67	
4353·10	+ 0·342 0 − 0·356	+ 1·80 0 − 1·87	
4341·21	+ 0·328 0 − 0·306	+ 1·74 0 − 1·62	
4333·05	+ 0·274 0 − 0·262	+ 1·45 0 − 1·39	
4298·23	.....	.....	Too weak and diffuse, and also the constituents widen into one another so that it was impossible to measure them.
4297·87	.....	.....	
4296·31	+ 0·282 0 − 0·290	+ 1·52 0 − 1·57	
4292·01	+ 0·282 0 − 0·290	+ 1·53 0 − 1·57	
4284·25	+ 0·135 0 − 0·135	+ 0·73 0 − 0·73	The middle constituent removed by calcite.
4277·14	+ 0·278 0 − 0·286	+ 1·51 0 − 1·55	
4271·68	+ 0·286 0 − 0·290	+ 1·56 0 − 1·58	
4268·83	+ 0·317 0 − 0·317	+ 1·74 0 − 1·74	
4235·47	.....	.....	These lines all widen and blur into each other; so that it was impossible to measure the respective constituents.
4234·71	.....	.....	
4233·12	.....	.....	
4232·66	.....	.....	
4232·20	.....	.....	

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λ	dλ	dλ/λ²	
4225·40	+ 0·200	+ 1·12	
	0	0	
	− 0·186	− 1·04	
4205·30	+ 0·240	+ 1·35	
	0	0	
	− 0·258	− 1·45	
4202·52	+ 0·285	+ 1·61	
	0	0	
	− 0·286	− 1·61	
4183·67	+ 0·275	+ 1·57	
	0	0	
	− 0·275	− 1·57	
4134·62	+ 0·364	+ 2·12	
	0	0	
	− 0·364	− 2·12	
4132·15	+ 0·364	+ 2·13	
	0	0	
	− 0·364	− 2·13	
4128·25	+ 0·346	+ 2·03	
	0	0	
	− 0·364	− 2·13	
4115·38	+ 0·380	+ 2·24	
	0	0	
	− 0·382	− 2·25	
4112·50	+ 0·382	+ 2·25	
	0	0	
	− 0·374	− 2·21	
4105·38	+ 0·352	+ 2·08	
	0	0	
	− 0·364	− 2·15	
4102·31	+ 0·189	+ 1·12	The middle constituent removed by calcite.
	0	0	
	− 0·189	− 1·12	
4100·00	+ 0·352	+ 2·09	
	0	0	
	− 0·344	− 2·04	
4095·66	+ 0·259	+ 1·54	
	0	0	
	− 0·258	− 1·54	
4092·86	+ 0·344	+ 2·06	
	0	0	
	− 0·352	− 2·11	
4090·79	+ 0·262	+ 1·56	
	0	0	
	− 0·262	− 1·56	

λ	dλ	dλ/λ²	
4065·21	+ 0·254	+ 1·53	The middle constituent removed by calcite.
	0	0	
	− 0·254	− 1·53	
4053·76	+ 0·213	+ 1·29	do.
	0	0	
	− 0·213	− 1·29	
4051·52	+ 0·238	+ 1·45	do.
	0	0	
	− 0·238	− 1·45	
4051·13	+ 0·213	+ 1·29	do.
	0	0	
	− 0·213	− 1·29	
4046·50	+ 0·255	+ 1·55	do.
	0	0	
	− 0·255	− 1·55	
4039·76	.....	.....	Too weak and diffuse to measure.
4035·82	+ 0·170	+ 1·04	The middle constituent removed by calcite.
	0	0	
	− 0·170	− 1·04	
4023·53	+ 0·256	+ 1·58	
	0	0	
	− 0·242	− 1·50	
4005·90	+ 0·257	+ 1·60	
	0	0	
	− 0·258	− 1·60	
3999·40	.....	.....	Too weak to measure and widens into 3998·9.
3998·90	+ 0·262	+ 1·64	The middle constituent removed by calcite.
	0	0	
	− 0·262	− 1·64	
3997·28	+ 0·232	+ 1·44	do.
	0	0	
	− 0·232	− 1·44	
3992·96	+ 0·268	+ 1·68	do.
	0	0	
	− 0·268	− 1·68	
3990·72	+ 0·200	+ 1·26	do.
	0	0	
	− 0·200	− 1·26	
3973·80	+ 0·200	+ 1·26	do.
	0	0	
	− 0·200	− 1·26	
3952·11	+ 0·250	+ 1·60	do.
	0	0	
	− 0·250	− 1·60	

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λ	dλ	dλ/λ²	
3934·20	+ 0·350 0 − 0·350	+ 2·26 0 − 2·26	The middle constituent removed by calcite.
3930·21	+ 0·230 0 − 0·230	+ 1·48 0 − 1·48	do.
3926·68	+ 0·236 0 − 0·236	+ 1·52 0 − 1·52	do.
3926·45	+ 0·192 0 − 0·192	+ 1·24 0 − 1·24	do.
3925·40	.....	.....	Too weak to measure and widens into 3924·86.
3924·86	+ 0·214 0 − 0·214	+ 1·38 0 − 1·38	The middle constituent removed by calcite.
3922·61	+ 0·294 0 − 0·294	+ 1·90 0 − 1·90	do.
3916·59	+ 0·208 0 − 0·208	+ 1·34 0 − 1·34	do.
3914·51	+ 0·214 0 − 0·214	+ 1·38 0 − 1·38	do.
3912·37	.....	.....	Too weak to measure.
3910·05	+ 0·294 0 − 0·294	+ 1·92 0 − 1·92	The middle constituent removed by calcite.
3903·50	+ 0·308 0 − 0·308	+ 2·02 0 − 2·02	do.
3902·70	+ 0·284 0 − 0·284	+ 1·86 0 − 1·86	do.
3899·32	+ 0·240 0 − 0·240	+ 1·57 0 − 1·57	do.
3893·03	+ 0·344 0 − 0·308	+ 2·26 0 − 2·03	
3878·85	+ 0·230 0 − 0·230	+ 1·53 0 − 1·53	do.
3876·03	+ 0·244 0 − 0·236	+ 1·62 0 − 1·57	
3871·21	.....	.....	Too weak to measure.
3867·75	+ 0·308 0 − 0·306	+ 2·05 0 − 2·04	
3866·90	+ 0·308 0 − 0·318	+ 2·06 0 − 2·12	
3865·02	+ 0·203 0 − 0·203	+ 1·35 0 − 1·35	The middle constituent removed by calcite.
3856·00	+ 0·251 0 − 0·251	+ 1·68 0 − 1·68	do.
3855·49	.....	.....	Too near 3856·00 to measure exactly.
3847·50	+ 0·225 0 − 0·225	+ 1·51 0 − 1·51	The middle constituent removed by calcite.
3844·60	+ 0·327 0 − 0·327	+ 2·21 0 − 2·21	do.
3840·92	+ 0·220 0 − 0·220	+ 1·49 0 − 1·49	do.
3828·72	+ 0·184 0 − 0·184	+ 1·25 0 − 1·25	do.
3827·13	+ 0·294 0 − 0·294	+ 2·00 0 − 2·00	do.
3823·37	.....	.....	Widens into 3823·50 and difficult to measure.
3823·50	.....	.....	Widens into 3823·37.
3822·21	.....	.....	Too weak and diffuse to measure.
3820·14	.....	.....	
3818·10	+ 0·104 0 − 0·104	+ 0·71 0 − 0·71	The middle constituent removed by calcite.
3815·50	+ 0·213 0 − 0·213	+ 1·46 0 − 1·46	do.

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λ	dλ	dλ/λ²	
3809·80	.....	.....	Too weak and diffuse to measure.
	+ 0·314	+ 2·17	
3803·64	0	0	
	− 0·314	− 2·17	
	+ 0·286	+ 1·98	
3800·07	0	0	
	− 0·282	− 1·95	
	+ 0·312	+ 2·20	
3795·08	0	0	
	− 0·322	− 2·23	
3794·49	.....	.....	Too weak and widens into 3795·08.
	+ 0·318	+ 2·21	
3790·64	0	0	
	− 0·322	− 2·24	
	+ 0·302	+ 2·12	
3787·39	0	0	
	− 0·274	− 2·00	
3778·82	.....	.....	Widen into each other and difficult to measure.
3778·50	.....	.....	
	+ 0·292	+ 2·05	
3773·14	0	0	
	− 0·270	− 2·00	
	+ 0·122	+ 0·85	Middle constituent removed by calcite.
3771·13	0	0	
	− 0·122	− 0·85	
3767·84	.....	.....	Too weak to measure.
	+ 0·318	+ 2·24	
3760·40	0	0	
	− 0·298	− 2·10	
	+ 0·242	+ 1·71	Middle constituent removed by calcite.
3759·41	0	0	
	− 0·242	− 1·71	
3750·43	.....	.....	Widens into 3750·10 and too weak to measure.
	+ 0·212	+ 1·51	Middle constituent removed by calcite.
3750·10	0	0	
	− 0·212	− 1·51	
	+ 0·162	+ 1·15	do.
3746·00	0	0	
	− 0·162	− 1·15	
	+ 0·130	+ 0·92	do.
3743·77	0	0	
	− 0·130	− 0·92	

λ	dλ	dλ/λ²	
	+ 0·277	+ 1·98	Middle component removed by calcite as the three constituents overlapped.
3736·16	0	0	
	− 0·277	− 1·98	
	+ 0·178	+ 1·28	Middle component removed by calcite.
3732·98	0	0	
	− 0·178	− 1·28	
	+ 0·256	+ 1·83	
3732·15	0	0	
	− 0·256	− 1·83	
	+ 0·120	+ 0·86	do.
3728·51	0	0	
	− 0·120	− 0·86	
	+ 0·238	+ 1·71	do.
3727·53	0	0	
	− 0·238	− 1·71	
3722·18	.....	.....	Widens into a weak line at 3722·39 and too weak to measure.
	+ 0·318	+ 2·29	
3718·35	0	0	
	− 0·302	− 2·18	
	+ 0·211	+ 1·53	Middle component removed by calcite.
3715·70	0	0	
	− 0·211	− 1·53	
	+ 0·262	+ 2·00	
3711·28	0	0	
	− 0·292	− 2·12	
3708·86	.....	.....	Too weak and diffuse to measure.
	+ 0·196	+ 1·42	Middle component removed by calcite.
3704·90	0	0	
	− 0·196	− 1·42	
	+ 0·252	+ 1·83	do.
3703·80	0	0	
	− 0·252	− 1·83	
3701·13	.....	.....	Too weak to measure and widens into 3700·5.
	+ 0·188	+ 1·37	Middle component removed by calcite.
3700·50	0	0	
	− 0·188	− 1·37	
	+ 0·406	+ 2·98	
3690·43	0	0	
	− 0·394	− 2·90	
	+ 0·322	+ 2·36	All three constituents seem to be equally strong.
3688·21	0	0	
	− 0·318	− 2·33	

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λ	dλ	dλ/λ²	
	+0·360	+2·65	
3683·25	0	0	
	−0·356	−2·62	
3680·15	.....	.....	Too weak to measure.
3675·83	.....	.....	Too weak and diffuse to measure.
	+0·370	+2·74	
3674·83	0	0	
	−0·368	−2·72	
	+0·320	+2·37	
3673·50	0	0	
	−0·329	−2·43	
3671·33	.....	.....	do.
	+0·203	+1·50	Middle constituent removed by calcite.
3669·53	0	0	
	−0·203	−1·50	
3667·84	.....	.....	Too weak and diffuse to measure.
3663·68	.....	.....	
	+0·159	+1·18	Middle constituent removed by calcite.
3661·53	0	0	
	−0·159	−1·18	
3646·02	.....	.....	Too weak to measure.
	+0·314	+2·38	
3627·83	0	0	
	−0·302	−2·30	
	+0·298	+2·26	
3625·71	0	0	
	−0·282	−2·14	
	+0·274	+2·08	
3621·35	0	0	
	−0·260	−2·00	
	+0·184	+1·40	Middle constituent removed by calcite.
3619·09	0	0	
	−0·184	−1·40	
	+0·220	+1·70	do.
3593·53	0	0	
	−0·220	−1·70	
	+0·194	+1·49	
3592·19	0	0	
	−0·190	−1·47	
	+0·157	+1·21	do.
3589·90	0	0	
	−0·157	−1·21	

λ	dλ	dλ/λ²	
3588·25	.....	.....	Too weak and diffuse to measure.
	+0·278	+2·17	
3574·51	0	0	
	−0·274	−2·14	
	+0·270	+2·11	
3573·21	0	0	
	−0·266	−2·10	
	+0·201	+1·58	
3556·93	0	0	
	−0·201	−1·58	
	+0·198	+1·54	Middle constituent removed by calcite.
3545·36	0	0	
	−0·198	−1·54	
3541·50	.....	.....	Too weak and diffuse to measure.
	+0·326	+2·60	
3538·88	0	0	
	−0·318	−2·53	
	+0·216	+1·72	
3533·86	0	0	
	−0·208	−1·66	
3532·45	.....	.....	
3531·63	.....	.....	
3530·96	.....	.....	
3522·02	.....	.....	do.
3517·46	.....	.....	
3509·18	.....	.....	
3507·69	.....	.....	
	+0·109	+0·88	
3504·58	0	0	
	−0·107	−0·87	
	+0·184	+1·50	
3497·23	0	0	
	−0·163	−1·32	
3493·27	.....	.....	do.
	+0·182	+1·52	
3457·30	0	0	
	−0·174	−1·45	
3453·23	.....	.....	
3404·60	.....	.....	do.
3346·08	.....	.....	

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λ	dλ	dλ/λ²	
3338·00	+0·116 0 −0·116	+1·04 0 −1·04	The middle constituent removed by calcite.
3321·72	+0·149 0 −0·147	+1·35 0 −1·33	
3319·05	.....	.....	Too weak and diffuse to measure.
3318·04	.....	.....	
3317·02	.....	.....	
3304·62	.....	.....	
3293·30	+0·126 0 −0·126	+1·16 0 −1·16	The middle constituent removed by calcite.
3288·47	.....	.....	Too diffuse and weak to measure.
3285·29	+0·177 0 −0·177	+1·64 0 −1·64	The middle constituent removed by calcite.
3282·69	+0·150 0 −0·147	+1·39 0 −1·36	
3281·92	+0·216 0 −0·200	+1·99 0 −1·90	
3281·26	+0·203 0 −0·195	+1·88 0 −1·81	
3280·02	+0·157 0 −0·166	+1·45 0 −1·54	
3277·88	+0·164 0 −0·164	+1·52 0 −1·52	do.
3277·55	+0·141 0 −0·141	+1·31 0 −1·31	do.
3276·25	+0·176 0 −0·172	+1·63 0 −1·60	The middle constituent is very little stronger than the two outside ones.
3274·65	.....	.....	Too weak and diffuse to measure.
3271·27	+0·155 0 −0·155	+1·44 0 −1·44	The three constituents seem to be equally strong: the middle one is only very slightly, if at all, stronger than the two outer ones.

λ	dλ	dλ/λ²	
3270·25	.....	.....	Too weak and diffuse to measure.
3267·84	+0·133 0 −0·127	+1·24 0 −1·18	The middle constituent is only very slightly, if at all, stronger than the two outside ones.
3266·06	+0·164 0 −0·159	+1·53 0 −1·50	
3258·02	+0·145 0 −0·153	+1·36 0 −1·44	
3254·90	+0·125 0 −0·122	+1·17 0 −1·15	
3252·01	+0·197 0 −0·203	+1·86 0 −1·91	
3250·90	+0·158 0 −0·135	+1·50 0 −1·30	
3238·08	+0·148 0 −0·148	+1·40 0 −1·40	
3233·98	+0·139 0 −0·139	+1·32 0 −1·32	The middle constituent removed by calcite.
3233·67	+0·125 0 −0·125	+1·19 0 −1·19	do.
3232·10	+0·100 0 −0·116	+1·00 0 −1·10	
3227·05	+0·125 0 −0·125	+1·20 0 −1·20	do.
3217·23	+0·170 0 −0·159	+1·64 0 −1·53	
3214·86	+0·240 0 −0·224	+2·32 0 −2·16	
3212·55	+0·153 0 −0·162	+1·45 0 −1·57	

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λ	dλ	dλ/λ²	
3208·46	+0·247 0 −0·228	+2·39 0 −2·21	The three constituents are very wide and diffuse into each other.
3205·70	+0·151 0 −0·160	+1·47 0 −1·55	
3196·66	.....	.....	Too weak and diffuse to measure.
3194·06	.....	.....	
3190·80	+0·203 0 −0·191	+1·99 0 −1·87	
3188·60	+0·164 0 −0·172	+1·61 0 −1·69	
3187·78	+0·110 0 −0·110	+1·08 0 −1·08	The middle constituent removed by calcite.
3185·46	+0·177 0 −0·172	+1·74 0 −1·70	
3184·04	+0·128 0 −0·128	+1·26 0 −1·26	
3183·48	+0·136 0 −0·136	+1·34 0 −1·34	
3174·61	+0·168 0 −0·176	+1·66 0 −1·74	
3174·17	+0·232 0 −0·230	+2·30 0 −2·30	
3168·21	+0·201 0 −0·201	+2·00 0 −2·00	do.
3167·55	+0·161 0 −0·170	+1·60 0 −1·69	The least refrangible constituent is wider and stronger than the most refrangible one, and it diffuses into the inner one; it seems to give some indications of being further divided, but the separation is not definitely clear.
3165·96	+0·220 0 −0·207	+2·19 0 −2·06	

λ	dλ	dλ/λ²	
3164·91	+0·234 0 −0·207	+2·33 0 −2·06	Very weak lines: almost too weak to measure, which may account for the want of symmetry.
3163·13	+0·207 0 −0·218	+2·06 0 −2·17	
3162·81	.....	.....	Too weak and diffuse to measure; and widens into 3163·13.
3161·42	+0·217 0 −0·207	+2·16 0 −2·07	
3158·01	+0·122 0 −0·122	+1·22 0 −1·22	The middle constituent removed by calcite.
3155·51	+0·139 0 −0·130	+1·39 0 −1·30	
3151·42	+0·089 0 −0·089	+0·89 0 −0·89	do.
3148·86	.....	.....	Too weak and diffuse to measure.
3146·40	+0·147 0 −0·143	+1·48 0 −1·44	
3143·61	.....	.....	do.
3142·67	+0·172 0 −0·159	+1·74 0 −1·61	
3142·33	.....	.....	Too weak to measure and widens into 3142·67.
3141·63	+0·151 0 −0·160	+1·53 0 −1·62	
3139·88	+0·118 0 −0·118	+1·19 0 −1·19	The middle constituent removed by calcite.
3138·17	+0·191 0 −0·184	+1·93 0 −1·86	
3136·64	+0·157 0 −0·147	+1·59 0 −1·49	
3135·08	+0·164 0 −0·164	+1·66 0 −1·66	

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λ	dλ	dλ/λ²	
3128·81	+0·191 0 −0·191	+1·95 0 −1·95	
3128·40	+0·191 0 −0·209	+2·00 0 −2·13	
3125·52	+0·070 0 −0·070	+0·72 0 −0·72	Calcite used to remove the middle constituent.
3125·20	.....	.....	Too weak to measure and widens into 3125·52.
3123·49	.....	.....	Too weak and diffuse to measure.
3123·01	+0·168 0 −0·166	+1·72 0 −1·70	
3120·36	+0·172 0 −0·172	+1·76 0 −1·76	
3119·44	.....	.....	do.
3118·51	+0·116 0 −0·116	+1·19 0 −1·19	Calcite used to remove the middle constituent. All three constituents seem to be equally strong.
3116·90	+0·139 0 −0·139	+1·43 0 −1·43	
3113·19	+0·130 0 −0·139	+1·32 0 −1·41	
3110·82	+0·135 0 −0·135	+1·39 0 −1·39	do.
3109·51	+0·149 0 −0·147	+1·54 0 −1·52	
3108·81	+0·229 0 −0·236	+2·36 0 −2·44	
3106·90	.....	.....	Too weak and diffuse to measure.
3105·03	+0·202 0 −0·205	+2·09 0 −2·12	
3102·39	+0·153 0 −0·153	+1·58 0 −1·58	Calcite used to remove the middle constituent. All three constituents seem to be equally strong.

λ	dλ	dλ/λ²	
3101·09	+0·096 0 −0·096	+0·99 0 −0·99	Calcite used to remove the middle constituent.
3094·33	+0·168 0 −0·154	+1·67 0 −1·60	
3093·23	+0·168 0 −0·168	+1·69 0 −1·69	All three constituents seem to be equally strong.
3086·61	+0·205 0 −0·205	+2·15 0 −2·15	Calcite used to remove the middle constituent.
3083·31	+0·166 0 −0·164	+1·75 0 −1·73	
3082·65	+0·213 0 −0·213	+2·24 0 −2·24	
3081·39	.....	.....	Too weak and diffuse to measure.
3070·31	.....	.....	
3067·20	+0·113 0 −0·113	+1·20 0 −1·20	Calcite used to remove the middle constituent.
3066·50	+0·186 0 −0·176	·1·97 0 −1·87	
3065·71	.....	.....	Too weak and diffuse to measure.
3063·30	+0·123 0 −0·123	+1·31 0 −1·31	Calcite used to remove the middle constituent.
3060·60	+0·174 0 −0·164	+1·85 0 −1·75	
3057·55	+0·143 0 −0·149	+1·53 0 −1·59	
3054·00	+0·174 0 −0·164	+1·86 0 −1·74	
3053·48	+0·135 0 −0·139	+1·44 0 −1·49	
3050·85	.....	.....	Too weak and diffuse to measure.

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λ	dλ	dλ/λ²	
	+0·160	+1·72	
3049·00	0	0	
	−0·126	−1·35	
	+0·143	+1·53	
3048·76	0	0	
	−0·137	−1·47	
	+0·097	+0·92	Calcite used to remove the middle constituent.
3042·39	0	0	
	−0·097	−0·92	
3038·63	.....	.....	Too weak and diffuse to measure.
	+0·159	+1·73	
3033·99	0	0	
	−0·168	−1·83	
	+0·147	+1·60	
3033·55	0	0	
	−0·159	−1·72	
3025·08	.....	.....	do.
	+0·337	+3·68	
3022·70	0	0	
	−0·341	−3·73	
	+0·219	+2·40	
3016·81	0	0	
	−0·217	−2·38	
3014·87	.....	.....	All three constituents are very broad and diffuse and run into each other, and are not easy to measure exactly.
	+0·234	+2·57	
3012·09	0	0	
	−0·221	−2·43	
	+0·298	+3·29	Very weak lines: almost too weak to measure, which may account for the want of symmetry.
3007·37	0	0	
	−0·275	−3·04	
3006·57	.....	.....	Too weak to measure.
3005·87	.....	.....	
	+0·228	+2·52	
3003·50	0	0	
	−0·225	−2·50	
	+0·180	+2·00	
3001·82	0	0	
	−0·182	−2·01	
	+0·209	+2·32	
3001·28	0	0	
	−0·213	−2·36	

λ	dλ	dλ/λ²	
	+0·244	+2·73	
2989·67	0	0	
	−0·261	−2·92	
	+0·203	+2·27	
2988·07	0	0	
	−0·205	−2·29	
	+0·131	+1·47	
2985·25	0	0	
	−0·133	−1·49	
	+0·172	+1·93	The most refrangible member is as strong as the middle one. The middle constituent removed by calcite.
2983·62	0	0	
	−0·172	−1·93	
	+0·283	+3·16	
2982·82	0	0	
	−0·275	−3·10	
	+0·141	+1·58	The middle constituent removed by calcite.
2981·27	0	0	
	−0·141	−1·58	
	+0·167	+1·88	do.
2976·55	0	0	
	−0·167	−1·88	
	+0·137	+1·54	
2974·06	0	0	
	−0·133	−1·50	
	+0·151	+1·70	
2972·31	0	0	
	−0·143	−1·61	
	+0·151	+1·71	
2968·40	0	0	
	−0·159	−1·80	
2958·68	.....	.....	Too weak and too diffuse to measure: the three constituents widen and diffuse into each other: the inner one is only a little stronger than the two outside ones.
	+0·167	+1·90	The three constituents are very wide and diffuse.
2957·54	0	0	
	−0·167	−1·90	
2955·65	.....	.....	Too weak and too diffuse to measure.
	+0·186	+2·13	
2952·12	0	0	
	−0·184	−2·11	
	+0·132	+1·51	
2949·24	0	0	
	−0·132	−1·51	

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λ	dλ	dλ/λ²	
	+0·137	+1·57	
2948·15	0	0	
	−0·130	−1·50	
	+0·203	+2·35	
2944·68	0	0	
	−0·199	−2·30	
2943·70	.....	.....	Too weak and too diffuse to measure.
2943·25	.....	.....	
2943·48	.....	.....	
	+0·201	+2·32	The least refrangible constituent is more diffuse than the most refrangible one.
2941·51	0	0	
	−0·199	−2·30	
2938·35	.....	.....	Too weak to measure.
2933·95	.....	.....	Too weak and too diffuse to measure.
	+0·153	+1·77	
2932·42	0	0	
	−0·161	−1·87	
	+0·164	+1·86	
2930·96	0	0	
	−0·155	−1·80	
2930·25	.....	.....	do.
	+0·071	+0·82	The middle constituent removed by calcite.
2926·50	0	0	
	−0·071	−0·82	
	+0·129	+1·50	
2925·40	0	0	
	−0·120	−1·40	
	+0·166	+1·94	
2924·79	0	0	
	−0·155	−1·81	
	+0·180	+2·10	
2924·14	0	0	
	−0·172	−2·01	
	+0·161	+1·83	All three very weak constituents.
2923·47	0	0	
	−0·155	−1·80	
	+0·203	+2·38	
2920·50	0	0	
	−0·200	−2·34	
	+0·175	+2·05	
2920·11	0	0	
	−0·166	−2·00	

λ	dλ	dλ/λ²	
2918·32	.....	.....	Too weak and diffuse to measure.
	+0·187	+2·19	The middle constituent removed by calcite. The three constituents are very wide and diffuse.
2917·41	0	0	
	−0·187	−2·19	
	+0·165	+1·94	The middle constituent removed by calcite.
2916·00	0	0	
	−0·165	−1·94	
2915·46	.....	.....	Too weak and diffuse to measure.
2914·97	.....	.....	
	+0·139	+1·60	
2914·40	0	0	
	−0·141	−1·66	
	+0·063	+0·74	
2911·78	0	0	
	−0·063	−0·74	
	+0·182	+2·19	
2911·17	0	0	
	−0·186	−2·19	
	+0·129	+1·52	
2910·15	0	0	
	−0·122	−1·44	
	+0·150	+1·74	Calcite used to remove the middle constituent.
2908·96	0	0	
	−0·150	−1·74	
2908·56	.....	.....	Widens into 2908·96 and too weak to measure.
	+0·182	+2·15	
2907·60	0	0	
	−0·191	−2·25	
	+0·168	+1·98	
2906·60	0	0	
	−0·161	−1·90	
	+0·172	+2·03	
2905·75	0	0	
	−0·160	−1·90	
	+0·146	+1·73	
2905·13	0	0	
	−0·154	−1·82	
	+0·229	+2·72	
2896·31	0	0	
	−0·227	−2·70	
	+0·154	+1·83	
2893·47	0	0	
	−0·146	−1·74	

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λ	dλ	dλ/λ²	
2892·82	+0·120 0 −0·120	+1·43 0 −1·43	Calcite used to remove the middle constituent.
2892·51	+0·145 0 −0·145	+1·73 0 −1·73	do.
2891·78	+0·089 0 −0·089	+1·07 0 −1·07	do.
2888·36	+0·132 0 −0·149	+1·58 0 −1·78	
2887·30 2887·08	....	....	Weak and widen into each other: too weak to measure.
2884·91	+0·172 0 −0·166	+2·07 *p* 0 *s* −2·00 *p*	The two outer members are a little stronger and wider than the middle one. The two outer constituents appear to vibrate parallel, and the inner one perpendicular to the lines of force.
2880·92	.....	.....	Too weak and diffuse to measure.
2879·26	+0·248 0 −0·246	+2·99 0 −2·95	
2877·80	+0·100 0 −0·100	+1·21 0 −1·21	Calcite used to remove the middle constituent.
2873·30	.....	.....	Too weak and diffuse to measure.
2869·22	+0·141 0 −0·139	+1·71 0 −1·70	
2864·60	+0·065 0 −0·065	+0·79 0 −0·79	The middle constituent removed by calcite.
2855·39	+0·095 0 −0·095	+1·16 0 −1·16	do.
2854·41	+0·118 0 −0·114	+1·44 0 −1·40	
2852·63	.....	.....	Too weak to measure.
2851·36	+0·116 0 −0·116	+1·42 0 −1·42	Calcite removed middle constituent.

λ	dλ	dλ/λ²	
2849·19	+0·094 0 −0·094	+1·15 0 −1·15	Calcite used to remove the middle constituent.
2847·65	+0·090 0 −0·090	+1·11 0 −1·11	do.
2845·37	+0·188 0 −0·186	+2·32 0 −2·30	
2841·20	+0·133 0 −0·129	+1·64 0 −1·60	
2836·62	+0·119 0 −0·119	+1·48 0 −1·48	do.
2830·52 2826·02	....	....	Too weak and too diffuse to measure.
2810·39	+0·137 0 −0·124	+1·73 0 −1·57	The least refrangible constituent is stronger than the most refrangible one and is almost as strong as the middle one. This may explain the want of symmetry.
2809·66 2808·39 2805·69	....	....	Too weak and diffuse to measure.
2803·60	+0·122 0 −0·122	+1·55 0 −1·55	
2802·93	+0·123 0 −0·123	+1·57 0 −1·57	Calcite used to remove the middle constituent.
2799·59	+0·076 0 −0·076	+0·97 0 −0·97	do.
2798·88	+0·127 0 −0·127	+1·62 0 −1·62	do.
2797·93	+0·114 0 −0·114	+1·45 0 −1·45	do.
2797·12	+0·088 0 −0·088	+1·13 0 −1·13	do.

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λ	dλ	dλ/λ²	
2781·69	.....	.....	Too weak and diffuse to measure.
2778·75	.....	.....	
2777·86	+0·149 0 −0·149	+1·93 0 −1·93	Calcite used to remove the middle constituent.
2774·81	+0·111 0 −0·111	+1·44 0 −1·44	do.
2774·40	+0·098 0 −0·098	+1·27 0 −1·27	do.
2768·69	+0·104 0 −0·104	+1·35 0 −1·35	do.
2766·59	+0·230 0 −0·227	+3·00 0 −2·98	
2765·81	+0·147 0 −0·143	+1·92 0 −1·86	
2760·62	+0·109 0 −0·109	+1·44 0 −1·44	do.
2760·26	+0·185 0 −0·185	+2·43 0 −2·43	do.
2753·54	+0·091 0 −0·091	+1·20 0 −1·20	do.
2747·55	+0·135 0 −0·135	+1·79 0 −1·79	do.
2729·81	+0·081 0 −0·081	+1·08 0 −1·08	do.
2715·80	+0·090 0 −0·090	+1·22 0 −1·22	do.
2714·31	.....	.....	Too weak and too diffuse to measure.
2711·88	+0·185 0 −0·188	+2·51 0 −2·55	

λ	dλ	dλ/λ²	
2706·34	+0·128 0 −0·128	+1·75 0 −1·75	Calcite used to remove the middle constituent.
2702·31	+0·153 0 −0·151	+2·09 0 −2·06	
2701·16	+0·112 0 −0·104	+1·53 0 −1·42	
2690·91	+0·164 0 −0·153	+2·26 0 −2·11	
2690·41	+0·159 0 −0·168	+2·20 0 −2·32	
2689·99	+0·172 0 −0·159	+2·37 0 −2·20	
2688·82	+0·168 0 −0·161	+2·32 0 −2·22	
2688·12	+0·151 0 −0·159	+2·10 0 −2·20	
2683·21	+0·174 0 −0·168	+2·41 0 −2·33	Widen into each other, and the least refrangible of one almost coincides with the most refrangible constituent of the other.
2682·98	+0·168 0 −0·157	+2·33 0 −2·20	
2679·39	+0·157 0 −0·164	+2·20 0 −2·29	
2678·66	+0·162 0 −0·159	+2·25 0 −2·21	
2677·91	+0·151 0 −0·149	+2·10 0 −2·07	
2672·11	+0·137 0 −0·135	+1·91 0 −1·90	
2663·42	+0·094 0 −0·094	+1·32 0 −1·32	The middle constituent removed by calcite.

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λ	dλ	dλ/λ²	
2655·82	.....	.....	Too weak and diffuse to measure.
2645·90	.....	.....	
2642·32	+0·130 0 −0·130	+1·86 0 −1·86	Very weak, middle constituent removed by calcite.
2630·72	+0·108 0 −0·108	+1·56 0 −1·56	do.
2595·20	+0·120 0 −0·120	+1·78 0 −1·78	do.
2593·18	.....	.....	Too weak and diffuse to measure.
2553·11	.....	.....	
2549·36	+0·123 0 −0·123	+1·89 0 −1·89	
2528·97	+0·087 0 −0·087	+1·34 0 −1·34	
2528·59	.....	.....	Constituents overlap and are too broad and diffuse to measure.
2528·00	+0·115 0 −0·115	+1·80 0 −1·80	
2524·07	.....	.....	Too weak and diffuse to measure.
2521·62	.....	.....	
2516·19	.....	.....	
2479·60	.....	.....	
2479·09	.....	.....	
2405·30	.....	.....	
2393·70	.....	.....	
2382·59	.....	.....	
2371·19	+0·080 0 −0·080	+1·42 0 −1·42	The middle constituent removed by calcite.
2366·40	.....	.....	Too weak and diffuse to measure.

The following lines became doublets, and most of them are very diffuse and weak; and it is probable that other constituents overlap. In each case the width between the centres of the aggregate of the two constituents has been measured.

3885·03	+0·164 0 −0·164	+1·08 0 −1·08	
3876·25	+0·203 0 −0·203	+1·35 0 −1·35	
3864·00	+0·153 0 −0·153	+1·02 0 −1·02	
3566·32	+0·180 0 −0·180	+1·41 0 −1·41	
3560·78	+0·148 0 −0·148	+1·16 0 −1·16	The two constituents are broad and strong.
3524·89	+0·387 0 −0·387	+3·11 0 −3·11	
3520·19	+0·179 0 −0·179	+1·44 0 −1·44	
3486·09	+0·172 0 −0·172	+1·41 0 −1·41	
3043·62	+0·109 0 −0·109	+1·18 0 −1·18	
3041·52	+0·120 0 −0·120	+1·29 0 −1·29	
3008·61	.....	.....	A very narrow separation: the constituents are very diffuse and the more refrangible one is stronger and sharper than the less refrangible one.

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The following lines do not seem to be either divided or widened; they appear to become a little sharper and weaker when vibrating in the magnetic field:—

4330·28	2910·50
3968·19	2889·71
3896·32	2775·89
2950·40	2739·80

There are numerous other lines which are widened, but they are too weak to examine or separate by the calcite; and others show a weak diffuse widened middle constituent, whilst the two outer constituents are too weak to affect the plate in the maximum time exposure of these experiments; they would probably become visible if an exposure of two or three hours were given.

Platinum.

The spark spectra of platinum and iridium have a large number of lines, most of which are weak. Only the strong lines are noted and whose images were photographed during a maximum exposure of 30 minutes when vibrating in the magnetic field. The lines were identified and compared with the descriptions of Exner and Haschek*.

The following lines are divided into triplets with the exception of those specially described in the notes.

λ	dλ	dλ/λ²	
4164·9	+0·289 0 −0·289	+1·66 0 −1·66	The triplet could be measured only by removing the constituents vibrating parallel to the lines of force by means of the calcite prism.
4118·9	+0·286 0 −0·286	+1·68 0 −1·68	do.
3966·4	+0·294 0 −0·294	+1·86 0 −1·86	do.
3925·6	+0·286 0 −0·286	+1·81 0 −1·81	do.
3818·9	+0·320 0 −0·320	+2·19 0 −2·19	do.
3628·3	+0·262 0 −0·262	+1·98 0 −1·98	do.
3408·3	+0·167 +0·122 0 −0·122 −0·167	+1·44 *s* +1·05 *p* 0 −1·05 *p* −1·44 *s*	The general appearance is that of a doublet with the inner edges sharp and strong, and the outer ones weak and diffuse. On analysis by the calcite, it appears to be a quadruplet. The two inner sharper constituents impinging upon and overlapping the two outer weaker ones.

λ	dλ	dλ/λ²	
3301·9	+0·186 0 −0·180	+1·70 0 −1·65	This has the appearance of a doublet not unlike that of λ 3408·30 with the inner edges sharp and the outer edges weak and diffuse. It is probably a quadruplet like 3408·30, but it is too weak to determine accurately.
3274·0	+0·146 0 −0·146	+1·36 0 −1·36	
3256·0	+0·159 0 −0·166	+1·50 0 −1·56	
3247·6	+0·157 0 −0·157	+1·48 0 −1·48	The middle constituent removed by calcite.
3204·2	+0·192 0 −0·195	+1·87 0 −1·89	
3201·0	+0·199 0 −0·195	+1·94 0 −1·90	
3139·5	+0·123 0 −0·123	+1·24 0 −1·24	This has the general appearance of a doublet like that of 3408·30, but the outer constituents are too weak and diffuse to measure. It is probably a quadruplet.
3064·8	+0·178 0 −0·178	+1·89 0 −1·89	

* *Sitz. d. k. Akad. Wien*, 1896, Vol. cv. II A Abth. pp. 523, 542.

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λ	dλ	dλ/λ²		λ	dλ	dλ/λ²	
3001·3	+ 0·105 0 − 0·105	+ 1·17 0 − 1·17	The same remark applies to 3001·3 as to 3139·5.	2771·6	+ 0·112 0 − 0·112	+ 1·45 0 − 1·45	The middle constituent was removed by the calcite prism.
2998·0	+ 0·178 0 − 0·184	+ 2·00 0 − 2·04		2733·9	+ 0·130 0 − 0·128	+ 1·73 0 − 1·71	
2929·9	+ 0·157 0 − 0·163	+ 1·82 0 − 1·89		2719·1	+ 0·149 0 − 0·159	+ 2·01 0 − 2·15	
2893·4	+ 0·186 0 − 0·189	+ 2·22 0 − 2·25		2705·9	+ 0·147 0 − 0·145	+ 2·00 0 − 1·98	
2830·3	+ 0·154 0 − 0·149	+ 1·92 0 − 1·86		2702·5	+ 0·141 0 − 0·140	+ 1·93 0 − 1·91	
2794·2	+ 0·135 0 − 0·132	+ 1·73 0 − 1·70		2659·5	+ 0·091 0 − 0·091	+ 0·67 0 − 0·67	do.
2774·8	+ 0·143 0 − 0·143	+ 1·85 0 − 1·85	The middle constituent was removed by the calcite prism.				

In addition to the lines described above there are other lines which are divided into triplets, but the constituents are too weak to measure with any degree of accuracy. Such lines are the following:—

4552·6	3639·0	2677·3
4498·9	3553·2	2628·1 ?
4442·7	3485·3 ?	2425·0
4192·7	3252·0	2377·3
3672·2	2890·5	2310·9 ?
3643·3	2755·0	

And there are a number of weak lines which are widened, but they are too weak to shew any separation when analysed by the calcite prism.

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Iridium.

λ	dλ	dλ/λ²	
	+0·298	+2·06 *s*	This has the general appearance of a doublet with the inner edges sharp and strong and the outer ones weak and diffuse. On analysis by the calcite it separates into a quadruplet, with the two inner constituents impinging upon the two outer ones.
	+0·197	+1·36 *p*	
3800·2	0	0	
	−0·197	−1·36 *p*	
	−0·298	−2·06 *s*	
	+0·266	+2·04	
3605·9	0	0	
	−0·254	−2·00	
	+0·316	+2·47	
3573·8	0	0	
	−0·306	−2·40	
	+0·314	+2·53	
3516·0	0	0	
	−0·310	−2·50	
	+0·192	+1·85	
3220·7	0	0	
	−0·189	−1·82	
	+0·194	+1·97	
3133·4	0	0	
	−0·192	−1·95	

λ	dλ	dλ/λ²	
	+0·192	+2·03	
3069·0	0	0	
	−0·189	−2·00	
	+0·118	+1·27	This has a general appearance similar to that of 3800·2: but the two outer constituents are too weak to analyse. It is probably a quadruplet.
3042·7	0	0	
	−0·118	−1·27	
	+0·150	+1·73	
2943·3	0	0	
	−0·141	−1·62	
	+0·184	+2·16	
2924·9	0	0	
	−0·184	−2·16	
	+0·157	+1·93	
2849·7	0	0	
	−0·154	−1·90	
	+0·174	+2·16	
2833·3	0	0	
	−0·167	−2·08	

In addition to the above lines there are others divided into triplets, but they are too weak to measure; such are:—

4070·3	3594·5	2774·9
3653·2	3573·8	2664·6
3628·8	2824·6	2368·1

There are also others which are widened, but they are too weak for analysis by the calcite.

As a rule in both platinum and iridium the middle constituent of each triplet is the strongest, but the constituents of the triplet of Pt 3064·8 seem to be equally intense. Also the constituents of the normally weak lines are always separated further from each other than those of the strong lines.

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GENERAL SUMMARY.

The more important conclusions which may be drawn from the preceding observations and measurements are:—

(1) The values of $d\lambda/\lambda^2$ for many lines are essentially identical. The general appearance of the corresponding constituents, and that of the normal lines, are also in these cases very similar. The most striking illustrations of similarity in the case of the vanadium lines are the sextuplets derived from 4123·7 and 4109·98, the quintuplets from 2934·48 and 2903·20, and the quadruplets from 4444·42 and 3696·00; and amongst the lines which are divided into four on analysis by the calcite there are several which are very similar in their appearance and in the values of $d\lambda/\lambda^2$. Also amongst the lines becoming triplets there are lines which can be grouped together in a similar way.

The method, therefore, may be a means of correlating and classifying into groups having the same general properties the spectral lines of elements which have not been classified in the same manner as those of the alkalis and the alkaline earths.

(2) The metals platinum and iridium are usually grouped together in the same family having the same general physical and chemical properties, and the method may be a means of correlating and classifying corresponding lines of elements belonging to the same family. Only the strongest lines of these two elements were photographed, but amongst them it will be noticed that there are several lines which have the same essential value of $d\lambda/\lambda^2$ for the constituents, and the general appearances both of the constituents and the normal lines are in these cases essentially the same.

(3) In some instances the values of $d\lambda/\lambda^2$ for the several constituents of the same line seem to be simple multiples of each other; and also the value of $d\lambda/\lambda^2$ for the constituents of some lines is a multiple of the value for the constituents of other lines.

The results of the measurements of other metals will be published later.

Finally I desire to convey my thanks to Professor Liveing for the use of the spectroscope, and to Professor Larmor for his sympathetic interest in the investigation.

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IX. *On the Asymptotic Expansion of the Integral Functions* $\sum_{n=0}^{\infty} \frac{x^n \Gamma(1+\alpha n)}{\Gamma(1+n)}$ *and* $\sum_{n=0}^{\infty} \frac{x^n \Gamma(1+n\theta)}{\Gamma(1+n+n\theta)}$.

By E. W. Barnes, M.A., Fellow of Trinity College, Cambridge.

[*Received and Read* March 12, 1906.]

§ 1. In a memoir "On the Asymptotic Expansion of Integral Functions defined by Taylor's Series*," the author has discussed the expansions of various wide classes of integral functions. The results obtained in the present paper were indicated, though no detailed investigation was given. Such an investigation is now supplied. The functions are of interest in that the *associated* functions have not finite radius of convergence. They may well be considered together as their asymptotic expansions are related. And the method employed to obtain their asymptotic expansions in a region including the positive half of the real axis differs from that which was used for any of the other functions considered, depending, as it does, on the use of Lagrange's series.

The function represented by the series $\sum_{n=0}^{\infty} \frac{x^n \Gamma(1+\alpha n)}{\Gamma(1+n)}$ we denote by $f(x, \alpha)$; that represented by $\sum_{n=1}^{\infty} \frac{x^n \Gamma(1+n\theta)}{\Gamma(1+n+n\theta)}$ by $\phi(x, \theta)$. We shall assume that $0 < \alpha < 1$, and that $\theta > 0$.

When $\alpha = 1$, $f(x, \alpha) = \frac{1}{1-x}$; when $\alpha = 0$, $f(x, \alpha) = e^x$. Both the functions $f(x, \alpha)$ and $1 + \phi(x, \theta)$ are particular cases of the more general function $\sum_{n=0}^{\infty} \frac{\Gamma(1+\alpha n)}{\Gamma(1+\beta n)} x^n$, $(\beta > \alpha > 0)$, which also includes as a particular case Mittag-Leffler's function $E_\alpha(x)$. This function the author hopes to consider on a future occasion.

§ 2. *The asymptotic expansion of* $f(x, \alpha)$ *when* $R(x) < 0$.

If αn be real and > -1, we know that

$$\Gamma(\alpha n + 1) = \int_0^\infty e^{-y} y^{\alpha n} dy,$$

the real positive value of $y^{\alpha n}$ being taken, and the integration being along the positive half of the real axis.

* Presented to the Royal Society in May 1905 and shortly to be published.

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Hence, if $0 < \alpha < 1$, the function $f(x, \alpha)$ can be represented by the integral

$$\int_0^\infty \exp\{-y + xy^\alpha\}\, dy.$$

Make now the substitution $t = y^\alpha$, in which t is real.

Then
$$f(-x, \alpha) = \frac{1}{\alpha}\int_0^\infty e^{-xt} e^{-t^{1/\alpha}} t^{1/\alpha - 1}\, dt$$
$$= \frac{1}{\alpha}\int_0^\infty e^{-xt} \sum_{n=0}^\infty \frac{(-)^n}{n!} t^{(n+1)/\alpha - 1}\, dt.$$

Suppose now that $|\arg x| < \pi/2$. Then in the latter integral we may make the substitution $xt = T$, and take the integral with respect to T also along the positive half of the real axis. For the difference between the two integrals is an integral along an arc of a great circle at infinity whose value is zero.

If then $|\arg x| < \pi/2$,
$$f(-x, \alpha) = \frac{1}{\alpha x^{1/\alpha}}\int_0^\infty \sum_{n=0}^\infty \frac{(-)^n}{n!\, x^{n/\alpha}} e^{-t} t^{(n+1)/\alpha - 1}\, dt.$$

Now I have shewn in my original memoir (§ 5) that if we replace the previous integral by the sum of the integrals of the various terms in the subject of integration we obtain a series which is truly asymptotic.

We therefore have
$$f(-x, \alpha) = \frac{1}{\alpha x^{1/\alpha}} \sum_{n=0}^N \frac{(-)^n \Gamma\{(n+1)/\alpha\}}{\Gamma(n+1)\, x^{n/\alpha}} + J_N,$$
where $|J_N x^{(N+1)/\alpha}|$ can be made as small as we please by making $|x|$ sufficiently large.

Finally therefore, if $R(x) < 0$, and $0 < \alpha < 1$, we have the asymptotic expansion
$$f(x, \alpha) = \frac{1}{\alpha(-x)^{1/\alpha}} \sum_{n=0}^\infty \frac{(-)^n \Gamma\{(n+1)/\alpha\}}{\Gamma(n+1)(-x)^{n/\alpha}},$$
the principal value of $(-x)^{n/\alpha}$, which is real when x is real and negative and which has a cross-cut along the positive half of the real axis, being taken.

§ 3. *The asymptotic expansion of $f(x, \alpha)$ when $\pi/2 \geqslant |\arg x| > (1-\alpha)\pi/2$.*

We now proceed to shew that the previous expansion is valid when
$$\pi/2 \geqslant |\arg x| > (1-\alpha)\pi/2.$$

Suppose that θ, the argument of x, is positive and equal to $\pi(1-\eta)/2$, where $0 \leqslant \eta < \alpha$.

We have seen that
$$f(x, \alpha) = \int_0^\infty \exp\{-y + xy^\alpha\}\, dy,$$
the integral being taken along the positive half of the real axis. Put $y = ze^{i\phi}$, and we have
$$f(x, \alpha) = e^{i\phi}\int_0^\infty (-\phi) \exp\{-ze^{i\phi} + xe^{\alpha i\phi} z^\alpha\}\, dz,$$
the integral now being taken along an axis of argument $(-\phi)$.

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Choose ϕ so that $\eta\pi/2\alpha < \phi < \pi/2$, as is always possible if $\eta < \alpha$. Then we have $\phi < \pi/2$ and $\alpha\phi + \pi(1-\eta)/2 > \pi/2$. Hence the subject of integration tends exponentially to zero along an arc of the great circle at infinity for which the argument lies between 0 and $-\phi$, the limits included. The integral can therefore be taken along the real axis, and we have

$$f(x, \alpha) = e^{i\phi} \int_0^\infty \exp\{xe^{\alpha i\phi} z^\alpha\} \sum_{n=0}^\infty \frac{(-z)^n e^{ni\phi}}{n!} dz$$

$$= \frac{e^{i\phi}}{\alpha} \sum_{n=0}^\infty \int_0^\infty e^{-y} \frac{(-)^n y^{(n+1)/\alpha - 1} e^{ni\phi}}{n! (-xe^{\alpha i\phi})^{(n+1)/\alpha}} dy, \text{ asymptotically.}$$

And thus we have the asymptotic expansion

$$f(x, \alpha) = \frac{1}{\alpha} \sum_{n=0}^\infty \frac{(-)^n \Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma(n+1)(-x)^{(n+1)/\alpha}}.$$

We may obviously employ a similar method when θ, the argument of x, is negative and $\pi/2 \geqslant |\theta| > (1-\alpha)\pi/2$.

§ 4. The asymptotic expansion for $f(x, \alpha)$ which has been obtained in §§ 2 and 3 is thus valid when $|\arg(-x)| < (1+\alpha)\pi/2$.

We proceed to verify this result by means of the contour integrals employed in the fundamental memoir.

Let us consider the contour integral

$$-\frac{1}{2\pi i} \int \Gamma(-s)\Gamma(\alpha s + 1)(-x)^s ds,$$

where the contour of integration embraces the positive half of the real axis and encloses the poles of $\Gamma(-s)$, but none of those of $\Gamma(\alpha s + 1)$. Its value, by Cauchy's theory of residues, is evidently $f(x, \alpha)$.

Now when $|\arg(-x)| < (1+\alpha)\pi/2$, the integral will vanish when taken round that part of an infinite contour for which $R(s)$ is greater than a finite negative quantity however large. This theorem is immediately evident from the known asymptotic expansion for $\Gamma(s)$, when $|s|$ is very large.

The integral may therefore be taken along a contour in the finite part of the plane which consists of a straight line parallel to the imaginary axis which passes between the poles $-k/\alpha$ and $-(k+1)/\alpha$, together with a loop which encloses the poles $-1/\alpha, \ldots, -k/\alpha$ of $\Gamma(\alpha s + 1)$.

We therefore have, if $|\arg(-x)| < (1+\alpha)\pi/2$,

$$f(x, \alpha) = \sum_{n=1}^k \Gamma(n/\alpha)(-x)^{-n/\alpha} \operatorname*{Lt}_{\epsilon=0} \epsilon\Gamma(-n+1+\alpha\epsilon) + J_k(x)$$

$$= \sum_{n=1}^k \frac{(-)^{n-1}\Gamma(n/\alpha)}{\alpha\Gamma(n)(-x)^{n/\alpha}} + J_k(x),$$

where $|J_k(x) x^{k/\alpha}|$ can be made as small as we please by taking $|x|$ sufficiently large.

This is equivalent to our former result.

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§ 5. *The asymptotic expansion of $f(x, \alpha)$ when $|\arg x| < \pi(1-\alpha)/2$.*

We will now shew that, when $|\arg x| < (1-\alpha)\pi/2$, we have the asymptotic expansion

$$f(x, \alpha) = \exp\{(1-\alpha)\alpha^{\alpha/(1-\alpha)} x^{1/(1-\alpha)}\} (\alpha x)^{1/2(1-\alpha)}$$
$$\times \left[\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\Gamma(2n+1)(\alpha x)^{n/(1-\alpha)}} \frac{d^{2n}}{dT^{2n}} \left\{\frac{1-\alpha T-(T+1)^{\alpha}}{\alpha T^2}\right\}_{T=0}^{-(2n+1)/2}\right].$$

We have seen that

$$f(x, \alpha) = \int_0^{\infty} \exp\{-y + xy^{\alpha}\}\, dy,$$

the integral being taken along the real axis.

Suppose now that we make the substitution

$$y = (\alpha x)^{1/(1-\alpha)} t.$$

Then the integral becomes

$$\int_0^{\infty} \exp[(\alpha x)^{1/(1-\alpha)} \{t^{\alpha}/\alpha - t\}] (\alpha x)^{1/(1-\alpha)}\, dt,$$

taken along an axis such that $\arg t = -\{\arg x\}/(1-\alpha)$.

If now $|\arg x|/(1-\alpha) < \pi/2$, this integral may be taken along the positive half of the real axis, for the difference between the two integrals will be an integral along an arc of the great circle at infinity which vanishes. If then we put $X = (\alpha x)^{1/(1-\alpha)}$, we have, if $|\arg x| < \pi(1-\alpha)/2$,

$$f(x, \alpha) = X \int_0^{\infty} \exp\{Xt^{\alpha}/\alpha - Xt\}\, dt,$$

the integral being taken along the real axis.

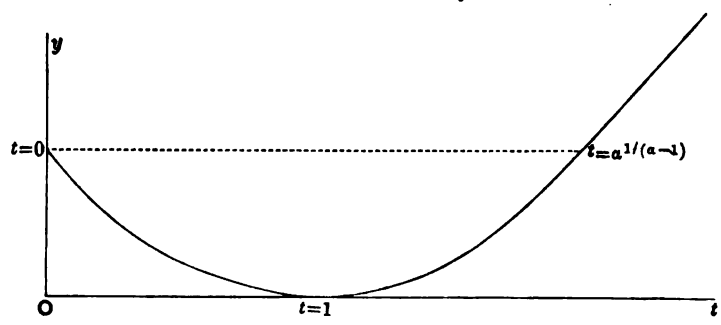
Make now the substitution

$$y = (1-t^{\alpha})/\alpha + t - 1,$$

all the quantities involved being real.

Since $\alpha < 1$, we see that, when t is very large, $y = t$. When $t = 0$, $y = 1/\alpha - 1$. When $t = 1$, $y = 0$, and we have the minimum value of y.

The curve which represents the connection between y and t will be given by the figure.



The line $y = 1/\alpha - 1$, cuts the curve again when $t = \alpha^{1/(\alpha-1)}$.

We obtain upon substitution

$$f(x, \alpha) \exp\{X(1-1/\alpha)\} X^{-1} = \int_0^{\infty} e^{-Xy} \left|\frac{dt}{dy}\right| dy + \int_0^{1/\alpha-1} e^{-Xy} \left|\frac{dt}{dy}\right| dy.$$

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§ 6. We have now to investigate $\left|\frac{dt}{dy}\right|$.

Putting $t = T+1$, we have

$$y = [1 + \alpha T - (T+1)^\alpha]/\alpha,$$

when that value of $(T+1)^\alpha$ is taken which corresponds to a dissection of the T-plane from $T = -1$ to $T = -\infty$.

When T is small the expansion of y in powers of T begins with T^2. Hence we may put

$$T = y^{1/2}\left\{\frac{1+\alpha T-(T+1)^\alpha}{\alpha T^2}\right\}^{-1/2},$$

and, by Lagrange's theorem, for such values of $|y|$ as make the series convergent,

$$T = \sum_{n=1}^{\infty} d_n y^{n/2}/n! \quad \text{.....(1)},$$

where $$d_n = \left[\frac{d^{n-1}}{dT^{n-1}}\left\{\frac{1+\alpha T-(T+1)^\alpha}{\alpha T^2}\right\}^{-n/2}\right]_{T=0}.$$

When $T > 0$ we must take the positive value of $y^{1/2}$, and when $T < 0$, the negative value.

Evidently $$d_1 = \sqrt{2/(1-\alpha)}.$$

To find the actual radius of convergence of the series (1), we must adopt the process of Nekrassoff* or some analogous method.

Let $\psi(z) = \left\{\frac{1+\alpha z-(1+z)^\alpha}{\alpha}\right\}^{-1/2}$, one or other of the two values of the root being definitely taken. Let T be a region of the z plane containing the origin within which the function $z\psi(z)$ is regular. Draw within T a contour S containing the origin but excluding the points for which z is real and < -1: such points lie on the cross-cut which serves to make $(1+z)^\alpha$ one-valued and for which $\psi(z)$ is not defined.

Then, as Cauchy first shewed, the equation

$$y^{1/2} = \sqrt{[1+\alpha T-(T+1)^\alpha]/\alpha},$$

will have one root T, which can be expanded in terms of $\sqrt{y}$ by the series (1), within the contour S, provided on the contour S, $|y^{1/2}\psi(z)| < 1$.

We must find that contour S on which $|\psi(z)|$ has the smallest possible value if we are to obtain the largest value of $|y^{1/2}|$ for which the series (1) is convergent. At each point of the domain T erect a perpendicular to the plane of length $M = |\psi(z)|$. We thus get a surface which is cut by a plane $M = M_0$ in a curve. When M_0 is very large this curve consists of a series of ovals round $z = 0$ and the other zeros of $1+\alpha z-(1+z)^\alpha$. As M decreases these ovals expand until finally the oval round $z = 0$ touches some other curve, let us say, when $M = k$.

* Nekrassoff, *Mathematische Annalen*, Vol. XXXI. pp. 337—358. Cauchy's original development will be found in the *Exercices d'Analyse, etc.*, 1840, T. II. pp. 41—98. For further references see Osgood, *Encyklopädie der Mathematischen Wissenschaften*, Bd II. 2, pp. 44—47.

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When $M \leqslant k$, we see that the corresponding contour S will enclose at least one other root of $1+\alpha z-(1+z)^\alpha=0$ besides $z^2=0$. And therefore $z\psi(z)$ will cease to be regular within S, so that the assumption that S lies within T will not hold.

When $M=k$ there will be a node on the curve $|\psi(z)|=M$, since two ovals touch.

Let $\left\{\dfrac{1}{\psi(z)}\right\}^2=\dfrac{1+\alpha z-(1+z)^\alpha}{\alpha}=u+\iota v$, where $z=x+\iota y$ and u and v are real functions of x and y. Then the ovals correspond to $u^2+v^2=1/M^4$.

At a node of such a curve $\dfrac{\partial u}{\partial x}=0=\dfrac{\partial v}{\partial x}$, and therefore $\dfrac{d}{dz}\left\{\dfrac{1}{\psi(z)}\right\}^2=0$, that is to say

$$(1+z)^{\alpha-1}=1.$$

We therefore have $$z=-1+e^{2r\pi\iota/(\alpha-1)}.$$

And therefore $$1+\alpha z-(1+z)^\alpha=(1-\alpha)\{1-e^{2r\pi\iota/(\alpha-1)}\}.$$

Hence $$k=|\psi(z)|=\left[\frac{2(1-\alpha)\sin r\pi/(1-\alpha)}{\alpha}\right]^{-1/2}.$$

We must exclude the value corresponding to $r=0$, for that indicates the origin at which we have taken a definite value of $z\psi(z)$. Otherwise we take the largest value for k which this expression can assume.

Beyond this we have to ensure that the oval round $z=0$ does not cut the surface in points which correspond to values of z which are real and $\leqslant -1$.

Now, when $z=-1-k$,

$$\psi(z)=\left\{\frac{1-\alpha-k\alpha-k^\alpha e^{\pm\pi\iota\alpha}}{\alpha}\right\}^{-1/2}.$$

When $0<\alpha<1$ and $k\geqslant 0$, this expression is nowhere infinite, since $e^{\pm\pi\iota\alpha}$ is not real.

There is therefore a quantity $\rho_1^{1/2}$ which is greater than zero and such that, when $|y^{1/2}|\leqslant\rho_1^{1/2}$, the expansion (1) is valid.

Hence, when $|y|\leqslant\rho<\rho_1$,

$$\frac{dt}{dy}=\frac{dT}{dy}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{d_n y^{n/2-1}}{(n-1)!},$$

the series being absolutely and uniformly convergent.

And therefore, if $\sqrt{y}$ denote the positive value of the root of y, y being real,

$$\left.\frac{dt}{dy}\right|=\frac{1}{2}\sum_{n=1}^{\infty}\frac{d_n(\sqrt{y})^{n-2}}{(n-1)!}\quad\text{if } T>0,$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}d_n\frac{(-\sqrt{y})^{n-2}}{(n-1)!}\quad\text{if } T<0.$$

§ 7. By the result of § 5 we now have

$$\exp\{X(\alpha-1)/\alpha\}\,X^{-1}f(x,\alpha)=\int_{1/\alpha-1}^{\infty}e^{-Xy}\left|\frac{dt}{dy}\right|dy+\int_0^{1/\alpha-1}e^{-Xy}\sum_{n=0}^{\infty}\frac{d_{2n+1}(\sqrt{y})^{2n-1}}{(2n)!}\,dy,$$

wherein, for values of y such that $\rho<y\leqslant 1/\alpha-1$, we substitute for the series the sum of the functions of which the series is the expansion when $|y|\leqslant\rho$.

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Write, for brevity, $1/\alpha - 1 = \phi$; $Q(y) = \sum_{n=0}^{\infty} \frac{d_{2n+1}}{(2n)!} y^n$, so that $Q(y)$ is defined by an absolutely and uniformly convergent series when $0 \leqslant |y| \leqslant \rho$, and is one-valued and finite when y is real and $\rho < y \leqslant \phi$. Then *I say that*

$$\int_0^{\phi} e^{(\phi-y)X} y^{-1/2} Q(y)\, dy$$

admits the asymptotic expansion

$$e^{\phi X}\left[\sum_{n=0}^{N-1} \frac{d_{2n+1}\,\Gamma(n+1/2)}{(2n)!\, X^{n+1/2}} + J_N\right],$$

where $|J_N X^N|$ *can be made as small as we please by taking* $|X|$ *sufficiently large, provided* $|\arg X| < \pi/2$.

For
$$\int_0^{\phi} e^{(\phi-y)X} y^{-1/2} Q(y)\, dy - \sum_{n=0}^{N-1} \int_0^{\infty} e^{(\phi-y)X} y^{-1/2} \frac{d_{2n+1}}{(2n)!} y^n dy$$
$$= \int_0^{\rho} e^{(\phi-y)X} y^{-1/2} \sum_{n=N}^{\infty} \frac{d_{2n+1}}{(2n)!} y^n dy$$
$$+ \int_{\rho}^{\phi} e^{(\phi-y)X} y^{-1/2} Q(y)\, dy - \sum_{n=0}^{N-1} \int_{\rho}^{\infty} e^{(\phi-y)X} y^{-1/2} \frac{d_{2n+1}}{(2n)!} y^n dy.$$

Therefore
$$|J_N| \leqslant \left| \frac{1}{X^{N+1/2}} \int_0^{\rho X} (\arg X)\, e^{-y} \sum_{n=0}^{\infty} \frac{d_{2n+2N+1}\, y^{n+N-1/2}}{X^n (2n+2N)!}\, dy \right|$$
$$+ \left| e^{-\rho X} \int_0^{\phi-\rho} e^{-yX} (y+\rho)^{-1/2} Q(\rho+y)\, dy \right|$$
$$+ \left| e^{-\rho X} \int_0^{\infty} e^{-yX} \sum_{n=0}^{N-1} \frac{d_{2n+1}}{(2n)!} (y+\rho)^{n-1/2}\, dy \right|,$$

the first integral being taken along an axis whose argument is equal to the argument of X which is $< \pi/2$.

Hence $|J_N X^N|$ can be made as small as we please by taking $|X|$ sufficiently large and $|\arg X| < \pi/2$. The theorem is therefore established.

§ 8. Now $\frac{dt}{dy} = \frac{1}{1 - t^{\alpha-1}}$, and this is finite on the range $1/\alpha - 1$ to ∞, and at infinity it behaves like $-y^{1-\alpha}$.

Hence $e^{(\phi-\epsilon)X} \int_{1/\alpha-1}^{\infty} e^{-Xy} \left| \frac{dt}{dy} \right| dy$, where $\epsilon > 0$, tends to zero as $|X|$ tends to infinity provided $|\arg X| < \pi/2$. Finally therefore if $|\arg x| < (1-\alpha)\pi/2$, we have the asymptotic expansion

$$f(x, \alpha) = X \exp\{X(1/\alpha - 1)\} \left\{ \sum_{n=0}^{N-1} \frac{d_{2n+1}\,\Gamma(n+1/2)}{\Gamma(2n+1)\, X^{n+1/2}} + \frac{J_N}{X^N} \right\},$$

where $|J_N|$ can be made as small as we please by taking $|X|$ sufficiently large, and $X = (\alpha x)^{1/(1-\alpha)}$.

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This result may be written

$$f(x, \alpha) = \exp\{(1-\alpha)\,\alpha^{\alpha/(1-\alpha)}\,x^{1/(1-\alpha)}\}\,(\alpha x)^{1/2(1-\alpha)}$$
$$\times\left[\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\Gamma(2n+1)(\alpha x)^{n/(1-\alpha)}}\frac{d^{2n}}{dT^{2n}}\left\{\frac{1+\alpha T-(T+1)^{\alpha}}{\alpha T^2}\right\}_{T=0}^{-n-1/2}\right].$$

The first term of the series is

$$\{2\pi/(1-\alpha)\}^{1/2}.$$

§ 9. *An interesting deduction from the previous formula may be noticed.*

The author has for a long time speculated as to whether it was possible to construct an elementary integral function which should tend to zero for all values of $|x|$ round infinity except those within a range of values of $\arg x$ as small as we please. being, of course, infinite within the latter range.

Such a function is given by

$$\sum_{n=0}^{\infty}\frac{x^n\,\Gamma(1+n-n/p)}{\Gamma(1+n)},$$

where p is a large positive number.

For if we put $\alpha = 1-1/p$ in the previous formula, we see that, within the range $|\arg x| < \pi/2p$, the function admits an asymptotic expansion whose dominant term is

$$\exp\left\{\frac{1}{p-1}(1-1/p)^p x^p\right\}\left(x\frac{p-1}{p}\right)^{1/2p}(2\pi p)^{1/2} \quad \text{.....(A).}$$

Outside this range the function admits an asymptotic expansion whose dominant term is

$$\Gamma\left(2+\frac{1}{p-1}\right)\Big/(-x)^{1+1/(p-1)} \quad \text{.....(B).}$$

By taking p as large as we please, we may narrow the range $|\arg x| < \pi/2p$ indefinitely.

When p is infinite the function ceases to be integral: it becomes $1/(1-x)$.

In this case the asymptotic value round infinity becomes to the first approximation $-\frac{1}{x}$, a result which accords with the formula (B). The formula (A) becomes infinite and indicates the existence of the finite singularity at the point $x=1$ on the positive direction of the real axis.

The special function $\sum_{n=0}^{\infty}\frac{x^n\,\Gamma(1+n/2)}{\Gamma(1+n)}$.

§ 10. In the preceding general formula we have only obtained the dominant exponential term in the region $|\arg x| < (1-\alpha)\pi/2$, and for all we know to the contrary other exponential terms of lower orders may exist.

In the special case when $\alpha = 1/2$, the corresponding series of Lagrange is finite, and we can shew that no such terms exist: we can also bring out more clearly the discontinuity in the asymptotic value of the function.

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Let
$$f(x) = \sum_{n=0}^{\infty} \frac{x^n \Gamma(1+n/2)}{\Gamma(1+n)}.$$

Then, as before, $f(x) = \int_0^\infty \exp\{-z + xz^{1/2}\}\, dz$, the integral being taken along the real axis from 0 to ∞, and the positive value of $z^{1/2}$ being taken.

The previous transformations in the special case now considered lead to

$$z^{1/2} = y^{1/2} + x/2 \qquad (1),$$

whence
$$-z + xz^{1/2} = -y + x^2/4 \text{ and } \frac{dz}{dy} = 1 + \frac{x}{2y^{1/2}}.$$

We thus have
$$f(x) = \int e^{x^2/4 - y} \left\{1 + \frac{x}{2y^{1/2}}\right\} dy.$$

The transformation (1) is the usual parabolic transformation.

If $x/2 = p + \iota q$ and $y = \xi + \iota\eta$, we have, corresponding to real values of z,
$$\eta^2 = 4q^2(\xi + q^2).$$

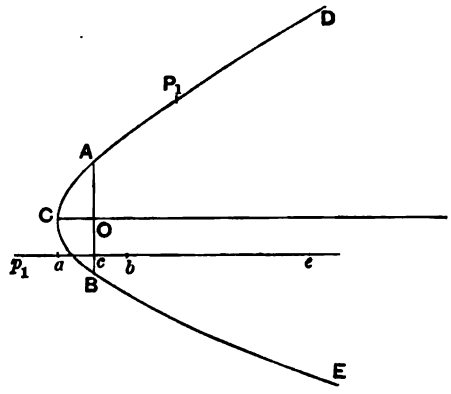
This is the parabola $DACBE$, with its focus at the origin, of the figure.

We will consider three cases

(α) when $0 < \arg x < \pi/4$,

(β) „ $\pi/4 < \arg x < 3\pi/4$,

(γ) „ $3\pi/4 < \arg x < \pi$.



The corresponding cases when $\arg x$ is negative will follow by symmetry.

In case (α) the point P_1, for which $y = x^2/4$, lies between D and A.

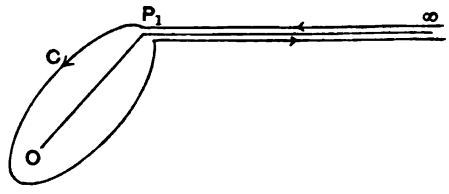
When $z^{1/2}$ describes the real axis from 0 to $+\infty$, y describes the curve P_1ACBE, and $y^{1/2}$ moves from $p_1 (= -x/2)$ through the corresponding points a, c, b, e.

In case (β) the point $P_2 (y = x^2/4)$ lies between A and B; and in case (γ) the point $P_3 (y = x^2/4)$ lies between B and E_∞.

§ 11. Thus, *in case* (α), $y^{1/2}$ is initially $-x/2$ and a circuit $PACBEP_1$ enclosing the origin would bring it back to P_1 with a value $+x/2$.

Hence, by Cauchy's Theorem, in case (α), $f(x)$ is represented by

$$\left[\int_C - \int_{\infty P_1}\right] e^{x^2/4 - y} \left\{1 + \frac{x}{2y^{1/2}}\right\} dy,$$



where C is a contour as in the figure with a broken cross-cut inside to make $y^{1/2}$ one-valued and ∞P_1 is the initial part of this contour.

On the contour C, if
$$y = re^{\iota\theta}\ (0 < \theta \leqslant 2\pi), \text{ we have } y^{1/2} = r^{1/2} e^{\iota(\theta - 2\pi)/2}.$$

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Hence
$$f(x) = \int_C e^{x^2/4-y}\{xy^{-1/2}/2\}\,dy + \int_{P_1\infty} e^{x^2/4-y}\left\{1+\frac{x}{2y^{1/2}}\right\}dy$$
$$= xe^{x^2/4}\int_0^\infty e^{-y}y^{-1/2}\,dy + \int_{x^2/4}^\infty e^{x^2/4-y}\left\{1-\frac{x}{2y^{1/2}}\right\}dy,$$

provided in the second integral $|\arg y^{1/2}| < \pi/4$.

The first integral is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

The second integral may be written
$$\int_0^\infty e^{-y}\{1-(1+4y/x^2)^{-1/2}\}\,dy,$$

that value being assigned to $(1+4y/x^2)^{-1/2}$ which, when $|4y/x^2| < 1$, is represented by the binomial series
$$\sum_{n=0}^\infty (-)^n \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\left(\frac{4y}{x^2}\right)^n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

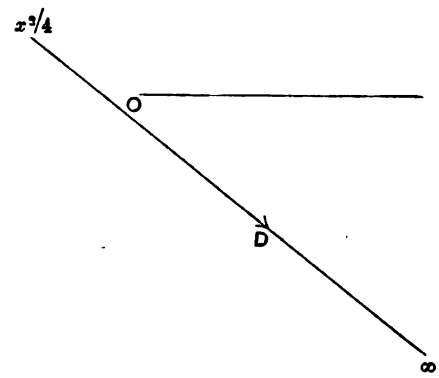
On the line of integration, when $|4y/x^2| \geqslant 1$, the series is divergent but summable. We may therefore substitute the series under the sign of integration to obtain the asymptotic value of the integral.

Therefore, when $|\arg x| < \pi/4$, we have the asymptotic formula
$$f(x) = \sqrt{\pi}\,xe^{x^2/4} + \frac{2}{x^2}\left\{\sum_{n=0}^N \frac{(-)^n\,\Gamma(2n+2)}{\Gamma(n+1)\,x^{2n}} + J_N\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A),$$

where $|J_N x^{2N}|$ tends to zero as $|x|$ tends to infinity.

§ 12. *Consider next cases* (β) *or* (γ), *when* $\pi/2 < \arg x^2/4 < 2\pi$.

By Cauchy's theorem the integral for $f(x)$ taken along the parabolic arcs P_2CBE_∞ or P_2E_∞ can be represented by the same integral taken along a line, from $x^2/4$ to ∞ in the positive half of the y-plane, so drawn as to pass to the left of the origin. This is the line D of the figure.



We therefore have
$$f(x) = \int_0^\infty e^{-y}\{1-(1+4y/x^2)^{-1/2}\}\,dy,$$

taken along a parallel to the former line, that value of $(1+4y/x^2)^{-1/2}$ being taken which when $|4y/x^2| < 1$ is represented by the binomial series (2).

Hence, when $|\arg x| \nless \pi/4$, we have the asymptotic formula
$$f(x) = \frac{2}{x^2}\left\{\sum_{n=0}^N \frac{(-)^n\,\Gamma(2n+2)}{\Gamma(n+1)\,x^{2n}} + J_N\right\}\ldots\ldots\ldots(B),$$

where $|J_N x^{2N}|$ tends to zero as $|x|$ tends to infinity.

The formulae (A) and (B) evidently agree with the previous results. They shew that the large zeros of $f(x)$ have arguments $\pm\pi/4$.

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The function $\phi(x, \theta) = \sum_{n=1}^{\infty} \frac{x^n \Gamma(1+n\theta)}{\Gamma(1+n+n\theta)}$.

§ 13. In the series by which we define the function $\phi(x, \theta)$ we suppose that θ is real and >0. Evidently when $\theta = 0$ the function becomes e^x.

To illustrate the method by which we now proceed to investigate the asymptotic expansion of $\phi(x, \theta)$ *when* $R(x) > 0$, *we will first consider the particular case when* $\theta = 1$.

In this case $$\phi(x, 1) = \sum_{n=1}^{\infty} \frac{x^n \Gamma(1+n)}{\Gamma(1+2n)}.$$

Since $\int_0^1 (1-x)^{n-1} x^{m-1}\, dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, if m and n be both >0, we have

$$\phi(x, 1) = \int_0^1 \sum_{n=1}^{\infty} \frac{x^n}{\Gamma(n)} y^n (1-y)^{n-1}\, dy = \int_0^1 xy \exp\{xy(1-y)\}\, dy.$$

Make now the substitution $z = y(1-y)$.

We see that z vanishes when $y=0$ or $y=1$. It has a maximum value $1/4$ when $y = 1/2$.

If then we put $z = 1/4 - \zeta$, $y = 1/2 + \eta$, we have $\zeta = \eta^2$ and therefore $\eta = \pm \zeta^{1/2}$.

We take the positive sign in this relation when $y > 1/2$, and the negative sign when $y < 1/2$.

Hence $$\phi(x, 1) = \int_0^{1/4} \frac{x}{2}(1/2 + \zeta^{1/2}) \exp\{x(1/4 - \zeta)\}\, \zeta^{-1/2}\, d\zeta$$

$$- \int_{1/4}^0 \frac{x}{2}(1/2 - \zeta^{1/2}) \exp\{x(1/4 - \zeta)\}\, \zeta^{-1/2}\, d\zeta$$

$$= \tfrac{1}{2} x e^{x/4} \int_0^{1/4} e^{-x\zeta} \zeta^{-1/2}\, d\zeta$$

$$= \tfrac{1}{2} e^{x/4} x^{1/2} \int_0^{x/4} e^{-y} y^{-1/2}\, dy.$$

Suppose now that $|\arg x| < \pi$. Draw a line I from $x/4$ to ∞t, where $|\arg t| < \pi/2$, this line not cutting the negative half of the real axis. Draw a line J parallel to this line from the origin.

Then by Cauchy's theorem

$$\phi(x, 1) = \tfrac{1}{2} e^{x/4} x^{1/2} \int_0^{\infty} (J)\, e^{-y} y^{-1/2}\, dy - \tfrac{1}{2} x^{1/2} \int_{x/4}^{\infty} (I)\, e^{x/4 - y} y^{-1/2}\, dy$$

$$= \tfrac{1}{2} e^{x/4} x^{1/2} \Gamma(1/2) - \int_0^{\infty} (J)\, e^{-y} (1 + 4y/x)^{-1/2}\, dy.$$

This formula shews us that, when $|\arg x| < \pi$,

$$|\phi(x, 1) - \tfrac{1}{2} e^{x/4} x^{1/2} \Gamma(1/2)|,$$

is finite for all values of $|x|$, however large.

We can readily obtain a complete asymptotic expansion valid when $|\arg x| < \pi$.

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For, if $|\arg x| < \pi$, the expansion of $(1+4y/x)^{-1/2}$ by the binomial theorem is either (1) convergent or (2) summable and divergent on the line of integration.

Hence, by a general theorem proved in my original memoir*, the integral last written admits the asymptotic expansion

$$\sum_{n=0}^{N} \int_0^\infty e^{-y} \frac{1.3\ldots(2n-1)}{2.4\ldots 2n} \left(-\frac{4y}{x}\right)^n dy + J_N,$$

where $|J_N x^N|$ can be made as small as we please for any value of N by taking $|x|$ sufficiently large.

We thus have, provided $|\arg x| < \pi$, the complete asymptotic formula

$$\phi(x, 1) = \frac{\sqrt{\pi}}{2} e^{x/4} x^{1/2} - \sum_{n=0}^{\infty} 1.3\ldots(2n-1)(-2/x)^n.$$

§ 14. It may be remarked that this formula can be otherwise obtained. *It is a particular case of the asymptotic expansions of generalised hypergeometric functions.*

If $F_\rho(x) = \sum_{n=1}^{\infty} \frac{\Gamma(\rho)}{\Gamma(n+\rho)} x^n$, then, as I propose to shew in a subsequent paper, if $|\arg x| < \pi$, we have the asymptotic equality

$$F_\rho(x) - e^x x^{1-\rho} \Gamma(\rho) = -\sum_{n=1}^{N} \frac{(\rho-1)\ldots(\rho-n)}{x^n} + J_N,$$

where $|J_N x^N|$ tends to zero as $|x|$ tends to infinity.

Now $\phi(x, 1) = F_{1/2}(x/4) - 1$, and we therefore have the expansion just obtained.

We see that $\phi(x, 1)$ satisfies the differential equation

$$4xy'' - (x-2)y' - 1 - y = 0.$$

A second independent solution is, in the notation of generalised hypergeometric functions,

$$\frac{x^{1/2}}{2} {}_1F_1\{-1/2;\ 3/2;\ x/4\} - 1.$$

§ 15. *We now proceed to apply the method used in the foregoing example to the general function*

$$\phi(x, \theta) = \sum_{n=1}^{\infty} \frac{x^n \Gamma(1+n\theta)}{\Gamma(1+n+n\theta)}, \qquad (\theta > 0).$$

As before we have

$$\phi(x, \theta) = x \int_0^1 y^\theta \exp\{xy^\theta(1-y)\}\, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We assume that y^θ is real and equal to $\exp\{\theta \log y\}$.

Make now the substitution

$$z = y^\theta(1-y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Then z vanishes when $y=0$ or $y=1$: it has its maximum value $\theta^\theta/(\theta+1)^{\theta+1}$ when $y = \theta/(\theta+1)$. And z is real for all values of y within the range of integration.

Put now $\qquad z = \theta^\theta/(\theta+1)^{\theta+1} - \zeta; \quad y = \theta/(\theta+1) + \eta.$

* § 5 of the memoir to which reference is made in § 1.

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The relation (1) becomes

$$\theta^\theta/(\theta+1)^{\theta+1}-\zeta=(\theta/(\theta+1)+\eta)^\theta(1/(\theta+1)-\eta),$$

or

$$\zeta=\frac{\theta^{\theta-1}}{2(\theta+1)^{\theta-2}}\eta^2+\frac{\theta^{\theta-2}(\theta-1)}{3(\theta+1)^{\theta-3}}\eta^3+\dots,$$

if

$$\left|\frac{\eta(\theta+1)}{\theta}\right|<1.$$

This relation may be written

$$\eta=\zeta^{1/2}\left\{\frac{\dfrac{\theta^\theta}{(\theta+1)^{\theta+1}}-\left(\dfrac{\theta}{\theta+1}+\eta\right)^\theta\left(\dfrac{1}{\theta+1}-\eta\right)}{\eta^2}\right\}^{-1/2}.$$

Hence, by Lagrange's theorem,

$$\eta=\sum_{n=1}^{\infty}c_n\zeta^{n/2},$$

where

$$c_n=\frac{1}{n!}\left[\frac{d^{n-1}}{d\eta^{n-1}}\left\{\frac{\dfrac{\theta^\theta}{(\theta+1)^{\theta+1}}-\left(\dfrac{\theta}{\theta+1}+\eta\right)^\theta\left(\dfrac{1}{\theta+1}-\eta\right)}{\eta^2}\right\}^{-n/2}\right]_{\eta=0}.$$

This series will be convergent for sufficiently small values of $|\zeta|$; say, when $|\zeta^{1/2}|<\sigma_1$.

Evidently

$$c_1=\sqrt{\frac{2(\theta+1)^{\theta-2}}{\theta^{\theta-1}}},\quad c_2=-\frac{2}{3}\frac{(\theta-1)(\theta+1)^{\theta-1}}{\theta^\theta}.$$

When $\eta>0$, we take the positive value of $\sqrt{\zeta}$ in the series, and when $\eta<0$, we take the negative value.

We have

$$\frac{dy}{dz}=-\frac{d\eta}{d\zeta}=-\sum_{n=1}^{\infty}c_n\frac{n}{2}\zeta^{\frac{n-2}{2}}.$$

On transforming the integral (1) by the substitution just employed we obtain

$$\phi(x,\theta)/x=\int_0^p y_1^\theta\exp\{xz\}\left|\frac{dy_1}{dz}\right|dz+\int_0^p y_2^\theta\exp\{xz\}\left|\frac{dy_2}{dz}\right|dz,$$

where $p\equiv\dfrac{\theta^\theta}{(\theta+1)^{\theta+1}}$, and y_1 is the value of y for which $\eta<0$, y_2 the value for which $\eta>0$.

Now $y_1^\theta=\left(\dfrac{\theta^\theta}{(\theta+1)^{\theta+1}}-\zeta\right)\Big/\left\{\dfrac{1}{\theta+1}-\eta\right\}$, and this when $|\zeta^{1/2}|$ is sufficiently small can be expanded in ascending powers of $|\zeta^{1/2}|$, the positive value of $\sqrt{\zeta}$ being taken when $\eta>0$ and the negative value when $\eta<0$.

We therefore have, if $|\zeta^{1/2}|\leqslant\sigma>0$,

$$y_2^\theta\left|\frac{dy_2}{d\zeta}\right|=\sum_{n=1}^{\infty}d_n\zeta^{n/2-1},$$

where

$$d_1=\left(\frac{\theta}{\theta+1}\right)^\theta\sqrt{\frac{(\theta+1)^{\theta-2}}{2\theta^{\theta-1}}}.$$

The corresponding value of $y_1\left|\dfrac{dy_1}{d\zeta}\right|$ is obtained by changing the sign of $\sqrt{\zeta}$.

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Hence $$\phi(x,\theta)/x = \int_0^p \exp\{x(p-\zeta)\} \sum_{n=1}^{\infty} d_n \zeta^{n/2-1}\, d\zeta$$
$$+ \int_0^p \exp\{x(p-\zeta)\} \sum_{n=1}^{\infty} d_n (-)^n \zeta^{n/2-1}\, d\zeta$$
$$= 2\int_0^p \exp\{x(p-\zeta)\}\, \zeta^{-1/2} \{d_1 + d_2\zeta + d_3\zeta^2 + \ldots\}\, d\zeta.$$

If $p > \sigma$, the series is divergent and must be replaced by the function from which it was obtained.

Now this function, which is the sum of the two values of $\left|\dfrac{1}{\theta+1} + \dfrac{\theta}{(\theta+1)^2\eta}\right|$, is finite within the range of integration.

The integral can therefore, by the theory previously established, be represented by the asymptotic expansion
$$2e^{px} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-x\zeta} d_{2n-1} \zeta^{n-3/2}\, d\zeta$$
when $R(x) > 0$.

If then $R(x) > 0$ we have the asymptotic expansion
$$\phi(x,\theta) = 2x \exp\left\{\frac{\theta^\theta}{(\theta+1)^{\theta+1}} x\right\} \sum_{n=1}^{\infty} \frac{d_{2n-1}\,\Gamma(n-1/2)}{x^{n-1/2}}.$$

The dominant term of this expansion is
$$\exp\left\{\frac{\theta^\theta}{(\theta+1)^{\theta+1}} x\right\} \left\{\frac{2\pi\theta^{\theta+1}x}{(\theta+1)^{\theta+2}}\right\}^{1/2}.$$

The coefficients in the expansion can be written in a symbolical form by the use of Lagrange's series. They can however be deduced from the coefficients in the asymptotic expansion of
$$f(x,\alpha) = \sum_{n=0}^{\infty} \frac{x^n\,\Gamma(1+\alpha n)}{\Gamma(1+n)},$$
by means of the following theorem.

§ 16. *If* $$|\arg x| < 3\pi/2 \quad \textit{and} \quad \theta = \frac{\alpha}{1-\alpha},$$
$$(1-\alpha) f(x^{1-\alpha},\alpha) - \phi(x,\theta) = 1 + \sum_{n=1}^{N} \frac{(-)^n\,\Gamma\{n/(1-\alpha)\}}{\Gamma\{n\alpha/(1-\alpha)\}\, x^n} + J_N,$$
where $|J_N x^N|$ *can be made as small as we please by taking* $|x|$ *sufficiently large.*

Consider the integral
$$-\frac{1}{2\pi\iota} \int \frac{\Gamma(-s)\,\pi x^s\, ds}{\Gamma(-\alpha s)\sin\pi(1-\alpha)s},$$
taken round a contour which embraces the positive half of the real axis and encloses the points
$$s = n \quad \text{and} \quad s = n/(1-\alpha), \qquad n = 0, 1, 2, \ldots \infty.$$

We assume of course that $0 < \alpha < 1$.

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The integral will be equal to

$$-\sum_{n=0}^{\infty} \frac{(-)^{n-1}\pi x^n}{\Gamma(n+1)\Gamma(-\alpha n)\sin\pi(1-\alpha)n} + \sum_{n=0}^{\infty} \frac{(-)^{n-1}\Gamma\left(-\frac{n}{1-\alpha}\right)x^{n/(1-\alpha)}}{\Gamma\left(-\frac{\alpha n}{1-\alpha}\right)(1-\alpha)}$$

$$= \sum_{n=0}^{\infty} \frac{x^n\Gamma(1+\alpha n)}{\Gamma(n+1)} - \frac{1}{1-\alpha}\sum_{n=0}^{\infty} \frac{\Gamma(1+n\theta)}{\Gamma(1+n+n\theta)} x^{n/(1-\alpha)},$$

where $\theta = \alpha/(1-\alpha)$,

$$= f(x, \alpha) - \frac{1}{1-\alpha}\{1 + \phi(x^{1/(1-\alpha)}, \theta)\}.$$

Now, if $|\arg x| < 3\pi(1-\alpha)/2$, the integral will vanish when taken along that part of an arc of a great circle at infinity for which $R(x) > -k$.

It is therefore equal to

$$\sum_{n=1}^{N} (-)^n \frac{\Gamma\left(\frac{n}{1-\alpha}\right)x^{-n/(1-\alpha)}}{\Gamma\left(\frac{\alpha n}{1-\alpha}\right)(1-\alpha)} + J_N,$$

where $|J_N x^{N/(1-\alpha)}|$ tends to zero as $|x|$ tends to infinity.

If then $|\arg x| < 3\pi(1-\alpha)/2$, we have

$$(1-\alpha)f(x, \alpha) - 1 - \phi(x^{1/(1-\alpha)}, \theta) = \sum_{n=1}^{N} \frac{(-)^n\Gamma\{n/(1-\alpha)\}}{\Gamma\{\alpha n/(1-\alpha)\}\, x^{n/(1-\alpha)}} + J_N.$$

Change now x into $x^{1-\alpha}$, and we obtain the given result.

§ 17. *Let us now apply this result to obtain the asymptotic expansion of* $\phi(x, \theta)$, *when* $R(x) > 0$.

We have seen that, if $|\arg x| < (1-\alpha)\pi/2$, we have the asymptotic equality

$$f(x, \alpha) = \exp\{(1-\alpha)\alpha^{\alpha/(1-\alpha)} x^{1/(1-\alpha)}\}(\alpha x)^{1/2(1-\alpha)} \left\{\sum_{n=0}^{\infty} d_n (\alpha x)^{-n/(1-\alpha)}\right\},$$

where
$$d_n = \frac{\Gamma(n+1/2)}{\Gamma(2n+1)}\left[\frac{d^{2n}}{d\tau^{2n}}\left\{\frac{1+\alpha\tau-(1+\tau)^\alpha}{\alpha\tau^2}\right\}^{-\frac{2n+1}{2}}\right]_{\tau=0}.$$

If then $|\arg x| < \pi/2$, we have the asymptotic expansion

$$\phi(x, \theta) = \exp\{\theta^\theta x/(\theta+1)^{\theta+1}\}\left\{\frac{\theta^{1+\theta}x}{(1+\theta)^{\theta+3}}\right\}^{1/2} \sum_{n=0}^{\infty} \frac{d_n}{x^n}\left(\frac{1+\theta}{\theta}\right)^{n(1+\theta)},$$

where
$$d_n = \frac{\Gamma(n+1/2)}{\Gamma(2n+1)}\left[\frac{d^{2n}}{d\tau^{2n}}\left\{\frac{1+\theta+\theta\tau-(1+\theta)(1+\tau)^{\theta/(1+\theta)}}{\theta\tau^2}\right\}^{-\frac{2n+1}{2}}\right]_{\tau=0}.$$

Since $d_0 = \{2\pi(1+\theta)\}^{1/2}$, we see that the dominant term of this expansion is

$$\exp\{\theta^\theta x/(\theta+1)^{\theta+1}\}\left\{\frac{2\pi x\theta^{\theta+1}}{(1+\theta)^{\theta+2}}\right\}^{1/2},$$

and this is in agreement with the former result.

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§ 18. It may be of interest to the reader if we actually obtain the asymptotic expansion of $\phi(x, \theta)$ when $R(x) > 0$ for a particular value of θ. We can then determine the actual radius of convergence of the Lagrange series employed under the sign of integration.

We will take the case when $\theta = 2$, and shew that, if $R(x) > 0$, we have the asymptotic equality

$$\sum_{n=1}^{\infty} \frac{x^n (2n)!}{(3n)!} = -\frac{2}{9} e^{4x/27} \sum_{n=0}^{\infty} \frac{\Gamma(3n - 1/2)}{\Gamma(2n+1)\, x^{n-1/2}}.$$

The function is represented by the integral

$$x \int_0^1 y^\theta \exp\{xy^\theta (1-y)\}\, dy.$$

We put $y = 2/3 + \eta$, and make the substitution

$$4/27 - \zeta = (2/3 + \eta)^2 (1/3 - \eta),$$

or

$$\zeta = \eta^2 + \eta^3.$$

The integral is then equal to

$$xe^{4x/27} \left\{ \int_0^{4/27} e^{-x\zeta} \left| (2/3 + \eta_1)^2 \frac{d\eta_1}{d\zeta} \right| d\zeta + \int_0^{4/27} e^{-x\zeta} \left| (2/3 + \eta_2)^2 \frac{d\eta_2}{d\zeta} \right| d\zeta \right\},$$

where η_1 and η_2 are the values of η which correspond to ζ within the range of integration.

Now, if $\eta = \sqrt{\zeta}\,(1+\eta)^{-1/2}$, we have

$$\tfrac{1}{3}(\tfrac{2}{3} + \eta)^2 = \sum_{n=1}^{\infty} \frac{\zeta^{n/2}}{n!} \left[\frac{d^{n-1}}{dy^{n-1}} (1+y)^{-n/2} (y + 2/3)^2 \right]_{y=0},$$

and therefore

$$(\tfrac{2}{3} + \eta)^2 \frac{d\eta}{d\zeta} = \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{\zeta^{n/2-1}}{\Gamma(n)} \left[\frac{d^{n-1}}{dy^{n-1}} (1+y)^{-n/2} (y + 2/3)^2 \right]_{y=0}.$$

When $\eta > 0$, we take the positive value of $\sqrt{\zeta}$, and when $\eta < 0$, we take the negative value.

On reduction we find that the above formula may be written

$$(\tfrac{2}{3} + \eta)^2 \frac{d\eta}{d\zeta} = \tfrac{1}{9} \sum_{n=1}^{\infty} (-)^n \frac{\Gamma(3n/2 - 2)}{\Gamma(n)\,\Gamma(n/2)} \zeta^{n/2-1}.$$

This series is evidently convergent if

$$\left| \frac{\zeta^{1/2} 3^{3/2}}{2} \right| < 1 \text{ or } |\zeta| < 4/27.$$

Thus the series is convergent over precisely the range of values for ζ for which we integrate.

We now have

$$\sum_{n=1}^{\infty} \frac{x^n (2n)!}{(3n)!} = -\frac{2x}{9} e^{4x/27} \int_0^{4/27} e^{-x\zeta} \sum_{n=1}^{\infty} \frac{\Gamma(3n - 7/2)}{\Gamma(2n-1)\,\Gamma(n - 1/2)} \zeta^{n-3/2} d\zeta.$$

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And, if $R(x) > 0$, this integral is asymptotically equal to

$$-\tfrac{2}{9}e^{4x/27}\sum_{n=1}^{\infty}\frac{\Gamma(3n-7/2)}{\Gamma(2n-1)\,x^{n-3/2}}.$$

We thus have the given result.

The first two terms of this expansion are

$$e^{4x/27}\left\{\frac{4\sqrt{\pi x}}{9}-\frac{\sqrt{\pi x}}{12x}\right\}.$$

It may be readily verified that these terms agree with those obtained from the general symbolical result of § 17.

For, when $\theta = 2$, $\qquad d_0 = \Gamma(\tfrac{1}{2})(\tfrac{1}{6}-\tfrac{2}{27}\tau)^{-3/2}_{\tau=0} = \sqrt{6\pi},$

and $$d_1 = \frac{\Gamma(\frac{3}{2})}{\Gamma(3)}\frac{d^2}{d\tau^2}\left\{\frac{1}{6}-\frac{2}{27}\tau+\frac{7}{6.27}\tau^2\right\}^{-3/2}_{\tau=0} = -\tfrac{1}{3}\sqrt{\pi/6}.$$

§ 19. *The asymptotic expansion of $\phi(x, \theta)$ when $R(x) < 0$.*

We have seen that, if $|\arg x| < 3\pi/2$ and $\theta = \alpha/(1-\alpha)$, we have the asymptotic equality

$$(1-\alpha)f(x^{1-\alpha}, \alpha) - \phi(x, \theta) = 1 + \sum_{n=1}^{\infty}\frac{(-)^n\,\Gamma\{n/(1-\alpha)\}}{\Gamma\{n\alpha/(1-\alpha)\}\,x^n}.$$

Also we deduce at once from the results of § 4, that, when $|\arg(-x)| < \pi/2$, we have asymptotically

$$f(x^{1-\alpha}, \alpha) = -\frac{1}{\alpha}\sum_{n=0}^{\infty}\frac{\Gamma\{(n+1)/\alpha\}}{\Gamma(n+1)(-x)^{(n+1)(1-\alpha)/\alpha}},$$

the principal value of $(-x)^{(1-\alpha)/\alpha}$ being taken.

Therefore, when $R(x) < 0$, we have the asymptotic expansion

$$\phi(x, \theta) = -\frac{1}{\theta}\sum_{n=0}^{\infty}\frac{\Gamma\{(n+1)(\theta+1)/\theta\}}{\Gamma(n+1)(-x)^{(n+1)/\theta}} - 1 - \sum_{n=1}^{\infty}\frac{(-)^n\,\Gamma(n+n\theta)}{\Gamma(n\theta)\,x^n},$$

terms of order less than that of any negative algebraic power of $|x|$ being neglected.

We see then that, when $R(x) < 0$, the function $\phi(x, \theta)$ needs two asymptotic series for its representation. There is, of course, no reason for surprise at this: it shews that $\phi(x, \theta)$ is a function whose behaviour is slightly more complex than the functions previously discussed.

§ 20. *We may obtain the previous result directly by considering the integral*

$$I = \frac{1}{2\pi\iota}\int\frac{x^s\,\Gamma(1+\theta s)}{\Gamma(1+s+\theta s)}\,\frac{\pi}{\sin\pi s}\,ds,$$

round a contour consisting of that part of the great circle at infinity for which $R(s) > -k$, where k is positive, finite and as large as we please, and the line $s = -k$.

If $|\arg x| < \pi/2$, we see that $|Ix^{k'}|$ can be made as small as we please by taking $|x|$ sufficiently large, when $k' < k$.

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By considering the residues of the subject of integration at its poles within the contour we see that the integral is equal to

$$\sum_{n=0}^{\infty} \frac{(-x)^n \Gamma(1+\theta n)}{\Gamma(1+n+\theta n)} + \sum_{n=1} \frac{(-x)^{-n} \Gamma(1-n\theta)}{\Gamma(1-n-\theta n)} + \frac{1}{\theta} \sum_{n=0} \frac{(-)^{n-1} x^{-(n+1)/\theta} \pi}{\Gamma(-n-(n+1)/\theta)\, \Gamma(n+1) \sin(n+1)\pi/\theta}$$

$$= 1 + \phi(-x, \theta) + \sum_{n=1} \frac{\Gamma(n+\theta n)}{\Gamma(\theta n)\, x^n} + \sum_{n=0} \frac{\Gamma\{(n+1)(\theta+1)/\theta\}}{\theta \Gamma(n+1)\, x^{(n+1)/\theta}}.$$

Those terms are to be omitted from the summation for which respectively $n > k$ or $(n+1)/\theta > k$.

We therefore have asymptotically, if $|\arg(-x)| < \pi/2$,

$$\phi(x, \theta) = -1 - \sum_{n=1}^{\infty} \frac{(-)^n \Gamma(n+n\theta)}{\Gamma(n\theta)\, x^n} - \frac{1}{\theta} \sum_{n=0}^{\infty} \frac{\Gamma\{(n+1)(\theta+1)/\theta\}}{\Gamma(n+1)(-x)^{(n+1)/\theta}},$$

which is equivalent to the former result.

§ 21. *The corresponding asymptotic expansions for the general function*

$$_{\alpha}E_{\beta}(x) = \sum_{n=0}^{\infty} \frac{\Gamma(1+\alpha n)}{\Gamma(1+\beta n)} x^n, \quad 0 < \alpha < \beta,$$

may be stated.

(1) If $\beta - \alpha < 2$, and $|\arg(-x)| < \pi \left\{1 - \frac{\beta-\alpha}{2}\right\}$, we have the asymptotic expansion

$$_{\alpha}E_{\beta}(x) = -\sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha n)}{\Gamma(1-\beta n)\, x^n} + \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-)^n}{(-x)^{(n+1)/\alpha}} \frac{\Gamma\{\beta(n+1)/\alpha\}}{\Gamma(n+1)} \frac{\sin\{\pi\beta(n+1)/\alpha\}}{\sin\{\pi(n+1)/\alpha\}},$$

the calculus of limits being employed for such particular values of α and β as give rise to infinite terms in the coefficients.

(2) In the second place put $p = \beta - \alpha$, $q = (\beta-\alpha)^{\beta-\alpha} \alpha^{\alpha}/\beta^{\beta}$, so that $p > 0$, $q > 0$.

Then if $|\arg x| < \pi(\beta-\alpha)/2$, we have the asymptotic expansion

$$_{\alpha}E_{\beta}(x) = e^{(qx)^{1/p}} (qx)^{1/2p} \sum_{r=0}^{\infty} \frac{d_r}{(qx)^{r/p}},$$

the quantities d_r being assignable functions of α and β.

The first two terms of the series are

$$\sqrt{\frac{2\pi\alpha}{p^3\beta}} \left[1 + \frac{p}{12(qx)^{1/p}} \left\{\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{2p}\right\}\right].$$

The first of these expansions is at once given by considering the contour integral

$$\frac{1}{2\pi\iota} \int \frac{\Gamma(1+\alpha s)}{\Gamma(1+\beta s)} \frac{\pi x^s}{\sin \pi s}\, ds.$$

The second may be obtained in two ways. We may employ an extension of the use of Lagrange's series given in the present memoir, or we may make the analysis depend upon the function defined, when $R(s) > 1/2$, by the series of gamma functions

$$\sum_{n=0}^{\infty} \frac{\Gamma(pn-s)\, \Gamma(1+\alpha n)}{\Gamma(1+\beta n)\, q^n}.$$

The latter method, by its analytical power, seems destined to play an important part in the further theory of expansions of more complex functions.

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X. *A class of Integral Equations.* By H. BATEMAN, B.A., Fellow of Trinity College, Cambridge.

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I. *Introduction.*

§ 1. THE subject of Integral Equations bears the same relation to the Integral Calculus as the theory of Differential Equations does to the Differential Calculus, and the equations

$$f(s)=\int_a^b k(s,t)\,\phi(t)\,dt \quad\text{.....(1),}$$

$$f(s)=\phi(s)-\lambda\int_a^b k(s,t)\,\phi(t)\,dt \quad\text{.....(2),}$$

in which ϕ is the unknown function, present themselves just as naturally in calculations arising from physical problems as do the corresponding differential equations

$$\frac{dy}{dx}=f(x), \qquad \frac{dy}{dx}-\lambda y=f(x),$$

but strange to say until quite recently this branch of mathematics was practically neglected*.

Now an integral equation retains many features of a problem which are lost when the description is given by means of differential equations, and the information which can be derived from it is as a rule fuller—the disadvantage is that the calculations are more difficult to make.

For instance if we replace an ordinary linear differential equation of the nth order by an integral equation of the form† (2), we may obtain a solution in the form of a converging series of integrals which will give us full information about the solution at any point in the plane when the value of the function and its first $(n-1)$ derivatives are known at a point which can be joined to the first point by means of a curve with continuous curvature, at every point of which the coefficients of the equation are finite and integrable. We thus obtain the information directly without having to resort to the method of continuation.

* Abel, Liouville and Rouché called attention to the new calculus and considerable progress was made later by Volterra and other Italian writers. A list of references will be found in a paper by the author, *Proc. London Math. Soc.* Ser. 2, Vol. 4, Part 2.

† The necessary transformation is given in the paper by the author just referred to.

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In the present paper I shall be occupied in studying a certain type of integral relation between two functions $h(s, t)$ and $k(s, t)$. The general form of this relation is

$$\int_a^b h(s, x) f(x, t)\, dx = \int_c^d f(s, y)\, k(y, t)\, dy \quad \text{.....(3)},$$

and it implies that the functions $h(s, x)$ and $k(y, t)$ possess a number of common properties which are of importance in the theory of integral equations.

As the subject will probably be new to the reader I shall commence by recapitulating a few of the known results. It must be clearly understood that none of the matter which is given in the remainder of this section is original, but I have thought it necessary to introduce it here in order that the account of my own work which is given in Sections II and III may be intelligible.

§ 2. An integral equation is an immediate generalisation of a system of linear equations and important progress in the theory was at once made when it began to be studied from this point of view*. The linear equations (1) and (2) which are the simplest equations of this kind† correspond to the system of linear equations

$$f_r = \sum_{s=1}^{n} \kappa_{rs} \phi_s \quad (r = 1, \dots m), \qquad f_r = \phi_r - \lambda \sum_{1}^{n} \kappa_{rs} \phi_s \quad (r = 1, \dots n) \quad \text{.....(4)},$$

and have been called integral equations of the first and second kind respectively‡.

The second equation is easier to deal with because the number of linear equations is equal to the number of unknowns and the known results for the system of linear equations apply almost word for word to the integral equation. Fredholm has in fact shown that the function $\phi(t)$ can be uniquely determined except when λ is a root of a certain whole function $\delta(\lambda)$ which corresponds to the determinant

$$\begin{vmatrix} 1 - \lambda\kappa_{11}, & -\lambda\kappa_{12}, & \dots \\ -\lambda\kappa_{21}, & \dots & \dots \\ \dots & \dots & 1 - \lambda\kappa_{nn} \end{vmatrix}.$$

This solution can be expressed in the form

$$\phi(s) = f(s) + \lambda \int_a^b K(s, t) f(t)\, dt \quad \text{.....(5)},$$

an equation which is similar in form to the original one, a fact from which many deductions can be made.

The function $K(s, t)$ is called the solving function of the integral equation; it is a uniform function of λ and can be expressed in the form

$$K(s, t) = -\frac{\Delta(\lambda;\, s, t)}{\delta(\lambda)} \quad \text{.....(6)},$$

* The new departure was made by Fredholm in 1900 ("Sur une nouvelle méthode pour la résolution du problème de Dirichlet." *Oefversigt af kongl. vet. akad. Förh.* Stockholm, 1900).

† The equations in which the upper limit b is replaced by the variable s may be included in the ones given above by defining $k(s, t)$ to be zero if $t > s$, a device due to Fredholm; the equations of this type have been studied by Volterra.

‡ Hilbert, *Gött. Nachr.* 1904, Heft I.

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where $\Delta(\lambda;\, s, t)$ and $\delta(\lambda)$ are whole functions of λ, their formal expressions being*

$$\left.\begin{aligned}\Delta(\lambda;\, s, t) &= -k(s, t) + \lambda\Delta_1(s, t) - \lambda^2\Delta_2(s, t) + \dots \\ \delta(\lambda) &= 1 - \delta_1\lambda + \delta_2\lambda^2 - \dots\end{aligned}\right\} \quad \dots\dots\dots\dots(7),$$

where $\Delta_h(s, t) = \dfrac{1}{h!}\displaystyle\int_a^b \dots \int_a^b \begin{vmatrix} k(s, t), & k(s, s_1) & k(s, s_h) \\ k(s_h, t), & \dots & k(s_h, s_h)\end{vmatrix} ds_1 \dots ds_h, \quad \delta_h = \dfrac{1}{h}\displaystyle\int_a^b \Delta_{h-1}(s, s)\, ds.$

If $\delta(\lambda) = 0$ the homogeneous equation

$$0 = \psi(s) - \lambda\int_a^b k(s, t)\,\psi(t)\, dt \quad \dots\dots\dots\dots(8),$$

will possess a solution $\psi(t)$ different from zero and conversely if a solution $\psi(t)$ exists we must have $\delta(\lambda) = 0$. Further the equation (2) will be impossible unless $f(s)$ satisfies one or more relations of the form

$$\int_a^b f(s)\,\chi(s)\, ds = 0.$$

We have here a phenomenon similar to that of resonance, for if the function $f(s)$ does not satisfy these conditions the solution of equation (2) will become very large as λ approaches a root of $\delta(\lambda) = 0$.

It has been remarked by Fredholm that if $k(t, s)$ is written instead of $k(s, t)$ the new solving function will be $K(t, s)$ and the value of $\delta(\lambda)$ will be unaltered, this shows that the equations

$$\left.\begin{aligned}0 &= \psi(s) - \lambda\int_a^b k(s, t)\,\psi(t)\, dt \\ 0 &= \chi(t) - \lambda\int_a^b k(s, t)\,\chi(s)\, ds\end{aligned}\right\} \quad \dots\dots\dots\dots(9),$$

can be satisfied for the same values of λ.

If λ_n is a p-fold root of $\delta(\lambda) = 0$, it can be shown that for values of λ very nearly equal to λ_r

$$K(s, t) = \frac{P(s, t)}{\lambda - \lambda_n} + F(s, t) \quad \dots\dots\dots\dots(10),$$

where $F(s, t)$ is finite for $\lambda = \lambda_n$ and $P(s, t)$ has the form

$$P(s, t) = \psi_1(s, \lambda_n)\,\chi_1(t, \lambda_n) + \dots \psi_p(s, \lambda_n)\,\chi_p(t, \lambda_n).$$

The functions ψ and χ are linearly independent solutions of the equations (9) for this value of λ and possess the important integral properties†

$$\int_a^b \psi_i(s, \lambda_m)\,\chi_k(s, \lambda_n)\, ds = \left\{\begin{matrix} 0 & m \neq n \\ 0 & m = n, \quad i \neq k \\ 1 & m = n, \quad i = k\end{matrix}\right\} \quad \dots\dots\dots\dots(11).$$

Also, in some cases the functions $k(s, t)$ and $K(s, t)$ can be represented by the series

$$\left.\begin{aligned}k(s, t) &= \sum_{p, n} \frac{\psi_p(s, \lambda_n)\,\chi_p(t, \lambda_n)}{\lambda_n} \\ K(s, t) &= \sum \frac{\psi_p(s, \lambda_n)\,\chi_p(t, \lambda_n)}{\lambda_n - \lambda}\end{aligned}\right\} \quad \dots\dots\dots\dots(12),$$

which are very useful for suggesting properties of the function $K(s, t)$.

* I am following the notation given by Hilbert in *Gött. Nachr.* The functions $k(s, t)$ and $f(s)$ are supposed to be finite and integrable for the given range of values of s and t.

† See Fredholm, *Acta Math.* 1903.

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We shall call the functions $\psi_r(s, \lambda_n)$ the fundamental functions* and the equation $\delta(\lambda) = 0$ the determinantal equation of the integral equation (2).

Many of the ideas that occur in the theory may be traced back to two papers by Poincaré†, where practically all the properties of the fundamental functions are predicted. The actual investigations have been carried out successfully by Hilbert who has established several important expansion theorems for the functions $\psi_r(s)$.

§ 3. An important advance made by Hilbert is the connection of the integral equation with the double integral

$$\int_a^b \int_a^b k(s, t)\, x(s)\, x(t)\, ds\, dt \qquad (13),$$

corresponding to the quadratic form $\Sigma\, k_{st} x_s x_t$. The function $k(s, t)$ is here supposed to be symmetrical in s and t but this restriction is not of very great consequence; the salient point is that many of the properties of the integral equation are suggested by those of the quadratic form. The determinantal equation $\delta(\lambda) = 0$ corresponds to the discriminant of the quadratic form and Sylvester's theorem that all the roots of the discriminant are real remains true, finally the resolution of the quadratic form into the sum of a number of squares suggests the general expansion theorem from which so many other expansion theorems are deduced.

When the double integral is positive for every function $x(s)$ the fundamental functions $\psi_1(s)$, $\psi_2(s)$ ... may be obtained as the values of $x(s)$ which make the double integral a maximum or minimum‡ subject to the condition

$$\int_a^b [x(s)]^2\, ds = 1,$$

the theory being exactly analogous to that developed by Poincaré in the study of the problem of Dirichlet.

The applications of the theory of the integral equation (2) are many and varied, at present the most important applications are to problems depending on finding solutions of linear differential equations to satisfy given boundary conditions.

§ 4. The theory of the integral equation

$$f(s) = \int_a^b k(s, t)\, \phi(t)\, dt,$$

is not so simple as that of equation (2) because there is no definite relation between the number of corresponding linear equations and the number of unknown quantities. In general in order that the solution may be possible it is necessary for the function $f(s)$ to satisfy all the linear equations in s that are satisfied by the function $k(s, t)$.

* In Germany they are called the Eigenfunktionen, the function $\kappa(s, t)$ is called the Kern and the quantities λ_n the Eigenwerte of the integral equation.

† "La méthode de Neumann et le problème de Dirichlet." *Acta Math.* Vol. xx., p. 120, 1897. "Sur les équations de la physique mathématique." *Pal. Rend.* t. VIII. (1894).

‡ The function $\psi_1(s)$ makes the double integral a maximum, when however the restriction $\int_a^b x(s)\, \psi_1(s)\, ds = 0$ is introduced the next function $\psi_2(s)$ is obtained. The variational problem is ascribed by Hilbert to Gauss.

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For instance if

$$\int_c^d \chi(s)\, k(s, t)\, dt = 0,$$

we should expect to have

$$\int_c^d f(s)\, \chi(s)\, ds = 0.$$

A complete determination of the necessary and sufficient conditions to be satisfied by the function $f(s)$ will be a matter of very great difficulty. A method of obtaining the function $\phi(t)$ has been suggested by the author*, but its field of application has not yet been fully determined.

It is important to distinguish two cases, first when a solution of the equation

$$0 = \int_a^b k(s, t)\, \phi(t)\, dt$$

exists, and secondly when no solution exists which is different from zero; in the first case the equation will be said to be special and in the second case general.

II. *A certain functional relation.*

§ 5. The study of a particular differential equation can often be facilitated by means of an appropriate transformation, accordingly it is natural to ask whether transformations can be profitably employed in the study of integral equations.

The type of transformation which we shall consider here is of a peculiar nature as it can only be applied to a certain class of functions determined by the generating function of the transformation. The characteristic relation on which the theory depends is of the form

$$\int_a^b h(s, x) f(x, t)\, dx = \int_c^d f(s, y)\, k(y, t)\, dy \quad \text{.....(1)},$$

the function $f(s, t)$ and the constants a, b, c, d are the elements of a transformation which associates a function $h(s, x)$ with another function $k(y, t)$.

The properties of the function $k(y, t)$ will depend partly on those of $h(s, x)$ and partly on those of $f(x, t)$; we now inquire what are the properties that depend only on those of $h(s, x)$.

§ 6. Let us suppose that the function $f(s, t)$ is such that no solutions (other than zero) of the equations

$$0 = \int_a^b a(x) f(x, t)\, dx, \quad 0 = \int_c^d f(s, y)\, b(y)\, dy,$$

exist, then we can prove that the quantities λ for which the homogeneous integral equations

$$0 = \phi(x) - \lambda \int_a^b h(x, t)\, \phi(t)\, dt,$$

$$0 = \chi(x) - \lambda \int_c^d k(x, t)\, \chi(t)\, dt,$$

can be satisfied are the same.

* *Proc. London Math. Soc.* Ser. 2, Vol. 4, Part 2.

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Suppose for instance that λ_n is a quantity for which a function $\chi_n(x)$ can be found such that

$$\chi_n(x) = \lambda_n \int_c^d k(x, t)\, \chi_n(t)\, dt,$$

we form the function

$$\theta_n(s) = \int_c^d f(s, x)\, \chi_n(x)\, dx \quad \text{.....(2).}$$

Then

$$\theta_n(s) = \lambda_n \int_c^d \int_c^d f(s, x)\, k(x, t)\, \chi_n(t)\, dx dt$$

$$= \lambda_n \int_a^b \int_c^d h(s, x)\, f(x; t)\, \chi_n(t)\, dx dt,$$

on account of relation (1), hence

$$\theta_n(s) = \lambda_n \int_a^b h(s, x)\, \theta_n(x)\, dx.$$

Now $\theta_n(s)$ cannot be identically zero for the equation

$$0 = \int_c^d f(s, x)\, \chi_n(x)\, dx$$

is impossible by hypothesis, hence a function $\theta_n(s)$ can be found to satisfy the homogeneous equation for $\lambda = \lambda_n$.

Moreover, if there are p linearly independent functions χ_n corresponding to this value of λ the formula (2) will give us p linearly independent functions θ_n, for a linear relation between the functions θ_n of the form $\Sigma a_n \theta_n = 0$ would imply the existence of a relation of the form

$$0 = \int_c^d f(s, x)\, \Sigma a_n \chi_n(x)\, dx,$$

and this is impossible by hypothesis.

Conversely, if λ_n is a quantity such that the equation

$$\theta_n(s) = \lambda_n \int_a^b h(s, x)\, \theta_n(x)\, dx$$

possesses a solution, then we know that for this value of λ a function $\phi_n(t)$ also exists for which

$$\phi_n(t) = \lambda_n \int_a^b h(s, t)\, \phi_n(s)\, ds.$$

Now let

$$\psi_n(x) = \int_a^b f(s, x)\, \phi_n(s)\, ds \quad \text{.....(3),}$$

then

$$\psi_n(x) = \lambda_n \int_a^b \int_a^b f(s, x)\, h(t, s)\, \phi_n(t)\, dt ds$$

$$= \lambda_n \int_c^d \int_a^b k(s, x) f(t, s)\, \phi_n(t)\, dt ds$$

$$= \lambda_n \int_c^d k(s, x)\, \psi_n(s)\, ds.$$

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And since the determinant $\delta(\lambda)$ is the same for $k(s, t)$ as for $k(t, s)$ we conclude that a function $\chi_n(x)$ exists for which

$$\chi_n(s) = \lambda_n \int_c^d k(s, x)\, \chi_n(x)\, dx.$$

We can show as before that if there are p linearly independent functions ϕ_n, the formula (3) will give us p linearly independent functions ψ_n. Now the number p is equal to the multiplicity of λ_n considered as a root of the determinant $\delta_1(\lambda)$ of the "h" equation, hence the root λ_n occurs to the same multiplicity in each equation and we can enunciate the following theorem.

Theorem I. If the function $f(s, t)$ is such that the equations

$$\int_c^d dt f(s, t)\, b(t) = 0 \quad \text{and} \quad \int_a^b f(s, t)\, a(s)\, ds = 0,$$

do not possess solutions, and if two functions $h(s, t)$ and $k(s, t)$ are connected by the relation

$$\int_a^b h(s, x) f(x, t)\, dx = \int_c^d f(s, x)\, k(x, t)\, dx,$$

then the roots of the determinantal equations for the integral equations

$$0 = \phi(x) - \lambda \int_a^b h(s, x)\, \phi(s)\, ds, \quad 0 = \chi(s) - \lambda \int_c^d k(s, x)\, \chi(x)\, dx,$$

are the same and occur to the same degree of multiplicity.

§ 7. The determinants will in general differ by an exponential factor, but in many cases they are the same; for instance if

$$h(s, x) = \int_c^d f(s, y)\, g(y, x)\, dy, \quad k(x, t) = \int_a^b g(x, y) f(y, t)\, dy,$$

the equation (1) is satisfied and we can easily verify that the quantities $\delta(\lambda)$ are the same. To do this, however, it is more convenient to use a formula for $\log \delta(\lambda)$ instead of that for $\delta(\lambda)$, this formula is*

$$-\log \delta(\lambda) = a_1\lambda + \tfrac{1}{2} a_2 \lambda^2 + \tfrac{1}{3} a_3 \lambda^3 + \dots,$$

where
$$a_n = \int_a^b \dots \int_a^b k(s_1, s_2)\, k(s_2, s_3) \dots k(s_n, s_1)\, ds_1 \dots ds_n.$$

Thus for the "h" equation we have

$$a_n = \int_a^b \int_c^d \dots \int_a^b \int_c^d f(s_1, t_1)\, g(t_1, s_2) f(s_2, t_2)\, g(t_2, s_3) \dots g(t_n, s_1)\, ds_1 \dots ds_n,\, dt_1 \dots dt_n,$$

and from the cyclical arrangement of the variables $s_1 \dots s_n$, $t_1 \dots t_n$, we see that the quantity a_n for the "k" equation will be the same.

It must not be thought that the above formulae will give all the functions $h(s, x)$ and $k(x, t)$ that are connected by equation (1) for this is not the case, the functions

* Fredholm, *Acta Math.* 1903.

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$h(s, x) = f(s, x)$, $k(x, t) = f(x, t)$ providing an exception, for it is not in general possible to express the function $f(s, x)$ by means of a definite integral of the form

$$f(s, x) = \int_c^d f(s, y)\, g(y, x)\, dy.$$

§ 8. A particular case of some interest occurs when the limits (a, b) and (c, d) are the same and the function $f(s, t)$ is equal to one when s is less than t and equal to nothing when s is greater than t. The relation (1) then takes the form

$$\int_a^t h(s, x)\, dx = \int_s^b k(x, t)\, dx.$$

It is clear that we must have $h(b, x) \equiv 0$, and so we may write

$$h(s, x) = \frac{d}{dx}[\phi(s, x) - \phi(b, x)],$$

and this gives $$\phi(s, t) - \phi(b, t) - \phi(s, a) + \phi(a, b) = \int_s^b k(x, t)\, dx.$$

Differentiating with regard to s we obtain

$$k(s, t) = \frac{d}{ds}[\phi(s, a) - \phi(s, t)].$$

It should be noticed however that this case is really included in the last, for

$$h(s, x) = -\int_s^b \frac{\partial^2}{\partial x \partial y} \phi(y, x) \,.\, dy, \qquad k(s, t) = -\int_a^t \frac{\partial^2}{\partial s \partial x} \phi(s, x) \,.\, dx,$$

and so we can verify as before that the determinants are the same.

§ 9. If the function $f(s, t)$ does not satisfy the two conditions laid down we cannot assert that all the roots of the two determinantal equations are the same, because the functions $\theta_n(s)$ and $\psi_n(x)$ may be identically zero and then the above proof breaks down. We can however say that some of the roots are the same except in the extreme case when both sides of equation (1) are identically zero.

It should be noticed that if a function $\omega(t)$ does exist for which

$$\int_c^d f(s, t)\, \omega(t)\, dt = 0,$$

then the relation (1) will still be satisfied if we replace $k(x, t)$ by $k(x, t) + \omega(x) F(t)$. Also if we multiply equation (1) by $\omega(t)$ and integrate between c and d we shall obtain

$$\int_c^d \int_c^d f(s, x)\, k(x, t)\, \omega(t)\, dt = 0.$$

The function $\alpha(x) = \int_c^d k(x, t)\, \omega(t)\, dt$ is therefore either identically zero or a multiple of $\omega(x)$; unless another function $\alpha(x)$ independent of $\omega(x)$ exists for which

$$\int_c^d f(s, x)\, \alpha(x)\, dx = 0.$$

In general the first alternative is true and so we see that the function $k(x, t)$ in general satisfies the same linear relations in t as the function $f(x, t)$.

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The fact that the equation (1) is satisfied by the two functions $k(x,t)$ and $k(x,t)+\omega(x)F(t)$ for the same function h suggests that there is some relation between the determinantal equations for these functions; accordingly we proceed to calculate them.

Let $$\int_c^d k(x,t)\,\omega(t)\,dt = \mu\omega(x), \qquad \int_c^d \omega(x)\,F(x)\,dx = c.$$

Then $$a_1 = \int_c^d k(x,x)\,dx, \qquad A_1 = \int_c^d \{k(x,x)+\omega(x)\,F(x)\}\,dx = a_1 + c,$$

$$a_2 = \int_c^d\int_c^d k(x,t)\,k(t,x)\,dxdt, \qquad A_2 = \int_c^d\int_c^d \{k(x,t)+\omega(x)\,F(t)\}\,\{k(t,x)+\omega(t)\,F(x)\}\,dxdt$$
$$= a_2 + 2c\mu + c^2,$$

and $$A_n = a_n + (\mu+c)^n - \mu^n.$$

Therefore $$-\log\Delta(\lambda) = \Sigma\frac{A_n}{n}\lambda^n = \Sigma\frac{a_n+(\mu+c)^n-\mu^n}{n}\lambda^n$$
$$= -\log\delta(\lambda) - \log\{1-\lambda(\mu+c)\} + \log\{1-\lambda\mu\},$$

or $$\Delta(\lambda) = \delta(\lambda)\frac{1-\lambda(\mu+c)}{1-\lambda\mu}.$$

The determinant $\Delta(\lambda)$ thus has the root $\dfrac{1}{\mu+c}$ in place of the root $\dfrac{1}{\mu}$ possessed by $\delta(\lambda)$.

§ 10. We shall now indicate another consequence of equation (1).

Theorem II. If $a(s)$ and $b(t)$ are two functions connected by the relation

$$a(s) = \int_c^d f(s,t)\,b(t)\,dt,$$

the functions $$A(s) = \int_a^b h(s,x)\,a(x)\,dx,$$

$$B(t) = \int_c^d k(t,x)\,b(x)\,dx,$$

are connected by a similar relation.

The proof is very simple, for if we substitute for $a(x)$ in the expression for A, we obtain

$$A(s) = \int_a^b\int_c^d h(s,x)\,f(x,t)\,b(t)\,dxdt$$
$$= \int_c^d\int_c^d f(s,x)\,k(x,t)\,b(t)\,dxdt$$
$$= \int_c^d f(s,x)\,B(x)\,dx.$$

If two functions $h(s,x)$ and $k(x,t)$ have been determined and the solution of the integral equation for a particular function $a(s)$ is known, then we may find the solutions for the system of functions obtained by repeating the operation

$$A(s) = \int_a^b h(s,x)\,a(x)\,dx,$$

any number of times.

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Again, suppose that we are required to solve the integral equation

$$A(s) = \int_a^b h(s, x)\, a(x)\, dx,$$

the process may sometimes be simplified by first determining a function B such that

$$A(s) = \int_c^d f(s, x)\, B(x)\, dx,$$

then obtaining the function b from the equation

$$B(t) = \int_c^d k(t, x)\, b(x)\, dx,$$

and finally calculating a by means of the formula

$$a(s) = \int_c^d f(s, x)\, b(x)\, dx.$$

§ 11. *Theorem III.* If $h(s, x)$, $k(x, t)$ is one pair of functions connected by the relation

$$\int_a^b h(s, x) f(x, t)\, dx = \int_c^d f(s, x)\, k(x, t)\, dx,$$

then

$$h_1(s, t) = \int_c^d f(s, x)\, h(x, t)\, dx,$$

$$k_1(s, t) = \int_c^d f(s, x)\, k(x, t)\, dx,$$

is another pair.

For

$$\int_a^b h_1(s, t) f(t, r)\, dt = \int_a^b \int_c^d f(s, x)\, h(x, t) f(t, r)\, dt\, dx$$

$$= \int_c^d \int_c^d f(s, x) f(x, t)\, k(t, r)\, dt\, dx$$

$$= \int_c^d f(s, x)\, k_1(x, r)\, dx.$$

§ 12. *Theorem IV.* If $H(s, x)$ and $K(x, t)$ are the solving functions of the integral equations

$$f(s) = \phi(s) - \lambda \int_a^b h(s, x)\, \phi(x)\, dx,$$

$$\psi(s) = \chi(s) - \lambda \int_c^d k(s, x)\, \chi(x)\, dx,$$

then $H(s, x)$ and $K(x, t)$ are also connected by the relation

$$\int_a^b H(s, x) f(x, t)\, dx = \int_c^d f(s, x)\, K(x, t)\, dx.$$

To prove this we recall the fact that $H(s, t)$ is the solution of the integral equation corresponding to $f(s) = h(s, t)$*, so that we have the relation

$$h(s, t) = H(s, t) - \lambda \int_a^b h(s, x)\, H(x, t)\, dx.$$

* Fredholm, *Acta Math.* 1903.

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Multiplying by $f(t, r)$ and integrating between a and b, we have

$$\int_a^b h(s, t) f(t, r)\, dt = \int_a^b H(s, t) f(t, r)\, dt - \lambda \int_a^b \int_a^b h(s, x) H(x, t) f(t, r)\, dx dt \quad \text{.....(A).}$$

Again, since $K(x, t)$ is also the solving function of the equation

$$a(t) = b(t) - \lambda \int_c^d k(t, x) b(x)\, dx,$$

we have
$$k(x, r) = K(x, r) - \lambda \int_c^d k(x, t) K(t, r)\, dt.$$

Multiplying by $f(s, x)$ and integrating between c and d, we have

$$\int_c^d f(s, x) k(x, r)\, dx = \int_c^d f(s, x) K(x, r)\, dx - \lambda \int_c^d \int_c^d f(s, x) k(x, t) K(t, r)\, dt dx,$$

but on account of equation (1) this may be written

$$\int_a^b h(s, t) f(t, r)\, dt = \int_c^d f(s, x) K(x, r)\, dx - \lambda \int_a^b \int_c^d h(s, x) f(x, t) K(t, r)\, dt dx \quad \text{...(B).}$$

Now the solution of the integral equation

$$\int_a^b h(s, t) f(t, r)\, dt = \phi(s) - \lambda \int_a^b h(s, x) \phi(x)\, dx,$$

is unique, hence we conclude on comparing (A) and (B) that

$$\int_a^b H(s, t) f(t, r)\, dt = \int_c^d f(s, x) K(x, r)\, dx.$$

This theorem completes our knowledge of the relations between the functions h and k, it shows us that when two functions h and k correspond to one another in the given transformation the whole system of solving functions for the different values of λ also correspond to one another. We may sometimes use this fact to calculate the function $K(x, r)$ when the corresponding function $H(s, t)$ is known.

§ 13. As an example* we shall consider the equation

$$\int_0^\xi h(s, x)\, dx = \int_s^1 k(x, \xi)\, dx,$$

where
$$k(x, \xi) = \begin{matrix} (1-\xi)\, x, & x \leqq \xi \\ (1-x)\, \xi, & x \geqq \xi \end{matrix}.$$

When $s > \xi$, we have $\displaystyle\int_0^\xi h(s, x)\, dx = \int_s^1 (1-x)\, \xi dx = \frac{(1-s)^2}{2} \xi,$

so that
$$h(s, \xi) = \frac{(1-s)^2}{2} \text{ for } s > \xi.$$

When $s < \xi$, we have

$$\int_0^\xi h(s, x)\, dx = \int_s^\xi (1-\xi) \times dx + \int_\xi^1 (1-x)\, \xi dx$$
$$= \frac{1-\xi}{2} (\xi^2 - s^2) + \frac{\xi (1-\xi)^2}{2} = \frac{(1-\xi)(\xi - s^2)}{2},$$

hence
$$h(s, \xi) = \frac{(1-s)^2}{2} + s - \xi \text{ for } s \leqslant \xi.$$

* I am indebted to the referee for the suggestion that examples of the theorems should be given. The actual construction of the examples has proved instructive and led to new results.

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Now the solving function for the integral equation

$$f(x) = \phi(x) - \lambda \int_0^1 k(x, \xi)\, \phi(\xi)\, d\xi$$

is known to be*

$$K(x, \xi) = \frac{\sin\{\sqrt{\lambda}(1-\xi)\}\sin(x\sqrt{\lambda})}{\sqrt{\lambda}\sin\sqrt{\lambda}} \qquad x \leqq \xi$$

$$= \frac{\sin\{\sqrt{\lambda}(1-x)\}\sin(\xi\sqrt{\lambda})}{\sqrt{\lambda}\sin\sqrt{\lambda}} \qquad x \geqq \xi.$$

Hence, when $s > \xi$, we have

$$\int_0^\xi H(s, x)\, dx = \int_s^1 \frac{\sin\{\sqrt{\lambda}(1-x)\}\sin(\xi\sqrt{\lambda})}{\sqrt{\lambda}\sin\sqrt{\lambda}}\, dx,$$

which gives
$$H(s, \xi) = \frac{[1 - \cos\{\sqrt{\lambda}(1-s)\}]\cos(\xi\sqrt{\lambda})}{\sqrt{\lambda}\sin\sqrt{\lambda}},$$

and when $s < \xi$, we have

$$\int_0^\xi H(s, x)\, dx = \int_s^\xi \frac{\sin\{\sqrt{\lambda}(1-\xi)\}\sin(x\sqrt{\lambda})}{\sqrt{\lambda}\sin\sqrt{\lambda}}\, dx + \int_\xi^1 \frac{\sin\{\sqrt{\lambda}(1-x)\}\sin(\xi\sqrt{\lambda})}{\sqrt{\lambda}\sin\sqrt{\lambda}}\, dx$$

$$= \frac{\sin\{\sqrt{\lambda}(1-\xi)\}(\cos s\sqrt{\lambda} - \cos\xi\sqrt{\lambda})}{\lambda\sin\sqrt{\lambda}} + \frac{\{1 - \cos\sqrt{\lambda}(1-\xi)\}\sin\xi\sqrt{\lambda}}{\lambda\sin\sqrt{\lambda}}$$

$$= \frac{\sin\sqrt{\lambda}(1-\xi)\cos s\sqrt{\lambda}}{\lambda\sin\sqrt{\lambda}} + \frac{\sin\xi\sqrt{\lambda}}{\lambda\sin\sqrt{\lambda}} - \frac{1}{\lambda}.$$

Therefore
$$H(x, \xi) = \frac{\cos\xi\sqrt{\lambda}}{\sqrt{\lambda}\sin\sqrt{\lambda}} - \frac{\cos s\sqrt{\lambda}\cos\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}\sin\sqrt{\lambda}}.$$

Hence we conclude that the solving function of the integral equation

$$f(s) = \phi(s) - \lambda \int_0^1 h(s, t)\, \phi(t)\, dt,$$

in which
$$h(s, t) = \frac{(1-s)^2}{2} \text{ if } s \geqslant t$$

$$= \frac{(1-s)^2}{2} + s - t \text{ if } s \leqslant t,$$

is given by
$$H(s, t) = \frac{\{1 - \cos(1-s)\sqrt{\lambda}\}\cos t\sqrt{\lambda}}{\sqrt{\lambda}\sin\sqrt{\lambda}} \qquad s \geqslant t$$

$$= \frac{\cos t\sqrt{\lambda} - \cos s\sqrt{\lambda}\cos(1-t)\sqrt{\lambda}}{\sqrt{\lambda}\sin\sqrt{\lambda}} \qquad s \leqslant t.$$

The chief interest in this example lies in the fact that $k(s, t)$ is the Green's function† for the differential equation $\frac{d^2u}{dx^2} = 0$, corresponding to the boundary conditions $u = 0$ at

* Hilbert, *Gött. Nachr.* 1904, Heft 3.

† Hilbert, *loc. cit.*

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$x=0$ and $x=1$. The characteristic property of such a function is that the solution of $\frac{d^2u}{dx^2}=f(x)$, that satisfies the given boundary conditions is given by

$$u=-\int_0^1 k(x,t)f(t)\,dt,$$

and the function $K(s,t)$ is the corresponding Green's function for the equation $\frac{d^2u}{dx^2}+\lambda u=0$.

Now the function $h(s,t)$ on the other hand is also a Green's function for the differential equation $\frac{d^2u}{dx^2}=0$, for if the function $f(t)$ satisfies the equation $\int_0^1 f(t)\,dt=0$, the solution of $\frac{d^2u}{dx^2}+f(x)=0$, which is such that u and $\frac{\partial u}{\partial x}$ both vanish at $x=1$, is given by

$$u=\int_0^1 h(x,s)f(s)\,ds.$$

Thus we have a method of passing from the Green's function for one set of boundary conditions to that for another.

The corresponding function for the equation $\frac{d^2u}{dx^2}+\lambda u=0$ will be found to be $H(s,t)$.

§ 14. The equation (1) may sometimes be used to construct a function $k(x,t)$ when the determinantal equation and the fundamental functions are given. In general the function $k(s,t)$ may be represented by the series

$$k(s,t)=\Sigma\frac{1}{\lambda_n}\psi(s,\lambda_n)\chi(t,\lambda_n),$$

but this may not be always true as there is some doubt about the convergence, moreover there may not be any obvious method of summing the series.

If however we introduce a new function

$$f(s,t)=\Sigma\frac{1}{\mu_n}\theta(s,\lambda_n)\chi(t,\lambda_n),$$

where $\theta(s,\mu_n)$ are the fundamental functions for $h(s,t)$ where the series

$$h(s,t)=\Sigma\frac{1}{\lambda_n}\theta(s,\lambda_n)\phi(t,\lambda_n)$$

is known to converge, and the quantities μ_n are chosen so that the series for $f(s,t)$ converges rapidly: we may be able to determine the function k by means of the relation

$$\int_a^b h(s,x)f(x,t)\,dx=\int_c^d f(s,x)k(x,t)\,dx \quad\text{.....(1 bis).}$$

To do this we must first calculate the integral on the left-hand side and try to find k by solving the integral equation of the first kind

$$J(s,t)=\int_c^d f(s,x)k(x,t)\,dx.$$

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Let us suppose that the functions $\theta(s, \lambda_n)$ and also the functions $\chi(t, \lambda_n)$ are linearly independent so that no solutions of

$$\int_a^b a(s) f(s, t)\, ds = 0 \text{ and } \int_c^d f(s, t)\, b(t) = 0 \text{ exist};$$

then the function k as determined by the above equation will be such that the integral equation

$$0 = \psi(s) - \lambda \int_c^d k(s, t)\, \psi(t)\, dt,$$

is satisfied for the values $(\lambda_1 \ldots \lambda_n \ldots)$, this follows from Theorem I. for the equation

$$0 = \theta(s) - \lambda \int_a^b h(s, t)\, \theta(t)\, dt,$$

is satisfied for these values of λ.

Also the fundamental functions are known to satisfy the relations

$$\int_c^d \psi(t, \lambda_m)\, \chi(t, \lambda_n)\, dt = \begin{matrix} 0 & \lambda_m \neq \lambda_n \\ 1 & \lambda_m = \lambda_n \end{matrix},$$

$$\int_a^b \theta(s, \lambda_m)\, \phi(s, \lambda_n)\, ds = \begin{matrix} 0 & \lambda_m \neq \lambda_n \\ 1 & \lambda_m = \lambda_n \end{matrix}.$$

Hence if we multiply equation (1) by $\psi(t, \lambda_n)$ and integrate we obtain

$$\int_c^d \int_c^d f(s, x)\, k(x, t)\, \psi(t, \lambda_n)\, dx\, dt = \int_c^d \int_a^b h(s, x) f(x, t)\, \psi(t, \lambda_n)\, dx\, dt$$

$$= \int_a^b h(s, x)\, \theta(x, \lambda_n)\, dx = \frac{1}{\lambda_n}\, \theta(s, \lambda_n)$$

$$= \frac{1}{\lambda_n} \int_c^d f(s, x)\, \psi(x, \lambda_n)\, dx.$$

Also the equation $\int_c^d f(s, x)\, \omega(x)\, dx = 0$ is only satisfied by $\omega(x) \equiv 0$, therefore we must have

$$\lambda_n \int_c^d k(x, t)\, \psi(t, \lambda_n)\, dt = \psi(x, \lambda_n),$$

and so the function $k(x, t)$ will possess the required properties.

III. *A partial integral equation.*

§ 15. The relation

$$\int_a^b h(s, x) f(x, t)\, dx = \int_c^d f(s, x)\, k(x, t)\, dx \quad \ldots\ldots\ldots\ldots (1),$$

may also be regarded as a partial integral equation to determine f when h and k are given, but we conclude at once from Theorem I. that it is useless to seek a solution except when the functions $h(s, x)$ and $k(x, t)$ are such that some of the roots of the corresponding determinantal equations are the same.

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This condition is evidently satisfied if the limits are the same and we take

$$h(s, t) = k(s, t);$$

the equation

$$\int_a^b k(s, x) f(x, t)\, dx = \int_a^b f(s, x)\, k(x, t)\, dx \quad \text{.....(2)},$$

is then closely connected with the integral equation of the second kind

$$f(s) = \phi(s) - \lambda \int_a^b k(s, t)\, \phi(t)\, dt.$$

Theorem V. If $f(s, t)$ and $g(s, t)$ are two solutions of the equation, then

$$F(s, t) = \int_a^b f(s, x)\, g(x, t)\, dx,$$

is also a solution.

For
$$\begin{aligned}\int_a^b k(s, x) F(x, t)\, dx &= \int_a^b \int_a^b k(s, x) f(x, y)\, g(y, t)\, dx\, dy \\ &= \int_a^b \int_a^b f(s, x)\, k(x, y)\, g(y, t)\, dx dy \\ &= \int_a^b \int_a^b f(s, x)\, g(x, y)\, k(y, t)\, dx dy \\ &= \int_a^b F(s, y)\, k(y, t)\, dy.\end{aligned}$$

The function $k(s, t)$ itself is evidently a solution of equation (2), accordingly the function

$$k_1(s, t) = \int_a^b k(s, x)\, k(x, t)\, dx,$$

is also a solution and we may build up a succession of solutions in this way.

§ 16. *Theorem VI. The solving function of the integral equation*

$$f(s) = \phi(s) - \lambda \int_a^b k(s, t)\, \phi(t)\, dt,$$

is a solution of equation (2).

This is a particular case of Theorem IV., it may be established at once however by comparing the relations

$$k(s, t) = K(s, t) - \lambda \int_a^b k(s, r)\, K(r, t)\, dr$$

$$k(s, t) = K(s, t) - \lambda \int_a^b k(r, t)\, K(s, r)\, dr.$$

§ 17. *Theorem VII. A function $f(s, t)$ which satisfies equation* (2) *has in general the same fundamental functions as $k(s, t)$.*

Let
$$\psi_n(s) = \lambda_n \int_a^b k(s, x)\, \psi_n(x)\, dx,$$

then if
$$\int_a^b k(s, x) f(x, t)\, dx = \int_a^b f(s, x)\, k(x, t)\, dx,$$

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we have
$$\int_a^b\int_a^b k(s, x) f(x, t)\psi_n(t)\,dx\,dt = \int_a^b\int_a^b f(s, x)\,k(x, t)\,\psi_n(t)\,dx\,dt$$
$$= \frac{1}{\lambda_n}\int_a^b f(s, x)\,\psi_n(x)\,dx.$$

Put
$$\phi_n(s) = \int_a^b f(s, x)\,\psi_n(x)\,dx,$$

then
$$\phi_n(s) = \lambda_n\int_a^b k(s, x)\,\phi_n(x)\,dx.$$

Hence if λ_n is a simple root of the determinantal equation for $k(s, t)$ we must have
$$\mu_n\phi_n(s) = \psi_n(s),$$
or
$$\mu_n\int_a^b f(s, x)\,\psi_n(x)\,dx = \psi_n(s),$$
showing that $\psi_n(x)$ is a fundamental function for $f(s, x)$. The case of $\mu_n = \infty$ may be considered to be a degenerate case.

If λ_n is a multiple root, this proof breaks down; but taking for simplicity the case in which there are two functions ϕ_n and ψ_n associated with the root λ_n, we can either have
$$\psi_n(s) = \mu_n\int_a^b f(s, x)\,\psi_n(x)\,dx, \quad \phi_n(s) = \nu_n\int_a^b f(s, x)\,\phi_n(x)\,dx,$$
or
$$\psi_n(s) = \mu_n\int_a^b f(s, x)\,\phi_n(x)\,dx, \quad \phi_n(s) = \nu_n\int_a^b f(s, x)\,\psi_n(x)\,dx.$$

In both cases the functions ψ_n and ϕ_n are fundamental for the function
$$f_1(s, t) = \int_a^b f(s, x)\,f(x, t)\,dx.$$

§ 18. As a particular case of this theorem we learn that if the equation (2) is satisfied by a function of the form
$$f(s, t) = \psi_n(s)\,\chi_n(t),$$
we must have simultaneously
$$\psi_n(s) = \lambda_n\int_a^b k(s, x)\,\psi_n(x)\,dx,$$
$$\chi_n(t) = \lambda_n\int_a^b \chi_n(x)\,k(x, t)\,dx,$$
this is easily verified by direct substitution.

We may build up a more general solution by adding together the products of this form for the different possible values of λ; thus if the series
$$\Sigma\, a_n\,\psi_n(s)\,\chi_n(t) = f(s, t),$$
is uniformly convergent it will be a solution of equation (2), but it is not always necessary for it to be uniformly convergent.

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§ 19. It is natural to inquire whether any applications can be made of the results obtained with regard to equations (1) and (2). At present the most interesting occur when $h(s, t)$ and $k(s, t)$ are the Green's functions for linear differential equations; this case will however be discussed in a paper shortly to be published by the author. We shall content ourselves here by considering another type of relation which leads to an equation of the form (2).

§ 20. Let us suppose that a function $F(s, x)$ exists such that

$$F(s, x+\omega) = \int_a^b k(s, t) F(t, x)\, dt \quad \text{.....(3)},$$

for a certain range of values of x, then if $\psi_n(s)$ is the fundamental function for which

$$\psi_n(t) = \lambda_n \int_a^b k(s, t) \psi_n(s)\, ds,$$

we should expect the function

$$\phi_n(x) = \int_a^b \psi_n(t) F(t, x)\, dt$$

to possess a special property; as a matter of fact it appears to be a periodic function of the second kind.

For
$$\begin{aligned}\phi_n(x+\omega) &= \int_a^b \psi_n(t) F(t, x+\omega)\, dt \\ &= \int_a^b \int_a^b \psi_n(t) k(t, r) F(r, x)\, dt dr \\ &= \frac{1}{\lambda_n} \int_a^b \psi_n(r) F(r, x)\, dr = \frac{1}{\lambda_n} \phi_n(x).\end{aligned}$$

If we suppose that the function $F(t, x)$ is given, the determination of the function $k(s, t)$ will depend upon the solution of an integral equation of the first kind; this may or may not be possible but it is sufficient to know that there are cases in which it can be done.

Let us assume that this can be done for more than one value of ω, so that we have

$$F(s, x+\omega) = \int_a^b k(s, t) F(t, x)\, dt,$$

$$F(s, x+\omega') = \int_a^b f(s, t) F(t, x)\, dt,$$

and that these equations still hold when $x+\omega$ or $x+\omega'$ is written instead of x. The two values of $F(s, x+\omega+\omega')$ that are obtained are

$$\int_a^b \int_a^b k(s, t) f(t, r) F(r, x)\, dt dr \text{ and } \int_a^b \int_a^b f(s, t) k(t, r) F(r, x)\, dt dr,$$

and these two integrals should be equal; hence if the function $F(r, x)$ is such that no solution of $\int_a^b F(r, x) \phi(r)\, dr = 0$ exists which is different from zero, we must have

$$\int_a^b k(s, t) f(t, r)\, dt = \int_a^b f(s, t) k(t, r)\, dt,$$

which is an equation of the type we have just studied.

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Now we have seen that this equation implies in general that the functions $k(s, t)$ and $f(t, r)$ have the same fundamental functions, hence we should expect that the function

$$\phi_n(x) = \int_a^b \psi_n(t) F(t, x)\, dt$$

would satisfy the two equations

$$\phi_n(x+\omega) = \frac{1}{\lambda_n}\phi_n(x),$$

$$\phi_n(x+\omega') = \frac{1}{\mu_n}\phi_n(x).$$

There are now several possibilities.

(1) $\phi_n(x)$ may be identically zero; this case is trivial.

(2) The ratio of ω and ω' may not be purely real; in this case these equations are possible and ϕ_n is a doubly periodic function of the second kind.

(3) ω and ω' may be multiples of the same quantity Ω.

(4) If ω and ω' are not of the form $n\Omega$ and $m\Omega$ these equations can only co-exist if $\phi_n(x)$ is of the form Ae^{-cx}, and in this case the quantities λ_n and μ_n are of the form $e^{c\omega}$ and $e^{c\omega'}$ respectively.

§ 21. For purposes of illustration it is convenient to consider a particular example, accordingly we shall endeavour to determine a function $k(s, t)$ such that

$$\frac{1}{1-2s(x+\omega)+(x+\omega)^2} = \int_{-1}^{+1} \frac{k(s, t)\, dt}{1-2tx+x^2}, \qquad |s| < 1,$$

on the supposition that $x > 1$ and that ω is positive.

Writing $x = \dfrac{1}{\xi}$ the equation becomes

$$\frac{1}{\xi^2(1-2\omega s+\omega^2)-2\xi(s-\omega)+1} = \int_{-1}^{+1} \frac{k(s, t)\, dt}{1-2t\xi+\xi^2}, \qquad \xi < 1.$$

Now the solution of the equation

$$f(\xi) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{1-2t\xi+\xi^2}, \qquad \xi < 1,$$

is given by*

$$\phi(t) = \frac{i}{\pi}\left[(t+i\sqrt{1-t^2})f(t+i\sqrt{1-t^2}) - (t-i\sqrt{1-t^2})f(t-i\sqrt{1-t^2})\right].$$

Hence we obtain (omitting the algebra)

$$\frac{1}{1-2s(x+\omega)+(x+\omega)^2}$$
$$= \frac{2}{\pi}\int_{-1}^{+1} \frac{dt}{1-2tx+x^2} \cdot \frac{\omega(\omega-2s)\sqrt{1-t^2}}{4(t-s+\omega)^2+4t(t-s+\omega)(\omega^2-2s\omega)+(\omega^2-2s\omega)^2},$$

so that $$k(s, t) = \frac{2}{\pi}\,\frac{\omega(\omega-2s)\sqrt{1-t^2}}{4(t-s+\omega)^2+4t(t-s+\omega)(\omega^2-2s\omega)+(\omega^2-2s\omega)^2}.$$

* This result will be proved in a paper shortly to be published by the author. If ξ is >1 it is sufficient to change the sign of ϕ in this formula.

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If now we give ω a different value we shall obtain another function $f(s, t)$ which will be connected with $k(s, t)$ by the relation (2).

Writing the above equation in the form

$$F(s, x+\omega) = \int_{-1}^{+1} k(s, t) F(t, x)\, dt,$$

we have
$$F(s, x+2\omega) = \int_{-1}^{+1}\int_{-1}^{+1} k(s, t) k(t, r) F(r, x)\, dt\, dr$$
$$= \int_{-1}^{+1} k_1(s, r) F(r, x)\, dr,$$

and
$$F(s, x+n\omega) = \int_{-1}^{+1} k_{n-1}(s, r) F(r, x)\, dr.$$

But for small values of λ the solving function $K(s, t)$ may be expanded in the form*

$$K(s, t) = k(s, t) + \lambda k_1(s, t) + \ldots \lambda^n k_n(s, t) + \ldots,$$

where
$$k_n(s, t) = \int_{-1}^{+1} \ldots \int_{-1}^{+1} k(s, s_1) k(s_1, s_2) \ldots k(s_n, t)\, ds_1 \ldots ds_n.$$

Hence if
$$G(s, x) = F(s, x+\omega) + \lambda F(s, x+2\omega) + \ldots,$$

we shall have
$$G(s, x) = \int_{-1}^{+1} K(s, t) F(t, x)\, dt.$$

Now
$$\frac{1}{1-2sy+y^2} = \frac{1}{1-s^2+(y-s)^2} = \frac{1}{\sqrt{1-s^2}} \int_0^\infty e^{-z(y-s)} . \sin z\sqrt{1-s^2}\, dz,$$

provided $s^2 < 1$ and $y > 1$.

Hence
$$\sum_1^\infty \frac{\lambda^{n-1}}{1-2s(x+n\omega)+(x+n\omega)^2} = \frac{1}{\sqrt{1-s^2}} \sum_1^\infty \lambda^{n-1} \int_0^\infty e^{-z(x+n\omega-s)} \sin z\sqrt{1-s^2} . dz$$
$$= \frac{1}{\sqrt{1-s^2}} \int_0^\infty \frac{e^{-z(x+\omega-s)}}{1-\lambda e^{-z\omega}} \sin z\sqrt{1-s^2} . dz, \qquad |\lambda| < 1.$$

This is our function $G(s, x)$; in order to obtain the corresponding value of $K(s, t)$ we must represent $G(s, x)$ by the definite integral

$$G(s, x) = \int_{-1}^{+1} \frac{K(s, t)}{1-2tx+x^2}\, dt,$$

and to do this we first represent e^{-zx} by a definite integral of this kind; our formula

$$\phi(t) = -\frac{i}{\pi} \{(t+i\sqrt{1-t^2}) f(t+i\sqrt{1-t^2}) - (t-i\sqrt{1-t^2}) f(t-i\sqrt{1-t^2})\}$$

gives
$$\phi(t) = -\frac{2}{\pi} e^{-zt} \{t \sin z\sqrt{1-t^2} - \sqrt{1-t^2} \cos z\sqrt{1-t^2}\}.$$

Hence we have

$$K(s, t) = -\frac{2}{\pi} . \frac{1}{\sqrt{1-s^2}} \int_0^\infty \frac{e^{-z(t+\omega-s)}}{1-\lambda e^{-z\omega}} \sin z\sqrt{1-s^2} . \{t \sin z\sqrt{1-t^2} - \sqrt{1-t^2} \cos z\sqrt{1-t^2}\}\, dz.$$

* Fredholm, *Acta Math.* 1903.

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The determinant $\delta(\lambda)$ may be calculated by means of the formula

$$\frac{\delta'(\lambda)}{\delta(\lambda)} = -\int_{-1}^{+1} K(s, s)\, ds.$$

Putting $s = \cos\theta$, we have

$$\int_{-1}^{+1} K(s, s)\, ds = -\frac{2}{\pi}\int_0^\pi \int_0^\infty \frac{e^{-z\omega}}{1-\lambda e^{-z\omega}} \sin(z\sin\theta)\sin(z\sin\theta - \theta)\, dz\, d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \int_0^\infty \frac{e^{-z\omega}}{1-\lambda e^{-z\omega}} [-\cos\theta + \cos(2z\sin\theta - \theta)]\, dz\, d\theta$$

$$= \int_0^\infty \frac{e^{-z\omega}}{1-\lambda e^{-z\omega}} J_1(2z)\, dz.$$

Now
$$\int_0^\infty e^{-z\mu} J_1(2z) = \frac{1}{2}\left[1 - \frac{\mu}{\sqrt{\mu^2+4}}\right],$$

therefore
$$-\frac{\delta'(\lambda)}{\delta(\lambda)} = \tfrac{1}{2}\sum_0^\infty \lambda^n \left[1 - \frac{(n+1)\omega}{\sqrt{\omega^2(n+1)^2+4}}\right].$$

The roots of the function $\delta(\lambda)$ are of the form $e^{\rho_i\omega}$ for we have

$$\sum_1^\infty \frac{1}{\lambda_i^n} = 1 - \frac{n\omega}{\sqrt{n^2\omega^2+4}},$$

and this is satisfied for all values of n if λ_i is of the form $e^{\rho_i\omega}$. Now this is exactly what was predicted at the end of § 20, accordingly we can say *à priori* that the fundamental functions are such that

$$\int_{-1}^{+1} \frac{\psi(t)\, dt}{1-2tx+x^2} = Ae^{-x\rho_i},$$

and as we have just seen, the corresponding value of ψ is

$$\psi(t) = -\frac{2A}{\pi} e^{-\rho_i t}[t\sin\rho_i\sqrt{1-t^2} - \sqrt{1-t^2}\cos\rho_i\sqrt{1-t^2}].$$

The quantities $e^{\omega\rho_i}$ are the roots of the function $\delta(\lambda)$ and are known to satisfy the relations

$$\sum_{i=1}^\infty \frac{1}{\lambda_i^n} = \sum_{i=1}^\infty e^{-n\omega\rho_i} = \frac{1}{2}\left[1 - \frac{n\omega}{\sqrt{n^2\omega^2+4}}\right].$$

The problem of finding the roots of $\delta(\lambda)$ is therefore reduced to that of expressing the function $\frac{1}{2}\left[1 - \frac{x}{\sqrt{x^2+4}}\right]$ by means of a series of the form

$$\frac{1}{2}\left[1 - \frac{x}{\sqrt{x^2+4}}\right] = \sum_{i=1}^\infty e^{-x\rho_i},$$

for positive values of x.

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TRANSACTIONS

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SIMPLE TYPES OF HYPERGEOMETRIC SERIES.

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XI. *On Functions defined by simple types of Hypergeometric Series.*

By E. W. Barnes, M.A., Fellow of Trinity College, Cambridge.

[*Received and Read* March 12, 1906.]

§ 1. In the present paper it is proposed to consider the asymptotic expansion of some simple cases of integral functions which are represented by generalised hypergeometric series.

The general type of such series is

$$1+\frac{\alpha_1\ldots\alpha_p}{1\,\rho_1\ldots\rho_q}x+\frac{\alpha_1(\alpha_1+1)\ldots\alpha_p(\alpha_p+1)}{1\,.\,2\,\rho_1(\rho_1+1)\ldots\rho_q(\rho_q+1)}x^2+\ldots$$
$$=\frac{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+n)\ldots\Gamma(\alpha_p+n)}{\Gamma(n+1)\,\Gamma(\rho_1+n)\ldots\Gamma(\rho_q+n)}x^n.$$

Such a series is an integral function if $p\leqslant q$. To this case we shall limit ourselves.

The series we shall denote by

$${}_pF_q\{\alpha_1,\ldots,\alpha_p;\ \rho_1,\ldots,\rho_q;\ x\},$$

or, more briefly, when the parameters are obvious, by ${}_pF_q(x)$.

The series satisfies the differential equation

$$\left\{(\vartheta+\alpha_1)\ldots(\vartheta+\alpha_p)-\frac{d}{dx}(\vartheta+\rho_1-1)\ldots(\vartheta+\rho_q-1)\right\}y=0,$$

wherein ϑ denotes $x\dfrac{d}{dx}$.

The differential equation is of order $(q+1)$. The other q solutions are given by

$$x^{1-\rho_1}\,{}_pF_q\{\alpha_1-\rho_1+1,\ldots,\alpha_p-\rho_1+1;\ 2-\rho_1,\rho_2-\rho_1+1,\ldots,\rho_q-\rho_1+1;\ x\},$$

and $(q-1)$ similar functions.

From one point of view the problem of obtaining asymptotic expansions of the functions is that of finding asymptotic solutions of the differential equation in the vicinity of $x=\infty$, and of connecting such solutions with those which exist in the finite part of

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the plane. Such an investigation for the case $p=q=1$ has been undertaken by Jacobsthal*. The direct problem has been attacked by Orr†, who in two papers suggested by the work of Stokes‡ has obtained the principal results by laborious inductive analysis.

§ 2. In this paper it is proposed to obtain elementary cases of the theory in a simplified form by the use of contour integrals involving gamma functions, and of the theory of factorial series. The theory is part of a general investigation on the *Asymptotic Expansion of Integral Functions defined by Taylor's Series.* In a paper with this title§ some of the following results were indicated‖ though no detailed investigation was given, and analogous developments appear in other papers by the author¶. Complete references to the history and literature of the subject appear in the paper first cited.

§ 3. Part I. of the present paper contains a complete discussion of the function ${}_1F_1\{\alpha;\ \rho;\ x\}$, which is the most simple function included in the general category.

It is shewn that

$$ {}_1F_1\{\alpha;\ \rho;\ x\} = e^x\, {}_1F_1\{\rho-\alpha;\ \rho;\ -x\}. $$

Then by means of a contour integral the asymptotic expansion of ${}_1F_1\{\alpha;\ \rho;\ x\}$ is obtained when $R(x)<0$. A combination of the two previous results gives the asymptotic expansion of ${}_1F_1\{\alpha;\rho;x\}$ when $R(x)>0$. The remainder of the theory then follows simply.

In Part II. the function ${}_0F_1\{\rho;x\}$ is discussed. This function is substantially Bessel's function, and its properties are made to depend on those of ${}_1F_1\{\alpha;\ \rho;\ x\}$ by means of the equality

$$ {}_0F_1\{\rho;\ x\} = e^{-2x^{1/2}}\, {}_1F_1\{\rho-1/2;\ 2\rho-1;\ 4x^{1/2}\} = e^{2x^{1/2}}\, {}_1F_1\{\rho-1/2;\ 2\rho-1;\ -4x^{1/2}\}. $$

The principal results of the general theory have been obtained, and will shortly be published.

PART I.

The series ${}_1F_1\{\alpha;\ \rho;\ x\}$.

§ 4. We consider first the simplest series of the hypergeometric type

$$ {}_1F_1\{\alpha;\ \rho;\ x\} = \frac{\Gamma(\rho)}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\, x^n}{\Gamma(n+1)\,\Gamma(\rho+n)}. $$

If we put $\alpha=\rho$, the function reduces to e^x.

We must obviously exclude the case when ρ is a negative integer (zero included). If α is a negative integer the series becomes a polynomial in x.

* Jacobsthal, *Mathematische Annalen*, Vol. LVI. pp. 129—154.

† Orr, *Cambridge Philosophical Transactions*, Vol. XVII. pp. 171—200 and pp. 283—290.

‡ Stokes, *Cambridge Philosophical Transactions*, Vol. X. pp. 105 *et seq.*, Vol. XI. pp. 412 *et seq.* *Cambridge Philosophical Proceedings*, Vol. VI. pp. 362 *et seq.*

§ *Philosophical Transactions Royal Society* (A) Vol. 206, pp. 249—297.

‖ *loc. cit.* Parts IX, X, XI.

¶ *Quarterly Journal of Mathematics*, Vol. XXXVII. pp. 289—313. *Proceedings of the London Mathematical Society* Ser. 2, Vol. IV. pp. 284—316. *Cambridge Philosophical Transactions*, Vol. XX. pp. 215—232. *Quarterly Journal of Mathematics*, Vol. XXXVIII. pp. 108—116 and pp. 116—140.

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The series satisfies the equation*

$$\left\{(\vartheta+\alpha)-\frac{d}{dx}(\vartheta+\rho-1)\right\}y=0.$$

or

$$x\frac{d^2y}{dx^2}-(x-\rho)\frac{dy}{dx}-\alpha y=0.$$

An independent solution of this equation is

$$x^{1-\rho}\,{}_1F_1(\alpha-\rho+1;\ 2-\rho;\ x).$$

§ 5. We will shew in the first place that, for all values of x of finite modulus,

$${}_1F_1\{\alpha;\ \rho;\ x\}=e^x\,{}_1F_1\{\rho-\alpha;\ \rho;\ -x\}.$$

Consider the integral

$$-\frac{1}{2\pi\iota}\int_C \Gamma(n-s)\,x^s\,ds$$

round a contour C embracing the positive half of the real axis and enclosing the origin.

By Cauchy's theory of residues it is equal to

$$\sum_{m=0}^{\infty}\frac{(-)^m x^{n+m}}{m!}=e^{-x}x^n.$$

Now

$$\frac{\Gamma(\alpha)}{\Gamma(\rho)}e^{-x}\,{}_1F_1\{\alpha;\ \rho;\ x\}=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\,e^{-x}x^n}{\Gamma(\rho+n)\,\Gamma(n+1)}.$$

Assume now that $R(\rho-\alpha)>\epsilon>0$. Then the last series is equal to

$$-\frac{1}{2\pi\iota}\int_C\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\,\Gamma(n-s)}{\Gamma(\rho+n)\,\Gamma(n+1)}x^s\,ds.$$

For the series under the integral sign when divided by $\Gamma(-s)$ is convergent if

$$R(\rho-\alpha+s)>0,$$

that is to say at all points on the contour of integration, if the contour passes very near the origin. And the integral is equal to the sum of the residues of the subject of integration inside the contour.

But by Gauss' formula

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\,\Gamma(n-s)}{\Gamma(\rho+n)\,\Gamma(n+1)}\,&\frac{\Gamma(\rho)}{\Gamma(\alpha)\,\Gamma(-s)}\\ &={}_2F_1(\alpha,\ -s;\ \rho;\ 1)\\ &=\frac{\Gamma(\rho)\,\Gamma(\rho-\alpha+s)}{\Gamma(\rho-\alpha)\,\Gamma(\rho+s)},\ \text{provided } R(\rho-\alpha+s)>0.\end{aligned}$$

We therefore have

$$\begin{aligned}\frac{\Gamma(\alpha)}{\Gamma(\rho)}e^{-x}\,{}_1F_1\{\alpha;\ \rho;\ x\}&=-\frac{1}{2\pi\iota}\int_C\frac{\Gamma(\rho-\alpha+s)\,\Gamma(\alpha)\,\Gamma(-s)}{\Gamma(\rho-\alpha)\,\Gamma(\rho+s)}x^s\,ds\\ &=\frac{\Gamma(\alpha)}{\Gamma(\rho-\alpha)}\sum_{n=0}^{\infty}\frac{(-)^n x^n\,\Gamma(\rho-\alpha-n)}{\Gamma(n+1)\,\Gamma(\rho-n)}\\ &\qquad\qquad\text{by Cauchy's theory of residues}\\ &=\frac{\Gamma(\alpha)}{\Gamma(\rho)}\,{}_1F_1\{\rho-\alpha;\ \rho;\ -x\}.\end{aligned}$$

* The equation has been considered by Pochhammer, *Mathematische Annalen*, Bd. XLVI. pp. 584—590.

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Thus, if $R(\rho-\alpha)>\epsilon>0$, we have the required identity

$$ {}_1F_1\{\alpha;\ \rho;\ x\} = e^x\,{}_1F_1\{\rho-\alpha;\ \rho;\ -x\}. $$

Since each side is continuous as a function of x for all values of α and ρ such that ρ is not zero or a negative integer we see that the theorem is true in general.

The important result of the present paragraph appears to be due to Orr*. A particular case had been previously obtained by Glaisher†.

As the method is one which will be constantly used, we will indicate a modified form of proof.

§ 6. We know that, when n is very large,

$$ \frac{\Gamma(\alpha+n)\,\Gamma(n-s)}{\Gamma(\rho+n)\,\Gamma(n+1)} = \frac{1}{n^{\rho-\alpha+1+s}}\left\{1+\frac{c_1}{n}+\ldots+\frac{c_R}{n^R}+\frac{J_R(n)}{n^R}\right\}, $$

the c's being assignable polynomials in ρ, α and s, and $J_R(n)$ being such that $|J_R(n)|$ can be made as small as we please by taking n sufficiently large.

Hence
$$ \frac{1}{\Gamma(-s)}\sum_{n=1}^{\infty}\left\{\frac{\Gamma(\alpha+n)\,\Gamma(n-s)}{\Gamma(\rho+n)\,\Gamma(n+1)}-\sum_{r=0}^{R}\frac{c_r}{n^{\rho-\alpha+1+r+s}}\right\}+\frac{\Gamma(\alpha)}{\Gamma(\rho)} $$

is convergent if $R(s)>R(\alpha-\rho)-R$.

For such values of s that $R(s)>R(\alpha-\rho)$ it is equal to

$$ \frac{\Gamma(\rho-\alpha+s)\,\Gamma(\alpha)}{\Gamma(\rho-\alpha)\,\Gamma(\rho+s)}-\sum_{r=0}^{R}\frac{c_r\,\zeta(\rho-\alpha+r+1+s)}{\Gamma(-s)}, $$

where $\zeta(s)$ denotes the simple Riemann ζ function.

This equality between analytic functions therefore holds provided $R(s)>R(\alpha-\rho)-R$. And by taking R as large as we please but finite we see that it will hold for all values of s such that $R(s)>$ a finite negative quantity.

Consider now the integral

$$ I=-\frac{1}{2\pi\iota}\int_{C_N}\left\{\frac{\Gamma(\rho-\alpha+s)\,\Gamma(\alpha)\,\Gamma(-s)}{\Gamma(\rho-\alpha)\,\Gamma(\rho+s)}-\sum_{r=0}^{R}c_r\,\zeta(\rho-\alpha+r+1+s)\right\}x^s\,ds $$

taken round a closed contour C_N which encloses the points $s=0, 1, 2, \ldots N$, and excludes the points $\alpha-\rho$, $\alpha-\rho-1$, ... which are poles of the Riemann ζ functions.

It is by Cauchy's theory of residues equal to

$$ \frac{\Gamma(\alpha)}{\Gamma(\rho)}\,{}_1F_1\{\rho-\alpha;\ \rho;\ -x\}_N, $$

the suffix N denoting that only the first $(N+1)$ terms of the series for ${}_1F_1\{\rho-\alpha;\ \rho;\ -x\}$ are taken.

* Orr, *Cambridge Philosophical Transactions*, Vol. XVII. p. 175.

† Glaisher, *Philosophical Transactions Royal Society*, (1881) Vol. 172, p. 774.

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But it is also equal to

$$-\frac{1}{2\pi\iota}\int_{C_N}\left[\frac{\Gamma(\alpha)\Gamma(-s)}{\Gamma(\rho)}+\sum_{n=1}^{\infty}\left\{\frac{\Gamma(\alpha+n)\Gamma(n-s)}{\Gamma(\rho+n)\Gamma(n+1)}-\sum_{r=0}^{R}\frac{c_r}{n^{\rho-\alpha+1+r+s}}\right\}\right]x^s\,ds$$

$$=-\frac{1}{2\pi\iota}\int_{C_N}\sum_{n=0}^{N}\left\{\frac{\Gamma(\alpha+n)\Gamma(n-s)}{\Gamma(\rho+n)\Gamma(n+1)}\right\}x^s\,ds$$

$$=\sum_{n=0}^{N}\frac{\Gamma(\alpha+n)}{\Gamma(\rho+n)\Gamma(n+1)}\sum_{m=0}^{N-n}\frac{(-)^m x^{n+m}}{m!}.$$

We therefore have

$$\frac{\Gamma(\alpha)}{\Gamma(\rho)}{}_1F_1\{\rho-\alpha;\ \rho;\ -x\}_N=\sum_n\sum_m\frac{\Gamma(\alpha+n)(-)^m x^{n+m}}{\Gamma(\rho+n)\Gamma(n+1)\Gamma(m+1)},$$

all values of n and m being taken which are positive or zero and such that $m+n\leqslant N$.

Consider now the effect of making N infinite.

Those terms of the double sum for which $m+n>N$ have evidently a sum which tends to zero as N tends to infinity. The double summation is absolutely convergent, and may be summed in any way we please.

Hence $$\frac{\Gamma(\alpha)}{\Gamma(\rho)}{}_1F_1\{\rho-\alpha;\ \rho;\ -x\}=e^{-x}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)x^n}{\Gamma(\rho+n)\Gamma(n+1)},$$

or $$_1F_1\{\alpha;\ \rho;\ x\}=e^x\,{}_1F_1\{\rho-\alpha;\ \rho;\ -x\}.$$

§ 7. We will next shew that, when $R(x)<0$, ${}_1F_1\{\alpha;\ \rho;\ x\}$ admits the asymptotic expansion

$$\frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}(-x)^{-\alpha}\,{}_2F_0\left\{\alpha,\ 1-\rho+\alpha;\ -\frac{1}{x}\right\}.$$

The error which results from stopping at the kth term of this series is at most of order $x^{-\alpha-k+1}$ when $|x|$ is large, and $(-x)^{-\alpha}=\exp\{-\alpha\log(-x)\}$, the logarithm having its principal value whose imaginary part lies between $\pm\pi/2$.

Consider the integral

$$-\frac{1}{2\pi\iota}\int\frac{\Gamma(-s)\Gamma(\alpha+s)(-x)^s}{\Gamma(\rho+s)}\,ds,$$

taken round a contour enclosing the poles of $\Gamma(-s)$ and no other poles of the subject of integration. In the integral $(-x)^s$ has its principal value.

The integral is evidently equal to

$$\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)x^n}{\Gamma(n+1)\Gamma(\rho+n)}=\frac{\Gamma(\alpha)}{\Gamma(\rho)}{}_1F_1\{x\}.$$

Now, if $R(x)<0$ so that $|\arg(-x)|<\frac{\pi}{2}$, the integral vanishes when taken round that part of the circle at infinity for which $R(s)>-k$.

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Hence asymptotically the integral is equal to

$$\sum_{n=0}^{\infty} \frac{(-)^n \Gamma(\alpha+n)(-x)^{-\alpha-n}}{\Gamma(n+1)\Gamma(\rho-\alpha-n)},$$

terms of order less than that of any algebraical power of $\frac{1}{|x|}$, when $|x|$ is large, being neglected.

The series may be written

$$(-x)^{-\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\alpha-\rho+1+n)}{\Gamma(n+1)(-x)^n} \frac{\sin \pi(\rho-\alpha)}{\pi}$$

$$= (-x)^{-\alpha} \frac{\Gamma(\alpha)}{\Gamma(\rho-\alpha)} {}_2F_0 \left\{\alpha, 1-\rho+\alpha; -\frac{1}{x}\right\}.$$

Thus, if $R(x) < 0$,

$${}_1F_1\{\alpha; \rho; x\} = \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)} (-x)^{-\alpha} {}_2F_0 \left\{\alpha, 1-\rho+\alpha; -\frac{1}{x}\right\}.$$

§ 8. Consider now the nature of the asymptotic value, when $|x|$ is large, and $R(x) < 0$, of ${}_1F_1\{\alpha; \rho; x\}$ when α and ρ tend to positive infinity, in such a manner that $\mathrm{Lt}(\alpha/\rho) = 1$.

We have, when $R(x) < 0$,

$${}_1F_1\{x\} = -\frac{1}{2\pi\iota} \int \Gamma(-s)(-x)^s \frac{\Gamma(\alpha+s)}{\Gamma(\alpha)} \frac{\Gamma(\rho)}{\Gamma(\rho+s)} ds,$$

and, however large $|\alpha|$ and $|\rho|$ may be, we may take this integral along a contour in the finite part of the plane parallel to the imaginary axis so that $-\alpha$ is to the left of the contour.

If then k be any finite quantity, and α and ρ tend to positive infinity in such a manner that $\mathrm{Lt}(\alpha/\rho) = 1$,

$${}_1F_1\{x\} = \frac{J(x)}{x^k},$$

where, however large k may be, we can make $|J(x)|$, when $|x|$ is sufficiently large, as small as we please.

§ 9. We may next shew that, if $R(x) > 0$,

$${}_1F_1\{\alpha; \rho; x\} = \frac{e^x \Gamma(\rho)}{\Gamma(\alpha) x^{\rho-\alpha}} {}_2F_0 \left\{\rho-\alpha, 1-\alpha; \frac{1}{x}\right\}.$$

The error obtained in stopping at the kth term of the series is, when $|x|$ is large, at most of order $\left|\frac{e^x}{x^{k-1+\rho-\alpha}}\right|$.

To prove this theorem we combine the two previous results. Then we see that if $R(x) < 0$, $e^x {}_1F_1\{\rho-\alpha; \rho; -x\}$ is asymptotically equal to

$$\frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)} (-x)^{-\alpha} {}_2F_0 \left\{\alpha, 1-\rho+\alpha; -\frac{1}{x}\right\}.$$

Change x into $-x$, $\rho-\alpha$ into α and therefore α into $\rho-\alpha$, and we have the result stated.

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Combining this result with that just obtained we see that, if $|\arg x| < \pi$,

$$ {}_1F_1\{\alpha;\ \rho;\ x\} = \frac{e^x\,\Gamma(\rho)}{\Gamma(\alpha)\,x^{\rho-\alpha}}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha;\ \frac{1}{x}\right\} $$
$$ + \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}(-x)^{-\alpha}\,{}_2F_0\left\{\alpha,\ 1-\rho+\alpha;\ -\frac{1}{x}\right\}. $$

This is the complete asymptotic expansion of ${}_1F_1\{\alpha;\ \rho;\ x\}$ in the vicinity of $|x| = \infty$. It shews us that the large zeros of ${}_1F_1\{\alpha;\ \rho;\ x\}$ ultimately lie along the imaginary axis.

§ 10. The nature of the asymptotic value of ${}_1F_1\{\alpha;\ \rho;\ x\}$ when $|x|$ is large and $R(x) > 0$, when α and ρ both tend to positive infinity in such a way that $\mathrm{Lt}(1-\alpha/\rho)$ tends to unity, can now be immediately deduced.

For, when $R(x) > 0$,

$$ {}_1F_1\{\alpha;\ \rho;\ x\} = e^x\,{}_1F_1\{\rho-\alpha;\ \rho;\ -x\}. $$

Hence if k be any finite quantity, and ρ and α tend to positive infinity in such a way that $\mathrm{Lt}(\alpha/\rho)$ tends to zero

$$ {}_1F_1\{\alpha;\ \rho;\ x\} = \frac{e^x J(x)}{x^k}, $$

where, however large k may be, we can make $|J(x)|$, when $|x|$ is sufficiently large, as small as we please.

§ 11. We have seen that the fundamental differential equation

$$ x\frac{d^2y}{dx^2} - (x-\rho)\frac{dy}{dx} - \alpha y = 0 $$

admits the two solutions

$$ {}_1F_1\{\alpha;\ \rho;\ x\} \quad \text{and} \quad {}_1F_1\{\alpha-\rho+1;\ 2-\rho;\ x\}\,x^{1-\rho}. $$

The idea therefore suggests itself that it is possible to take such a linear combination of these two integral functions as will admit all round $x = \infty$ the single asymptotic expansion

$$ (-x)^{-\alpha}\,{}_2F_0\left\{\alpha,\ 1-\rho+\alpha;\ -\frac{1}{x}\right\} $$

and that equally we can take another combination which will admit the single asymptotic expansion

$$ e^x x^{\alpha-\rho}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha;\ \frac{1}{x}\right\}. $$

We proceed to prove that such expectations are correct.

We will first shew that*, if $|\arg x| < \frac{3\pi}{2}$,

$$ \Gamma(\alpha)\,\Gamma(1-\rho)\,{}_1F_1\{\alpha;\ \rho;\ x\} + \Gamma(\alpha+1-\rho)\,\Gamma(\rho-1)\,x^{1-\rho}\,{}_1F_1(\alpha-\rho+1;\ 2-\rho;\ x) $$
$$ = \Gamma(\alpha)\,\Gamma(1+\alpha-\rho)\,x^{-\alpha}\,{}_2F_0\left\{\alpha,\ 1+\alpha-\rho;\ -\frac{1}{x}\right\}. $$

* Compare Orr, *loc. cit.* p. 178.

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Consider the integral

$$\frac{1}{2\pi\iota}\int \Gamma(-s)\,\Gamma(1-\rho-s)\,\Gamma(\alpha+s)\,x^s\,ds$$

taken round that part of the circle at infinity for which $R(s) > -k$, and the line $s = -k$.

The integral along the circle will vanish if

$$|\arg x| < \frac{3\pi}{2}.$$

The integral along the straight line is of order $\frac{1}{|x|^k}$.

Hence we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-)^{n-1} x^n}{n!}\,\Gamma(1-\rho-n)\,\Gamma(\alpha+n) \\
&\quad + \sum_{n=0}^{\infty} \frac{(-)^{n-1} x^{1-\rho+n}}{n!}\,\Gamma(\rho-n-1)\,\Gamma(\alpha+1-\rho+n) \\
&\quad + \sum_{n=0}^{l} x^{-\alpha-n}\,\frac{(-)^n}{n!}\,\Gamma(\alpha+n)\,\Gamma(1+\alpha+n-\rho) \\
&\quad = J_k,
\end{aligned}$$

where $R(\alpha)+l+1 > k > R(\alpha)+l$ and $|J_k|$ is, when $|x|$ is large, of order $\frac{1}{|x|^k}$.

Thus asymptotically, when $|\arg x| < \frac{3\pi}{2}$,

$$\begin{aligned}
&\Gamma(\alpha)\,\Gamma(1-\rho)\,{}_1F_1\{\alpha;\ \rho;\ x\} + \Gamma(\alpha+1-\rho)\,\Gamma(\rho-1)\,x^{1-\rho}\,{}_1F_1\{\alpha-\rho+1;\ 2-\rho;\ x\} \\
&\qquad = x^{-\alpha}\,{}_2F_0\left\{\alpha,\ 1+\alpha-\rho;\ -\frac{1}{x}\right\}\Gamma(\alpha)\,\Gamma(1+\alpha-\rho). \qquad \text{(A)}
\end{aligned}$$

§ 12. The previous result has been established when $|\arg x| < \frac{3}{2}\pi$. Therefore when $R(x) < 0$ it is equivalent to two different results. And by eliminating the function ${}_1F_1\left\{\alpha-\rho+1;\ 2-\rho;\ -\frac{1}{x}\right\}$ we can obtain the asymptotic expansion of ${}_1F_1\{\alpha;\ \rho;\ x\}$ when $R(x) < 0$. This is readily done as follows.

Let

$$\Gamma(\alpha)\,\Gamma(1-\rho)\,{}_1F_1\{\alpha;\ \rho;\ x\} = P,$$

$$\Gamma(\alpha+1-\rho)\,\Gamma(\rho-1)\,{}_1F_1\{\alpha-\rho+1;\ 2-\rho;\ x\} = Q,$$

$$\Gamma(\alpha)\,\Gamma(1+\alpha-\rho)\,{}_2F_0\left\{\alpha,\ 1+\alpha-\rho;\ -\frac{1}{x}\right\} = R,$$

and suppose that $\arg x$ lies between $\frac{1}{2}\pi$ and π.

Then the previous result is equivalent to the two relations

$$P + x^{1-\rho}Q = x^{-\alpha}R,$$

$$P + x^{1-\rho}e^{-2\pi\iota(1-\rho)}Q = x^{-\alpha}e^{2\pi\iota\alpha}R,$$

the principal values of $x^{1-\rho}$ and $x^{-\alpha}$ being taken.

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Hence
$$P(e^{2\pi i\rho}-1)=x^{-\alpha}R(e^{2\pi i\rho}-e^{2\pi i\alpha}),$$
or
$$P\sin\pi\rho=e^{\pi i\alpha}x^{-\alpha}R\sin\pi(\rho-\alpha)$$
$$=(-x)^{-\alpha}R\sin\pi(\rho-\alpha),$$
where $(-x)^{-\alpha}$ has its principal value, the logarithm by which it is defined being real when x is real and negative and having a cross-cut along the positive half of the real axis.

Similarly, if $\arg x$ lies between $-\frac{\pi}{2}$ and $-\pi$, we shall have
$$P\sin\pi\rho=e^{-\pi i\alpha}x^{-\alpha}R\sin\pi(\rho-\alpha).$$
Thus in either case
$$P\sin\pi\rho=(-x)^{-\alpha}R\sin\pi(\rho-\alpha).$$
Hence, if $R(x)<0$,
$${}_1F_1\{\alpha;\ \rho;\ x\}\frac{\Gamma(\alpha)}{\Gamma(\rho)}=\frac{\Gamma(\alpha)}{\Gamma(\rho-\alpha)}(-x)^{-\alpha}\,{}_2F_0\left\{\alpha,\ 1+\alpha-\rho;\ -\frac{1}{x}\right\},$$
or
$${}_1F_1\{\alpha;\ \rho;\ x\}=\frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}(-x)^{-\alpha}\,{}_2F_0\left\{\alpha,\ 1+\alpha-\rho;\ -\frac{1}{x}\right\}.$$
We have thus obtained the previous result.

§ 13. From the result (A) of the last paragraph but one, we can immediately shew that, if $|\arg(-x)|<\frac{3\pi}{2}$,
$$\frac{\Gamma(1-\rho)}{\Gamma(1-\alpha)}{}_1F_1\{\alpha;\ \rho;\ x\}+\frac{\Gamma(\rho-1)}{\Gamma(\rho-\alpha)}(-x)^{1-\rho}\,{}_1F_1\{\alpha-\rho+1;\ 2-\rho;\ x\}$$
$$=e^x(-x)^{\alpha-\rho}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha;\ \frac{1}{x}\right\}.$$
For putting $\rho-\alpha$ for α in the result (A) we have, if $|\arg x|<\frac{3\pi}{2}$,
$$\frac{\Gamma(1-\rho)}{\Gamma(1-\alpha)}{}_1F_1\{\rho-\alpha;\ \rho;\ x\}+\frac{\Gamma(\rho-1)}{\Gamma(\rho-\alpha)}{}_1F_1\{1-\alpha;\ 2-\rho;\ x\}\,x^{1-\rho}$$
$$=x^{\alpha-\rho}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha;\ -\frac{1}{x}\right\}.$$
Now
$${}_1F_1\{\rho-\alpha;\ \rho;\ x\}=e^x\,{}_1F_1\{\alpha;\ \rho;\ -x\},$$
and
$${}_1F_1\{1-\alpha;\ 2-\rho;\ x\}=e^x\,{}_1F_1\{1-\rho+\alpha;\ 2-\rho;\ -x\}.$$
Thus, if $|\arg x|<\frac{3\pi}{2}$, we have
$$\frac{\Gamma(1-\rho)}{\Gamma(1-\alpha)}{}_1F_1\{\alpha;\ \rho;\ -x\}+\frac{\Gamma(\rho-1)}{\Gamma(\rho-\alpha)}x^{1-\rho}\,{}_1F_1\{1-\rho+\alpha;\ 2-\rho;\ -x\}$$
$$=e^{-x}x^{\alpha-\rho}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha;\ -\frac{1}{x}\right\}.$$
Changing x into $-x$ we have the given result.

When $R(x)>0$, this result is equivalent to two different relations. By eliminating the function ${}_1F_1\{\alpha-\rho+1;\ 2-\rho;\ x\}$ we can obtain the asymptotic expansion previously given for ${}_1F_1\{\alpha;\ \rho;\ x\}$ when $R(x)>0$.

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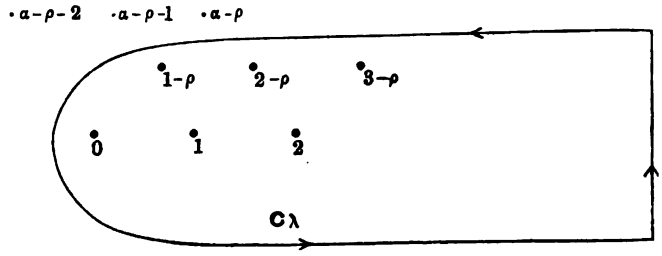
§ 14. We proceed to give a direct proof of the asymptotic equality of § 13.



Let C_λ be a contour as in the figure which encloses the points

$$0,\ 1,\ 2,\ 3, \ldots$$

$$1-\rho,\ 2-\rho,\ 3-\rho,\ \ldots,$$

and excludes the points $\alpha-\rho$, $\alpha-\rho-1$, $\alpha-\rho-2$, ..., and which is that part of a contour embracing the real axis which is included within a circle of radius λ together with an arc of this circle.

Consider
$$-\frac{1}{2\pi\iota}\sum_{t=0}^{\infty}\int_{C_\lambda}\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}x^s\,ds.$$

It is equal to

$$-\frac{1}{2\pi\iota}\int_{C_\lambda}\sum_{t=0}^{N-1}\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}x^s\,ds-\frac{1}{2\pi\iota}\sum_{t=N}^{\infty}\int_{C_\lambda}\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}x^s\,ds$$

$$=-\frac{1}{2\pi\iota}\int_{C_\lambda}\sum_{t=0}^{N-1}{}''\left\{\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}-\sum_{r=0}^{R}\frac{c_r(s)}{t^{s+\rho-\alpha+r+1}}\right\}x^s\,ds$$

$$-\frac{1}{2\pi\iota}\sum_{t=N}^{\infty}\int_{C_{\lambda-t}}\frac{\Gamma(-s)\,\Gamma(1+\rho-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s)}x^{s+t}\,ds,$$

the double accent denoting that the summation $\sum_{r=0}^{R}$ does not exist when $t=0$. The coefficients $c_r(s)$ are analogous to the polynomials introduced in § 6, and are defined by the equality (1) below.

The integral is thus equal to

$$-\frac{1}{2\pi\iota}\int_{C_\lambda}\sum_{t=0}^{\infty}{}''\left\{\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}-\sum_{r=0}^{R}\frac{c_r(s)}{t^{s+\rho-\alpha+r+1}}\right\}x^s\,ds$$

$$-\frac{1}{2\pi\iota}\sum_{t=N}^{\infty}\frac{x^t}{\Gamma(t+1)}\int_{C_{\lambda-t}}\frac{\Gamma(-s)\,\Gamma(1+\rho-s)}{\Gamma(1-\alpha-s)}x^s\,ds+I_N,$$

where
$$I_N=\frac{1}{2\pi\iota}\int_{C_\lambda}\sum_{t=N}^{\infty}\left\{\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}-\sum_{r=0}^{R}\frac{c_r(s)}{t^{s+\rho-\alpha+r+1}}\right\}x^s\,ds.$$

Now, if t/λ is large,

$$\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}-\sum_{r=0}^{R}\frac{c_r(s)}{t^{s+\rho-\alpha+r+1}}=\frac{J_R(t)}{t^{s+\rho-\alpha+R+1}},\tag{1}$$

where $|J_R(t)|$ can be made as small as we please, by taking t sufficiently large, for all finite values of R.

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Hence
$$I_N = \frac{1}{2\pi\iota}\int_{C_\lambda}\sum_{t=N}^{\infty}\frac{J_R(t)}{t^{s+\rho-\alpha+R+1}}x^s\,ds.$$

And therefore, since we can always take N so large that $|J_R(s,t)|<\eta$ if $t>N$ and t/λ is large, η being a positive quantity as small as we please, we have
$$|I_N| < \frac{\eta}{2\pi}\int_{C_\lambda}\sum_{t=N}^{\infty}\left|\frac{x^s}{t^{s+\rho-\alpha+R+1}}\right||ds|.$$

Making N tend to infinity and λ also tend to infinity in such a way that λ/N is small we see that $|I_N|$ tends to zero, if R be a sufficiently large finite positive integer.

Again, when N is very much larger than λ, the terms of the series
$$\sum_{t=N}^{\infty}\frac{x^t}{\Gamma(t+1)}\int_{C_{\lambda-t}}\frac{\Gamma(-s)\,\Gamma(1+\rho-s)}{\Gamma(1-\alpha-s)}x^s\,ds$$
all vanish.

Hence if C be the limit of the contour C_λ when λ tends to infinity, we have
$$-\frac{1}{2\pi\iota}\sum_{t=0}^{\infty}\int_C\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}x^s\,ds$$
$$= -\frac{1}{2\pi\iota}\int_C\left[S(s)-\sum_{r=0}^{R}c_r(s)\,\zeta(s+\rho-\alpha+r+1)\right]x^s\,ds,$$
where $S(s)$ is the function defined when $R(s)>R(\alpha-\rho)$ by the series
$$\sum_{t=0}^{\infty}\frac{\Gamma(-s+t)\,\Gamma(1-\rho+t-s)}{\Gamma(t+1)\,\Gamma(1-\alpha-s+t)}.$$

Thus, since $\zeta(s+\rho-\alpha+r+1)$ has no poles within the contour C, we have
$$-\frac{1}{2\pi\iota}e^{x}\int_C\frac{\Gamma(-s)\,\Gamma(1-\rho-s)}{\Gamma(1-\alpha-s)}x^s\,ds = -\frac{1}{2\pi\iota}\int_C S(s)\,x^s\,ds.$$

§ 15. We have incidentally seen that
$$\left[S(s)-\sum_{r=0}^{R}c_r(s)\,\zeta(s+\rho-\alpha+r+1)\right]\frac{1}{\Gamma(-s)\,\Gamma(-s+1-\rho)}$$
can be represented by an absolutely convergent series valid when $R(s)>R(\alpha-\rho)-R$.

Thus the sole finite singularities of $S(s)$ are poles at the points
$$0,\ 1,\ 2,\ \ldots\ldots$$
$$1-\rho,\ 2-\rho,\ 3-\rho,\ \ldots\ldots$$
$$\alpha-\rho,\ \alpha-\rho-1,\ \alpha-\rho-2,\ \ldots\ldots$$

And, if $R(s)>R(\alpha-\rho)-R$, and the coefficients a_r be suitably determined,
$$\frac{S(s)}{\Gamma(-s)\,\Gamma(-s+1-\rho)}-\sum_{r=0}^{R}\frac{a_r}{s+\rho-\alpha+r}=F(s),$$
where $F(s)$ is an integral function.

By Gauss' theorem we know that
$$\frac{S(s)}{\Gamma(-s)\,\Gamma(-s+1-\rho)}=\frac{\Gamma(s+\rho-\alpha)}{\Gamma(1-\alpha)\,\Gamma(\rho-\alpha)}.$$

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We thus verify the previous result and shew that

$$a_r = \frac{(-)^r}{r!\,\Gamma(1-\alpha)\,\Gamma(\rho-\alpha)}.$$

We now have

$$-\frac{1}{2\pi\iota}\int_C \frac{\Gamma(-s)\,\Gamma(1-\rho-s)}{\Gamma(1-\alpha-s)}\,x^s ds = -\frac{1}{2\pi\iota}e^{-x}\int_C \frac{\Gamma(-s)\,\Gamma(-s+1-\rho)\,\Gamma(s+\rho-\alpha)}{\Gamma(1-\alpha)\,\Gamma(\rho-\alpha)}\,x^s ds.$$

The first integral is equal to the sum of the residues of the subject of integration inside the contour C.

It is therefore equal to

$$\sum_{n=0}^{\infty}\frac{\Gamma(1-\rho-n)}{\Gamma(1-\alpha-n)}\frac{(-x)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-)^n x^{1-\rho+n}\,\Gamma(\rho-1-n)}{n!\,\Gamma(\rho-\alpha-n)}$$

$$= \frac{\Gamma(1-\rho)}{\Gamma(1-\alpha)}\,{}_1F_1\{\alpha;\ \rho;\ -x\} + \frac{\Gamma(\rho-1)}{\Gamma(\rho-\alpha)}\,x^{1-\rho}\,{}_1F_1\{\alpha-\rho+1;\ 2-\rho;\ -x\}.$$

The second integral is zero along a contour at infinity for which the real part of s is greater than a finite negative quantity, provided $|\arg x| < 3\pi/2$.

Hence it is equal to

$$x^{\alpha-\rho}\left\{\sum_{n=0}^{k}\frac{(-)^n}{n!}\frac{\Gamma(n-\alpha+\rho)\,\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)\,\Gamma(\rho-\alpha)}\,x^{-n} + \frac{J_k}{x^k}\right\},$$

where $|J_k|$ tends to zero as $|x|$ tends to infinity.

It is thus asymptotically equal to

$$x^{\alpha-\rho}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha;\ -\frac{1}{x}\right\}.$$

Finally therefore, if $|\arg(-x)| < 3\pi/2$,

$$\frac{\Gamma(1-\rho)}{\Gamma(1-\alpha)}\,{}_1F_1\{\alpha;\ \rho;\ x\} + \frac{\Gamma(\rho-1)}{\Gamma(\rho-\alpha)}(-x)^{1-\rho}\,{}_1F_1\{\alpha-\rho+1;\ 2-\rho;\ x\}$$

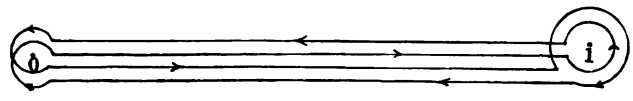
$$= e^x(-x)^{\alpha-\rho}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha;\ \frac{1}{x}\right\}.$$

§ 16. To verify the complete asymptotic expansion for ${}_1F_1(\alpha;\ \rho;\ x)$, which was given in § 9, we will obtain the result directly by the use of Pochhammer's contours and their deformation.

By his result* we know that for all values of p and q

$$\int_P v^{p-1}(1-v)^{q-1}\,dv = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\,\frac{1}{(1-e^{2\pi\iota p})(1-e^{2\pi\iota q})},$$

the integral being taken round the contour P of the figure.



* Pochhammer, *Mathematische Annalen*, Bd. xxxv. pp. 470—526.

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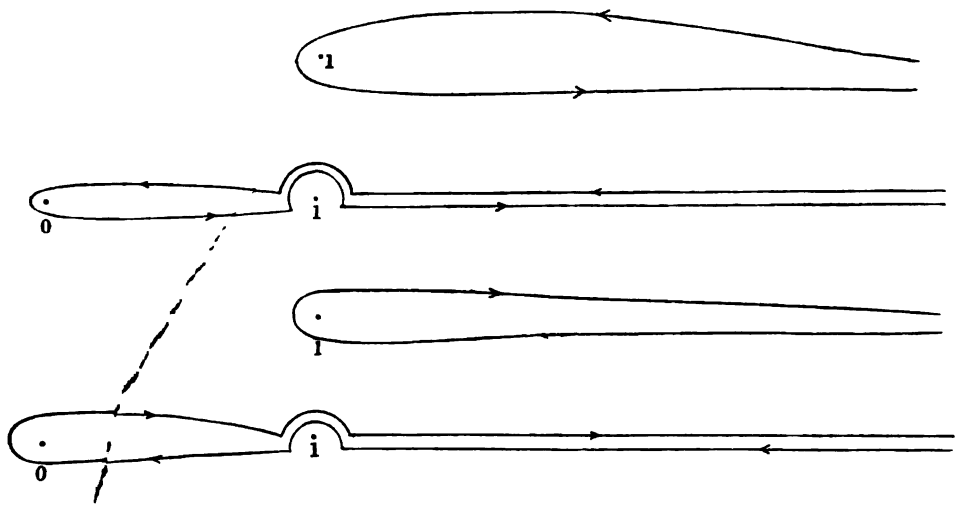
The argument of $v^{p-1}(1-v)^{q-1}$ on the initial line is zero. Then we make a positive circuit round 1, so that the argument of the return line is $e^{2\pi i q}$. Then follows a positive circuit of the point 0, so that the argument on the following direct line is $e^{2\pi i (p+q)}$. Then we have a negative circuit of the point 1, so that the argument on the second return line is $e^{2\pi i p}$. Finally a negative circuit of the point 0 brings us back to the starting-point with zero argument.

Now
$$ {}_1F_1\{\alpha;\ \rho;\ -x\}\frac{\Gamma(\alpha)}{\Gamma(\rho)} = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(n+1)\,\Gamma(\rho+n)}(-x)^n $$
$$ = \frac{1}{\Gamma(\rho-\alpha)(e^{2\pi i\alpha}-1)(e^{2\pi i(\rho-\alpha)}-1)}\int_P \sum_{n=0}^{\infty}\frac{(-x)^n}{\Gamma(n+1)}v^{\alpha+n-1}(1-v)^{\rho-\alpha-1}\,dv $$
$$ = \frac{1}{\Gamma(\rho-\alpha)(e^{2\pi i\alpha}-1)(e^{2\pi i(\rho-\alpha)}-1)}\int_P e^{-xv}\,v^{p-1}(1-v)^{q-1}\,dv, $$
where $p=\alpha$, $q=\rho-\alpha$.

Suppose that $R(x)>0$. Then we may modify the Pochhammer contour till it consists of the four contours of the figure.



On the first contour the argument of $v^{p-1}(1-v)^{q-1}$ is initially $e^{-\pi i(q-1)}$ and finally $e^{\pi i(q-1)}$.

On the second contour the argument is initially $e^{\pi i(q-1)}$. After the semi-circle round the point 1 it is $e^{2\pi i(q-1)}$. After the loop round the origin it is $e^{2\pi i(p+q-2)}$, and finally it is $e^{\pi i(2p+q-3)}$.

On the third contour the argument is initially $e^{\pi i(2p+q-3)}$, and finally, after the negative loop round the point 1, it is $e^{\pi i(2p-q-1)}$.

On the fourth contour the argument is initially $e^{\pi i(2p-q-1)}$. After the positive semi-circle round the point 1, it is $e^{\pi i(2p-2)}$. After the loop round the origin it is $e^{0\pi i}$, and finally it is $e^{-\pi i(q-1)}$.

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If now we add together the integrals round the four contours we obtain

$$\int_P e^{-xv} v^{p-1} (1-v)^{q-1} dv$$

$$= (1 - e^{2\pi i p}) \int e^{-x(1+v)} (1+v)^{p-1} (-v)^{q-1} dv$$

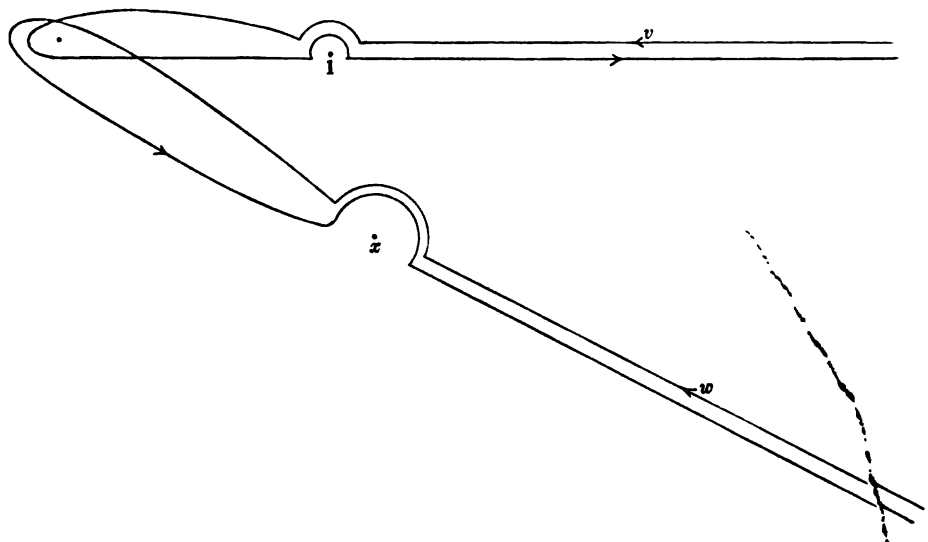
$$+ e^{\pi i p} (1 - e^{2\pi i q}) \int e^{-xv} (-v)^{p-1} (1-v)^{q-1} dv,$$

the integrals being taken round the usual contour G for the gamma function deformed in the second case so as to pass above the point $v = 1$, and $(-v)^{q-1}$ and $(-v)^{p-1}$ having their principal values which are real when v is real and negative.

Suppose now that $\arg x$ lies between $-\frac{\pi}{2}$ and 0. Then by the substitution $xv = w$ the second integral becomes

$$\int e^{-w} \left(-\frac{w}{x}\right)^{p-1} \left(1 - \frac{w}{x}\right)^{q-1} \frac{dw}{x}$$

round the new contour of the figure.



Suppose now that we expand $\left(1 - \frac{w}{x}\right)^{q-1}$ by the summable divergent series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1-q)}{\Gamma(1-q)\,\Gamma(n+1)} \left(\frac{w}{x}\right)^n.$$

This will be valid in the plane dissected by the line passing from x in a direction away from the origin to infinity. Therefore when this series replaces $\left(1 - \frac{w}{x}\right)^{q-1}$ under the integral sign, the value of the integral along the w contour will be equal to that of the integral along the contour G which embraces the real axis if $\arg x$ lies between $-\frac{\pi}{2}$ and 0, which is the case represented in the figure.

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Similarly in the first integral we may employ the summable divergent series for $(1+v)^{p-1}$, though in this case since $R(x)>0$ no such difficulties arise.

Therefore
$$\int_P e^{-xv} v^{p-1} (1-v)^{q-1}\, dv$$
$$= (1-e^{2\pi\iota p})\, e^{-x} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-p)}{\Gamma(1-p)\,\Gamma(n+1)\,x^{n+q}} \cdot \frac{2\pi}{\iota\Gamma(1-n-q)}$$
$$+ e^{\pi\iota p}(1-e^{2\pi\iota q}) \sum_{n=0}^{\infty} \frac{\Gamma(n+1-q)\,(-)^n}{\Gamma(1-q)\,\Gamma(n+1)\,x^{n+p}} \cdot \frac{2\pi}{\iota\Gamma(1-n-p)},$$

the principal values of x^{n+q} and x^{n+p} being taken, and the series being asymptotic in Poincaré's sense†.

Thus if $R(x)>0$ and if also $\arg x$ lies between $-\frac{\pi}{2}$ and 0,

$${}_1F_1(\alpha;\ \rho;\ -x)\frac{\Gamma(\alpha)\,\Gamma(\rho-\alpha)}{\Gamma(\rho)} = e^{-x}e^{-\pi\iota q}\sum_{n=0}^{\infty}(-)^n \frac{\Gamma(n+1-p)\,\Gamma(n+q)}{\Gamma(1-p)\,\Gamma(n+1)\,x^{q+n}}$$
$$+ \sum_{n=0}^{\infty} \frac{\Gamma(n+p)\,\Gamma(n+1-q)}{\Gamma(1-q)\,\Gamma(n+1)\,x^{n+p}},$$

or
$${}_1F_1(\alpha;\ \rho;\ -x) = \frac{e^{-\pi\iota(\rho-\alpha)}}{x^{\rho-\alpha}}\frac{\Gamma(\rho)}{\Gamma(\alpha)} e^{-x}\, {}_2F_0\left(1-\alpha,\ \rho-\alpha;\ -\frac{1}{x}\right)$$
$$+ \frac{\Gamma(\rho)}{x^{\alpha}\,\Gamma(\rho-\alpha)}\, {}_2F_0\left(\alpha,\ 1-\rho+\alpha;\ \frac{1}{x}\right)$$
$$= e^{-x}\frac{\Gamma(\rho)}{\Gamma(\alpha)}(-x)^{\alpha-\rho}\, {}_2F_0\left(1-\alpha,\ \rho-\alpha;\ -\frac{1}{x}\right)$$
$$+ \frac{\Gamma(\rho)}{x^{\alpha}\,\Gamma(\rho-\alpha)}\, {}_2F_0\left(\alpha,\ 1-\rho+\alpha;\ \frac{1}{x}\right), \qquad (1)$$

where $(-x)^{\alpha-\rho}$ has its principal value whose argument lies between $\frac{1}{2}\pi(\alpha-\rho)$ and $\pi(\alpha-\rho)$.

The reader will readily supply the deformation which it is convenient to take for the original Pochhammer contour in the case when $\arg x$ lies between 0 and $\frac{1}{2}\pi$. By it we establish that the formula (1) is valid whenever $R(x)>0$.

Utilise now the identity ${}_1F_1\{\alpha;\ \rho;\ x\} = e^x\, {}_1F_1\{\rho-\alpha;\ \rho;\ -x\}$ or if it is preferred modify again the contours of the original Pochhammer circuit and we establish that asymptotically

$${}_1F_1(\alpha;\ \rho;\ x) = e^x x^{\alpha-\rho}\frac{\Gamma(\rho)}{\Gamma(\alpha)}\, {}_2F_0\left(\rho-\alpha,\ 1-\alpha;\ \frac{1}{x}\right)$$
$$+ \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}(-x)^{-\alpha}\, {}_2F_0\left(1-\rho+\alpha,\ \alpha;\ -\frac{1}{x}\right),$$

whether $R(x)$ be positive or negative. In this equality $(-x)^{-\alpha}$ has its argument between $-\alpha\pi$ and $\alpha\pi$, and $x^{\alpha-\rho}$ has its argument between $-(\alpha-\rho)\pi$ and $(\alpha-\rho)\pi$.

This is the result previously obtained.

† See § 5 of the first memoir by the author quoted in § 1 of the present paper.

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§ 17. We have now completed the discussion of the asymptotic behaviour of the function ${}_1F_1\{\alpha;\ \rho;\ x\}$.

But before proceeding to other investigations we may note the connection between our analysis and the theory of linear differential equations.

We saw that

$${}_1F_1\{\alpha;\ \rho;\ x\} \text{ and } x^{1-\rho}\,{}_1F_1\{\alpha-\rho+1;\ 2-\rho;\ x\}$$

were the two independent solutions of the linear differential equation

$$x\frac{d^2y}{dx^2}-(x-\rho)\frac{dy}{dx}-\alpha y=0.$$

Hence from our results we also see that

$$x^{-\alpha}\,{}_2F_0\left\{\alpha,\ 1+\alpha-\rho;\ -\frac{1}{x}\right\},$$

and

$$e^{-x}\,x^{\alpha-\rho}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha;\ -\frac{1}{x}\right\},$$

are two independent asymptotic solutions of the type first introduced to analysis by Poincaré. These asymptotic solutions for this particular equation have been obtained by Jacobsthal*. If the associated series were convergent one integral would be regular and the other normal†.

The function $F_\rho(x)=\sum\limits_{n=0}^{\infty}\dfrac{x^n\,\Gamma(\rho)}{\Gamma(\rho+n)}$.

§ 18. If we put $\alpha=1$ in the asymptotic expansion which has been obtained for ${}_1F_1\{\alpha;\ \rho;\ x\}$ we obtain

$$F_\rho(x)=e^x x^{1-\rho}\,\Gamma(\rho)\,{}_2F_0\left\{\rho-1,\ 0;\ \frac{1}{x}\right\}+(\rho-1)(-x)^{-1}\,{}_2F_0\left\{2-\rho,\ 1;\ -\frac{1}{x}\right\}.$$

In this equality $x^{1-\rho}$ is such that $|\arg x|<\pi$, that is to say, $x^{1-\rho}$ has its principal value with respect to a cross-cut along the negative half of the real axis.

Evidently

$${}_2F_0\left\{\rho-1,\ 0;\ \frac{1}{x}\right\}=1,$$

and

$${}_2F_0\left\{2-\rho,\ 1;\ -\frac{1}{x}\right\}=1+\sum_{n=1}^{\infty}\frac{(\rho-2)(\rho-3)\ldots(\rho-1-n)}{x^n}.$$

Hence, if $|\arg x|<\pi$, we have the asymptotic equality

$$F_\rho(x)-e^x x^{1-\rho}\,\Gamma(\rho)=-\sum_{n=1}^{N}\frac{(\rho-1)\ldots(\rho-n)}{x^n}+J_N, \tag{A}$$

where $|J_N x^N|$ tends to zero as $|x|$ tends to infinity.

We can immediately verify this result when $\rho=1$ or 2.

For $\quad F_\rho(x)=e^x$ when $\rho=1$, and $F_\rho(x)=\dfrac{1}{x}(e^x-1)$ when $\rho=2$.

* Jacobsthal, *Mathematische Annalen*, Vol. LVI. pp. 129—154.

† Forsyth, *Linear Differential Equations* [*Theory of Differential Equations*, Part III. Vol. IV.], Chapter VII.

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§ 19. The previous asymptotic expansion (A) can be readily obtained *de novo* as follows, when $R(\rho) > 0$.

We have
$$F_\rho(x) = 1 + x \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\Gamma(n+1)\Gamma(\rho)}{\Gamma(\rho+n+1)}$$
$$= 1 + x \sum_{n=0}^{\infty} \int_0^1 \frac{x^n (1-y)^n}{n!} y^{\rho-1}\, dy$$
$$= 1 + xe^x \int_0^1 e^{-xy} y^{\rho-1}\, dy.$$

The process of summation under the sign of integration is justified by the fact that $F_\rho(x)$ is uniformly convergent for all finite values of $|x|$.

Hence
$$F_\rho(x) = 1 + e^x x^{1-\rho} \int_0^x e^{-\eta} \eta^{\rho-1}\, d\eta.$$

From the origin and the point x respectively draw two parallel lines A and B to infinity making an acute angle with the positive direction of the real axis. When $R(x) < 0$, the line from x must not cut the negative half of the real axis.

Then
$$\int_0^x + \int_x^{\infty(B)} = \int_0^{\infty(A)}.$$

Hence, if $|\arg x| < \pi$,
$$F_\rho(x) = 1 + e^x x^{1-\rho} \left\{ \int_0^{\infty(A)} e^{-\eta} \eta^{\rho-1} d\eta - \int_x^{\infty(B)} e^{-\eta} \eta^{\rho-1}\, d\eta \right\}$$
$$= 1 + e^x x^{1-\rho} \Gamma(\rho) - e^x x^{1-\rho} \int_x^{\infty(B)} e^{-\eta} \eta^{\rho-1}\, d\eta$$
$$= 1 + e^x x^{1-\rho} \Gamma(\rho) - x^{1-\rho} \int_0^{\infty(A)} e^{-y} (x+y)^{\rho-1}\, dy.$$

Thus
$$F_\rho(x) - e^x x^{1-\rho} \Gamma(\rho) = -\int_0^{\infty(A)} e^{-y} \left\{ \left(1 + \frac{y}{x}\right)^{\rho-1} - 1 \right\} dy.$$

If now we expand $\left(1 + \frac{y}{x}\right)^{\rho-1} - 1$ in the form, valid when $\left|\frac{y}{x}\right| < 1$,
$$\sum_{n=1}^{\infty} \frac{(\rho-1)\ldots(\rho-n)}{n!} \left(\frac{y}{x}\right)^n, \tag{1}$$

and substitute under the sign of integration, we shall obtain an asymptotic expansion of the integral when $|x|$ is very large and $|\arg x| < \pi$ since the expansion (1) is summable over the area of the plane dissected by a line passing from the point $-x$ away from the origin to infinity*.

Finally therefore we have the asymptotic expansion (A) of the previous paragraph, if $R(\rho) > 0$.

When $R(\rho) < 0$, and $\rho + n\,(n = 0, 1, \ldots \infty)$ is not zero, we can at once establish the same result by substituting contour integrals for the line integrals which have been employed.

* See the general theorem proved in § 5 of my original memoir.

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PART II.

The function ${}_0F_1\{\rho;\ x\}$.

§ 20. We now proceed to consider the function

$$ {}_0F_1\{\rho;\ x\} = 1 + \frac{x}{1.\rho} + \frac{x^2}{1.2.\rho(\rho+1)} + \ldots = \Gamma(\rho) \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)\,\Gamma(\rho+n)}, $$

wherein ρ may have any value real or complex which is not zero or a negative integer.

The function satisfies the differential equation

$$ x\frac{d^2y}{dx^2} + \rho\frac{dy}{dx} - y = 0. $$

A second independent solution is $x^{1-\rho}\,{}_0F_1\{2-\rho;\ x\}$. The differential equation has been considered by Pochhammer*.

The reader may feel surprise that the equation was not considered before the discussion just completed. The reason for this, however, is that by a relation due originally to Kummer† it is possible to make the theory of the function ${}_0F_1\{\rho;\ x\}$ a particular case of that of the function ${}_1F_1\{\alpha;\ \rho;\ x\}$. And this method of treatment seems to be more simple than any other which we could have adopted.

The reader will notice the connection between ${}_0F_1\{\rho;\ x\}$ and Bessel's function.

For
$$ J_n(z) = \sum_{\mu=0}^{\infty} \frac{(-)^\mu \left(\frac{z}{2}\right)^{2\mu+n}}{\Gamma(\mu+1)\,\Gamma(\mu+n+1)}, $$

and therefore

$$ \left(\frac{z}{2}\right)^{-n} J_n(z) = \sum_{\mu=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^\mu}{\Gamma(\mu+1)\,\Gamma(\mu+n+1)} = \frac{1}{\Gamma(n+1)}\,{}_0F_1\left\{n+1,\ -\frac{z^2}{4}\right\}. $$

All the following investigation can therefore be translated into an investigation of the nature of the asymptotic expansions of Bessel's function $J_n(z)$, when $|z|$ is large, for general complex values of the parameter n, negative integral values being excluded.

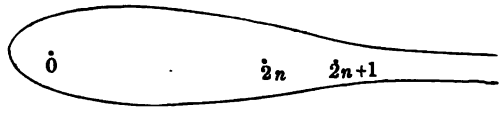
§ 21. We proceed now to prove Kummer's result

$$ {}_0F_1\{\rho;\ x\} = e^{-2x^{\frac{1}{2}}}\,{}_1F_1\{\rho-\tfrac{1}{2};\ 2\rho-1;\ 4x^{\frac{1}{2}}\} = e^{2x^{\frac{1}{2}}}\,{}_1F_1\{\rho-\tfrac{1}{2};\ 2\rho-1;\ -4x^{\frac{1}{2}}\}. $$

We have
$$ e^{-2x^{\frac{1}{2}}}\,{}_0F_1\{\rho;\ x\} = \Gamma(\rho) \sum_{n=0}^{\infty} \frac{e^{-2x^{\frac{1}{2}}}\,x^n}{\Gamma(n+1)\,\Gamma(\rho+n)}. $$

Now consider
$$ -\frac{1}{2\pi i}\int_C \Gamma(2n-s)\,(4x)^{\frac{s}{2}}\,ds $$

taken round a contour C enclosing the points $s = 2n,\ 2n+1,\ 2n+2\ldots$ as in the figure.



* Pochhammer, *Mathematische Annalen*, Vol. XXXVIII. pp. 228—240 and Vol. XLI. pp. 174—178.

† Kummer, *Crelle*, Bd. XV. p. 139.

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By Cauchy's theory of residues it is equal to

$$\sum_{r=0}^{\infty} \frac{(-)^r (4x)^{\frac{2n+r}{2}}}{r!} = e^{-2x^{\frac{1}{2}}} . (4x)^n.$$

Hence $$e^{-2x^{1/2}} {}_0F_1\{\rho;\, x\} = -\frac{\Gamma(\rho)}{2\pi\iota} \sum_{n=0}^{\infty} \int_C \frac{\Gamma(2n-s)(4x)^{s/2}}{4^n \Gamma(n+1)\Gamma(\rho+n)}\, ds.$$

Now by the multiplication theorem for the gamma function

$$\Gamma(2n-s) = \frac{2^{2n-s-1}}{\sqrt{\pi}} \Gamma\left(n-\frac{s}{2}\right) \Gamma\left(n-\frac{s}{2}+\frac{1}{2}\right).$$

Hence $$e^{-2x^{1/2}} {}_0F_1\{\rho;\, x\} = -\frac{\Gamma(\rho)}{2\pi\iota} \sum_{n=0}^{\infty} \int_C \frac{x^{s/2}}{2\sqrt{\pi}} \frac{\Gamma\left(n-\frac{s}{2}\right)\Gamma\left(n-\frac{s}{2}+\frac{1}{2}\right)}{\Gamma(n+1)\Gamma(\rho+n)}\, ds,$$

and this integral may by a similar argument to that employed in § 14 be written, if $R(\rho+s) > 1/2$,

$$-\frac{1}{2\pi\iota} \int_C \frac{x^{s/2}}{2\sqrt{\pi}} {}_2F_1\left\{-\frac{s}{2},\, -\frac{s}{2}+\frac{1}{2};\, \rho;\, 1\right\} \Gamma\left(-\frac{s}{2}\right) \Gamma\left(\frac{1}{2}-\frac{s}{2}\right) ds$$
$$= -\frac{1}{2\pi\iota} \int \Gamma(-s)\, 2^s x^{s/2} {}_2F_1\left\{-\frac{s}{2},\, -\frac{s}{2}+\frac{1}{2};\, \rho;\, 1\right\} ds.$$

Now by Gauss' formula

$${}_2F_1\left\{-\frac{s}{2},\, -\frac{s}{2}+\frac{1}{2};\, \rho;\, 1\right\} = \frac{\Gamma(\rho)\Gamma(\rho+s-\frac{1}{2})}{\Gamma\left(\rho+\frac{s}{2}\right)\Gamma\left(\rho+\frac{s}{2}-\frac{1}{2}\right)} \text{ if } R(\rho+s) > \tfrac{1}{2}$$

$$= \frac{\Gamma(\rho)\Gamma(\rho+s-\frac{1}{2})\, 2^{2\rho+s-1}}{\Gamma(2\rho+s-1)\, 2\sqrt{\pi}} \text{ by the multiplication formula.}$$

Hence if at all points on the contour of integration $R(\rho+s) > \frac{1}{2}$, we have

$$e^{-2x^{1/2}} {}_0F_1\{\rho;\, x\} = -\frac{1}{2\pi\iota} \int_C \Gamma(-s)\, x^{s/2} \frac{2^{2\rho+2s-2}\Gamma(\rho)\Gamma(\rho+s-\frac{1}{2})}{\Gamma(2\rho+s-1)\sqrt{\pi}}\, ds$$
$$= \sum_{m=0}^{\infty} \frac{(-)^m}{m!} \frac{x^{\frac{m}{2}}}{\sqrt{\pi}} 2^{2\rho+2m-2} \frac{\Gamma(\rho)\Gamma(\rho+m-\frac{1}{2})}{\Gamma(2\rho+m-1)}$$
$$= \sum_{m=0}^{\infty} \frac{(-4x^{1/2})^m}{\Gamma(m+1)} \cdot \frac{\Gamma(\rho+m-\frac{1}{2})}{\Gamma(2\rho+m-1)} \frac{2^{2\rho-2}\Gamma(\rho)}{\sqrt{\pi}}$$
$$= {}_1F_1\{\rho-\tfrac{1}{2};\, 2\rho-1;\, -4x^{1/2}\}.$$

We have thus shewn that

$${}_0F_1\{\rho;\, x\} = e^{2x^{1/2}} {}_1F_1\{\rho-\tfrac{1}{2};\, 2\rho-1;\, -4x^{1/2}\},$$

provided $\frac{1}{2} - R(\rho) < \epsilon < 0$. But it is evident that, except for those real values of ρ which introduce infinities into the equality, both sides are continuous. The formula is therefore true in general. And this at once appears if we modify the foregoing proof by introducing simple Riemann ζ functions as in § 14.

Since $\arg x$ is unrestricted in the proof we have equally

$${}_0F_1\{\rho;\, x\} = e^{-2x^{1/2}} {}_1F_1\{\rho-\tfrac{1}{2};\, 2\rho-1;\, 4x^{1/2}\}.$$

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§ 22. On account of the importance of the foregoing relation we will give an alternative proof, which however, though more simple, is not so intimately connected with the general plan of this paper.

We will first shew that, if S_ν denotes the series

$$\frac{(-2)^\nu \Gamma(2\rho-1)}{\Gamma(2\rho+\nu-1)} \sum_{m=0}^{\nu} (-2)^m \frac{\Gamma(\rho-1/2+m)}{\Gamma(\rho-1/2)\, m!} \cdot \frac{\Gamma(2\rho+\nu-1)}{(\nu-m)!\, \Gamma(2\rho+m-1)},$$

then $$S_\nu = 0 \text{ if } \nu \text{ be odd,}$$

$$= \frac{\Gamma(\rho)}{\Gamma(\rho+\nu/2)\, \Gamma(\nu/2+1)} \text{ if } \nu \text{ be even.}$$

Write Σ_ν for the series $\sum_{m=0}^{\nu} (-2)^m \frac{\Gamma(\rho-1/2+m)}{\Gamma(\rho-1/2)\, m!} \frac{\Gamma(2\rho+\nu-1)}{(\nu-m)!\, \Gamma(2\rho+m-1)}$.

Then it is evident that Σ_ν is the coefficient of x^ν in the expansion of

$$(1+2x)^{-\rho+1/2} (1+x)^{2\rho+\nu-2}$$

in ascending powers of x.

Thus $$\Sigma_\nu = \frac{1}{2\pi\iota} \int \frac{(1+x)^{\nu+2\rho-2}}{x^{\nu+1}} (1+2x)^{-\rho+1/2}\, dx,$$

the integral being taken round a small circle enclosing the origin.

Make now the substitution $x/(1+x) = \xi$, so that

$$1-\xi = 1/(1+x) \text{ and } (1+\xi)/(1-\xi) = 1+2x.$$

Then we have $$\Sigma_\nu = \frac{1}{2\pi\iota} \int \frac{d\xi}{\xi^{\nu+1}} (1-\xi^2)^{-\rho+1/2},$$

the new contour being a small oval enclosing $\xi = 0$.

Thus Σ_ν is the coefficient of ξ^ν in the expansion of $(1-\xi^2)^{-\rho+1/2}$.

Therefore $$\Sigma_\nu = 0 \text{ if } \nu \text{ be odd,}$$

$$= \frac{\Gamma(\rho-1/2+\nu/2)}{\Gamma(\rho-1/2)\, \Gamma(\nu/2+1)} \text{ if } \nu \text{ be even.}$$

Thus $S_\nu = 0$ if ν be odd; and, if ν be even,

$$S_\nu = \frac{\Gamma(2\rho-1)}{\Gamma(\rho-1/2)} \cdot \frac{\Gamma(\rho-1/2+\nu/2)}{\Gamma(2\rho-1+\nu)} \frac{(-2)^\nu}{\Gamma(\nu/2+1)}.$$

But by the multiplication theorem for the gamma function

$$\frac{\Gamma(2\rho-1)}{\Gamma(\rho-1/2)} = \frac{2^{2\rho-2}}{\sqrt{\pi}} \Gamma(\rho).$$

Hence, when ν is even, $$S_\nu = \frac{\Gamma(\rho)}{\Gamma(\nu/2+1)\, \Gamma(\rho+\nu/2)}.$$

Consider now the product

$$e^{-2x^{1/2}} {}_1F_1 \{\rho-1/2;\ 2\rho-1;\ 4x^{1/2}\}$$

$$= \sum_{n=0}^{\infty} \frac{(-2x^{1/2})^n}{n!} \sum_{m=0}^{\infty} \frac{\Gamma(\rho+m-1/2)\, \Gamma(2\rho-1)\, (4x^{1/2})^m}{\Gamma(\rho-1/2)\, \Gamma(2\rho+m-1)\, \Gamma(m+1)}.$$

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This double series is unconditionally convergent: we may therefore rearrange it in a series proceeding in ascending powers of $x^{1/2}$. And the coefficient of $x^{\nu/2}$ is S_ν.

Thus the product is equal to

$$\sum_{n=0}^{\infty} \frac{\Gamma(\rho)}{\Gamma(\rho+n)\,\Gamma(n+1)} x^n = {}_0F_1\{\rho;\ x\}.$$

The theorem is therefore established.

§ 23. We may now obtain the asymptotic expansion of ${}_0F_1\{\rho;\ x\}$ from that of

$${}_1F_1\{\alpha;\ \rho;\ x\}.$$

For by the previous result (§ 16) we have

$$\begin{aligned}
&{}_1F_1\{\rho-1/2;\ 2\rho-1;\ 4x^{1/2}\} \\
&\quad = e^{4x^{1/2}} \frac{\Gamma(2\rho-1)}{\Gamma(\rho-1/2)} (4x^{1/2})^{-\rho+1/2}\, {}_2F_0\{\rho-1/2,\ 3/2-\rho;\ 1/(4x^{1/2})\} \\
&\quad + \frac{\Gamma(2\rho-1)}{\Gamma(\rho-1/2)} (-4x^{1/2})^{1/2-\rho}\, {}_2F_0\{\rho-1/2,\ 3/2-\rho;\ -1/(4x^{1/2})\}.
\end{aligned}$$

This result is valid if $|\arg x| < 2\pi$: we then have $|\arg x^{1/2}| < \pi$. If $\arg x = \theta$ and $0 < |\theta| < 2\pi$, we have $\arg(x^{1/2}) = \theta/2$, and we must take such a value of $(4x^{1/2})^{1/2-\rho}$ as is equal to $2^{1-2\rho} x^{1/4-\rho/2}$, where $\arg(x^{1/4-\rho/2}) = (1/4-\rho/2)\,\theta$.

We also have $\quad \arg(-x^{1/2}) = \theta/2 - \pi$ if $\theta/2$ be positive,

$\qquad\qquad\qquad\qquad = \theta/2 + \pi$ if $\theta/2$ be negative;

and therefore $\quad (-4x^{1/2})^{1/2-\rho} = 2^{1-2\rho} x^{1/4-\rho/2}$,

where $\quad \arg(x^{1/4-\rho/2}) = (\theta - 2\pi)(1/4-\rho/2)$ if $\theta/2$ be positive,

$\qquad\qquad\qquad\qquad = (\theta + 2\pi)(1/4-\rho/2)$ if $\theta/2$ be negative.

Thus, since $\quad \Gamma(2\rho-1) = \Gamma(\rho-1/2)\,\Gamma(\rho)\,2^{2\rho-2}/\sqrt{\pi}$,

$${}_0F_1\{\rho;\ x\} = x^{1/4-\rho/2} \frac{\Gamma(\rho)}{2\sqrt{\pi}} \left[e^{2x^{1/2}}\, {}_2F_0\{\rho-1/2,\ 3/2-\rho;\ 1/(4x^{1/2})\} + P e^{-2x^{1/2}}\, {}_2F_0\{\rho-1/2,\ 3/2-\rho;\ 1/(-4x^{1/2})\}\right],$$

where $P = e^{\pm\pi i(1/2-\rho)}$, the + or − sign being taken according as $\theta/2$ is negative or positive, and where $x^{1/4-\rho/2} = (x^{1/2})^{1/2-\rho}$, and $|\arg x^{1/2}| < \pi$ *.

We see that the complete asymptotic expansion for ${}_0F_1\{\rho;\ x\}$ persists in form as x makes a complete circuit round the origin, except when x crosses the positive direction of the real axis, when there is a sudden change in value in the second (and then negligible) series.

§ 24. Consider now the asymptotic expansion of $J_n(z)$ when $|z|$ is large, and n has any finite value real or complex except negative integral values. The quantity n may be zero.

We have $$J_n(z) = \left(\frac{z}{2}\right)^n \frac{1}{\Gamma(n+1)}\, {}_0F_1\left\{n+1,\ -\frac{z^2}{4}\right\}.$$

* This result is due to Stokes, and appears substantially in his long-forgotten paper 'On the discontinuity of arbitrary constants which appear in divergent developments,' *Cambridge Philosophical Transactions*, Vol. XI. (1868), pp. 414–418.

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We assume as part of the definition* of $J_n(z)$ that when n is complex $z^n = \exp\{n \log z\}$ and that the logarithm has its principal value whose argument lies between $-\pi$ and π.

Putting $2x^{\frac{1}{2}} = \iota z$ in the formula of § 21 we have

$$J_n(z) = \left(\frac{z}{2}\right)^n \frac{1}{\Gamma(n+1)} e^{-\iota z}\, {}_1F_1\{n+\tfrac{1}{2};\ 2n+1;\ 2\iota z\}.$$

Therefore for all values of $\arg z$ we have asymptotically

$$J_n(z) = \left(\frac{z}{2}\right)^n \frac{\Gamma(2n+1)}{\Gamma(n+1)\,\Gamma(n+\frac{1}{2})} \left[e^{\iota z}(2\iota z)^{-n-\frac{1}{2}}\, {}_2F_0\left\{n+\tfrac{1}{2},\ \tfrac{1}{2}-n;\ \frac{1}{2\iota z}\right\}\right.$$
$$\left. + e^{-\iota z}(-2\iota z)^{-n-\frac{1}{2}}\, {}_2F_0\left\{n+\tfrac{1}{2},\ \tfrac{1}{2}-n;\ -\frac{1}{2\iota z}\right\}\right],$$

or

$$J_n(z) = \frac{z^n}{\sqrt{2\pi}} \left[e^{\iota z}(\iota z)^{-n-\frac{1}{2}}\, {}_2F_0\left\{n+\tfrac{1}{2},\ \tfrac{1}{2}-n;\ \frac{1}{2\iota z}\right\}\right.$$
$$\left. + e^{-\iota z}(-\iota z)^{-n-\frac{1}{2}}\, {}_2F_0\left\{n+\tfrac{1}{2},\ \tfrac{1}{2}-n;\ -\frac{1}{2\iota z}\right\}\right]. \qquad \text{(A)}$$

Now the argument of $(\iota z)^{-n-\frac{1}{2}}$ lies between $\pm(n+\frac{1}{2})\pi$ and therefore we may put

$$(\iota z)^{-n-\frac{1}{2}} = e^{-\frac{\iota\pi}{2}(n+\frac{1}{2})}\, z^{-n-\frac{1}{2}},$$

where $z^{-n-\frac{1}{2}}$ is made uniform by a cross-cut along the positive half of the imaginary axis, and has an argument which lies between $\frac{3}{2}\pi(n+\frac{1}{2})$ and $-\frac{1}{2}\pi(n+\frac{1}{2})$.

Similarly the argument of $(-\iota z)^{-n-\frac{1}{2}}$ lies between $\pm(n+\frac{1}{2})\pi$, and we may put

$$(-\iota z)^{-n-\frac{1}{2}} = e^{\frac{\iota\pi}{2}(n+\frac{1}{2})}\, z^{-n-\frac{1}{2}},$$

where $z^{-n-\frac{1}{2}}$ is made uniform by a cross-cut along the negative half of the imaginary axis, and has an argument which lies between $-\frac{3\pi}{2}(n+\frac{1}{2})$ and $\frac{\pi}{2}(n+\frac{1}{2})$.

If now $z^{-\frac{1}{2}}$ has its principal value whose argument lies between $\pm\frac{\pi}{2}$, and with the previous convention for the factor z^n which intervenes in the definition of $J_n(z)$ we have, putting

$${}_2F_0\left\{n+\tfrac{1}{2},\ \tfrac{1}{2}-n;\ \frac{1}{2\iota z}\right\} = P,$$

$${}_2F_0\left\{n+\tfrac{1}{2},\ \tfrac{1}{2}-n;\ -\frac{1}{2\iota z}\right\} = Q,$$

1. When $R(z) > 0$,

$$J_n(z) = \frac{1}{\sqrt{2\pi}\, z^{\frac{1}{2}}} \left[e^{\iota z - \frac{\iota\pi}{2}(n+\frac{1}{2})} P + e^{-\iota z + \frac{\iota\pi}{2}(n+\frac{1}{2})} Q\right].$$

2. When $R(z) < 0$ and $I(z) > 0$,

$$J_n(z) = \frac{1}{\sqrt{2\pi}\, z^{\frac{1}{2}}} \left[e^{\iota z + \frac{3\iota\pi}{2}(n+\frac{1}{2})} P + e^{-\iota z + \frac{\iota\pi}{2}(n+\frac{1}{2})} Q\right].$$

3. When $R(z) < 0$ and $I(z) < 0$,

$$J_n(z) = \frac{1}{\sqrt{2\pi}\, z^{\frac{1}{2}}} \left[e^{\iota z - \frac{\iota\pi}{2}(n+\frac{1}{2})} P + e^{-\iota z - \frac{3\iota\pi}{2}(n+\frac{1}{2})} Q\right].$$

These are the complete asymptotic expansions for $J_n(z)$.

* A different convention is adopted by Nielsen, *Cylinderfunktionen*, p. 19.

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If we ignore series which are infinitely small compared with those retained we have:

1. When $I(z) > 0$,

$$J_n(z) = \frac{1}{\sqrt{2\pi}\, z^{\frac{1}{2}}} e^{-\iota z + \frac{\iota\pi}{2}(n+\frac{1}{2})} Q.$$

2. When $I(z) < 0$,

$$J_n(z) = \frac{1}{\sqrt{2\pi}\, z^{\frac{1}{2}}} e^{-\iota z + \frac{\iota\pi}{2}(n+\frac{1}{2})} P.$$

When $R(z) > 0$ the dominant term in the expansion of $J_n(z)$ may be written

$$\frac{1}{\sqrt{2\pi}\, z^{\frac{1}{2}}} 2\cos\left[z - (n+\tfrac{1}{2})\frac{\pi}{2}\right].$$

More completely we have, when $R(z) > 0$,

$$J_n(z) = \sqrt{\frac{2}{\pi z}}\left\{A\cos\left[z - \frac{\pi}{2}(n+\tfrac{1}{2})\right] + \frac{B}{2z}\sin\left[z - \frac{\pi}{2}(n+\tfrac{1}{2})\right]\right\},$$

where
$$A = \sum_{m=0}^{k} \frac{\Gamma(n+2m+\frac{1}{2})\,\Gamma(-n+2m+\frac{1}{2})}{\Gamma(n+\frac{1}{2})\,\Gamma(-n+\frac{1}{2})\,\Gamma(2m+1)\,(2\iota z)^{2m}} + \frac{I_k}{z^{2k}},$$

and
$$B = \sum_{m=0}^{k} \frac{\Gamma(n+2m+\frac{3}{2})\,\Gamma(-n+2m+\frac{3}{2})}{\Gamma(n+\frac{1}{2})\,\Gamma(-n+\frac{1}{2})\,\Gamma(2m+2)\,(2\iota z)^{2m}} + \frac{J_k}{z^{2k}}.$$

The quantities I_k and J_k are such that their moduli tend to zero as $|z|$ tends to infinity, for any finite value of k.

This is substantially Hankel's result. When z is not real, the complete series, of course, is not such that the modulus of the difference between $J_n(z)$ and the first k terms of the series is of order less than $\frac{1}{|z|^k}$ when $|z|$ is large*.

§ 25. We will next consider the nature of the asymptotic value of ${}_0F_1\{\rho;\, x\}$ when $|x|$ is large and $|\arg x| < \pi$, as ρ tends to positive infinity.

We have
$${}_0F_1\{\rho;\, x\} = e^{2x^{1/2}}\, {}_1F_1\{\rho - 1/2;\, 2\rho - 1;\, -4x^{1/2}\}.$$

If now $R(x^{1/2}) > 0$, we have (§ 7)

$${}_1F_1\{\rho - 1/2;\, 2\rho - 1;\, -4x^{1/2}\}$$
$$= -\frac{1}{2\pi\iota}\int \Gamma(-s)\,\frac{\Gamma(2\rho-1)}{\Gamma(2\rho-1+s)}\,\frac{\Gamma(\rho-1/2+s)}{\Gamma(\rho-1/2)}\,(4x^{1/2})^s\, ds; \qquad (1)$$

and we may take the contour of the integral along a parallel to the imaginary axis through the point $s = 1 - m$, if $R(\rho)$ be large positive and $\geqslant m$, m being finite real and positive.

Let us now consider what happens to the integral when $R(\rho)$ tends to positive infinity.

* With this series the reader may compare Forsyth, *Linear Differential Equations* [*Theory of Differential Equations*, Part III. Vol. IV.] p. 331, and also Nielsen, *Cylinderfunktionen*, pp. 154—156, &c. The history of the investigation has been considered in the introduction to my fundamental memoir.

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As regards that part of the integral for which $|s| \leqslant L$, where L is a positive quantity as large as we please but such that L/ρ is small, it is evident that

$$\frac{\Gamma(2\rho-1)}{\Gamma(2\rho-1+s)}\,\frac{\Gamma(\rho-1/2+s)}{\Gamma(\rho-1/2)}=\frac{1}{2^s}.$$

When $|s|>L$, where L/ρ is small, we may put $|I(s)|=\mu\rho$, and on the corresponding parts of the integral μ will take all values from $\epsilon(>0)$ to ∞.

And now, if $R(s)=u$, we have, when ρ is large,

$$\frac{\Gamma(2\rho-1)}{\Gamma(2\rho-1+s)}\,\frac{\Gamma(\rho-1/2+s)}{\Gamma(\rho-1/2)}=\frac{\Gamma(2\rho-1)\,\Gamma\{\rho(1\pm\iota\mu)+u-1/2\}}{\Gamma(\rho-1/2)\,\Gamma\{\rho(2\pm\iota\mu)+u-1\}}$$

$$=\frac{(2\rho)^{2\rho-3/2}\,e^{-2\rho}}{\rho^{\rho-1}\,e^{-\rho}}\,\frac{[\rho(1\pm\iota\mu)]^{\rho(1\pm\iota\mu)+u-1}\,e^{-\rho(1\pm\iota\mu)}}{[\rho(2\pm\iota\mu)]^{\rho(2\pm\iota\mu)+u-3/2}\,e^{-\rho(2\pm\iota\mu)}},\text{ approximately,}$$

the upper or lower signs being taken together.

Hence $$\left|\frac{\Gamma(2\rho-1)\,\Gamma(\rho-1/2+s)}{\Gamma(2\rho-1+s)\,\Gamma(\rho-1/2)}\right|=2^{2\rho-3/2}\,\rho^{\rho-1/2}\,\frac{[\rho\sqrt{1+\mu^2}]^{\rho+u-1}\,e^{-\mu\rho\tan^{-1}\mu}}{[\rho\sqrt{4+\mu^2}]^{2\rho+u-3/2}\,e^{-\mu\rho\tan^{-1}\mu/2}}$$

$$=2^{-3/2}\frac{(1+\mu^2)^{(u-1)/2}}{(4+\mu^2)^{u/2-3/4}}\exp\{\rho[\mu\tan^{-1}\mu/2-\mu\tan^{-1}\mu+\log\sqrt{1+\mu^2}-2\log\sqrt{1+\mu^2/4}]\}.$$

Suppose now that

$$y=\mu\tan^{-1}\mu/2-\mu\tan^{-1}\mu+\log\sqrt{1+\mu^2}-2\log\sqrt{1+\mu^2/4}.$$

Then $$\frac{dy}{d\mu}=\frac{\mu/2}{1+\mu^2/4}-\frac{\mu}{1+\mu^2}+\frac{\mu}{1+\mu^2}-\frac{\mu/2}{1+\mu^2/4}+\tan^{-1}\mu/2-\tan^{-1}\mu$$

$$=-\tan^{-1}\mu/(2+\mu^2).$$

Hence $dy/d\mu$ is always negative and therefore y constantly decreases as μ increases. Now, when $\mu=0$, $y=0$. Hence y is always negative if $\mu>0$.

We thus see that, if m be finite and if $|\arg x^{1/2}|<\pi/2$, that part of the integral (1) for which $|s|>L$, where L is as large as we please but such that L/ρ is small, tends to zero as ρ tends to infinity.

We now have, if $|\arg x|<\pi$,

$${}_0F_1\{\rho;\ x\}=e^{2\sqrt{x}}\,\frac{I_\rho(x)}{x^{(m-1)/2}},$$

where $I_\rho(x)$ tends to a definite finite limit as $R(\rho)$ tends to positive infinity for any finite value of $|x|$ however large, and where $|I_\rho(x)|$ can be made for all values of ρ (such that $R(\rho)>m$) as small as we please by taking $|x|$ sufficiently large.

The first part of this theorem can, of course, be stated in the simple form that, if $|x|$ be finite, $\operatorname*{Lt}_{\rho=\infty}{}_0F_1\{\rho;\ x\}=1$.

§ 26. We have seen that ${}_0F_1\{\rho;\ x\}$ and $x^{1-\rho}\,{}_0F_1\{2-\rho,\ x\}$ are solutions of the differential equation

$$x\frac{d^2y}{dx^2}+\rho\frac{dy}{dx}-y=0.$$

Equally each of the asymptotic series which occurs in the complete asymptotic expansion of ${}_0F_1\{\rho;\ x\}$ is a solution in the neighbourhood of infinity. It is natural then to seek a combination of the two functions which shall be completely represented by one of the asymptotic series.

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We will prove that asymptotically, if $|\arg x| < \pi$,

$$\Gamma(1-\rho)\,{}_0F_1\{\rho;\ x\} + x^{1-\rho}\Gamma(\rho-1)\,{}_0F_1\{2-\rho;\ x\} = \sqrt{\pi}\,x^{\frac{1-2\rho}{4}}\,e^{-2x^{1/2}}\,{}_2F_0\left\{\tfrac{3}{2}-\rho;\ \rho-\tfrac{1}{2};\ \frac{1}{-4x^{1/2}}\right\}.$$

In the first place we may shew that the combination on the left-hand side is, when $|x|$ is large and $|\arg x| < \pi$, smaller than any positive power of $1/|x|$.

For consider the integral

$$-\frac{1}{2\pi i}\int_C \Gamma(-s)\,\Gamma(1-\rho-s)\,x^s\,ds,$$

round a contour which embraces the positive half of the real axis and encloses all the poles of the subject of integration.

It is evidently equal to

$$\sum_{n=0}^{\infty}\frac{(-x)^n\,\Gamma(1-\rho-n)}{\Gamma(n+1)} + \sum_{n=0}^{\infty}(-)^n\frac{x^{1-\rho+n}\,\Gamma(-n-1+\rho)}{\Gamma(n+1)}.$$

This in turn may be written

$$\sum_{n=0}^{\infty}\frac{\pi x^n}{\Gamma(n+1)\,\Gamma(\rho+n)\sin\pi\rho} + \sum_{n=0}^{\infty}\frac{\pi x^{1-\rho+n}}{\Gamma(n+1)\,\Gamma(2-\rho+n)\sin(\rho-1)\pi}$$
$$= \Gamma(1-\rho)\,{}_0F_1(\rho;\ x) + x^{1-\rho}\Gamma(\rho-1)\,{}_0F_1(2-\rho;\ x).$$

Now, if $|\arg x| < \pi$, the integral vanishes round that part of a great circle at infinity for which $R(s) > -k$ where k is a finite positive quantity. Hence when $|x|$ is large it is smaller in value than any positive power of $\dfrac{1}{|x|}$.

To prove the asymptotic expansion given we use Kummer's result.

We have (§ 21) $\quad \Gamma(1-\rho)\,{}_0F_1(\rho;\ x) + x^{1-\rho}\Gamma(\rho-1)\,{}_0F_1(2-\rho;\ x)$

$$= e^{-2x^{1/2}}\{\Gamma(1-\rho)\,{}_1F_1(\rho-\tfrac{1}{2};\ 2\rho-1;\ 4x^{1/2}) + x^{1-\rho}\Gamma(\rho-1)\,{}_1F_1(\tfrac{3}{2}-\rho;\ 3-2\rho;\ 4x^{1/2})\},$$

and this expression is by § 9 asymptotically equal to

$$e^{-2x^{1/2}}\left[\frac{\Gamma(1-\rho)\,\Gamma(2\rho-1)}{\Gamma(\rho-\frac{1}{2})\,(-4x^{1/2})^{\rho-\frac{1}{2}}} + \frac{x^{1-\rho}\,\Gamma(\rho-1)\,\Gamma(3-2\rho)}{\Gamma(\frac{3}{2}-\rho)\,(-4x^{1/2})^{3/2-\rho}}\right]{}_0F_0\left(\rho-\tfrac{1}{2},\ \tfrac{3}{2}-\rho;\ -\frac{1}{4x^{1/2}}\right)$$
$$+ e^{2x^{1/2}}\left[\frac{\Gamma(1-\rho)\,\Gamma(2\rho-1)}{\Gamma(\rho-\frac{1}{2})\,(4x^{1/2})^{\rho-\frac{1}{2}}} + \frac{\Gamma(3-2\rho)\,\Gamma(\rho-1)\,x^{1-\rho}}{\Gamma(\frac{3}{2}-\rho)\,(4x^{1/2})^{3/2-\rho}}\right]{}_2F_0\left\{\tfrac{3}{2}-\rho,\ \rho-\tfrac{1}{2};\ \frac{1}{4x^{1/2}}\right\}.$$

Now by the multiplication formula for the gamma function

$$\frac{\Gamma(2z)}{\Gamma(z)} = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z+\tfrac{1}{2});$$

hence it may be readily seen that

$$\frac{\Gamma(1-\rho)\,\Gamma(2\rho-1)}{\Gamma(\rho-\frac{1}{2})} + \frac{\Gamma(3-2\rho)\,\Gamma(\rho-1)}{\Gamma(\frac{3}{2}-\rho)\,4^{2-2\rho}} = 0,$$

and
$$\frac{\Gamma(1-\rho)\,\Gamma(2\rho-1)}{\Gamma(\rho-\frac{1}{2})\,(-4)^{\rho-\frac{1}{2}}} + \frac{\Gamma(\rho-1)\,\Gamma(3-2\rho)}{\Gamma(\frac{3}{2}-\rho)\,(-4)^{3/2-\rho}} = \sqrt{\pi},$$

remembering the prescription which we must give to the functions $(-4)^{\rho-\frac{1}{2}}$ and $(-4)^{3/2-\rho}$.

Hence finally we obtain the given result.

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§ 27. We proceed now to give a direct proof of the previous theorem and to shew that, if $|\arg x| < 3\pi$,

$$\Gamma(1-\rho)\,{}_0F_1\{\rho;\ x\} + x^{1-\rho}\Gamma(\rho-1)\,{}_0F_1\{2-\rho;\ x\} = \sqrt{\pi}\,x^{\frac{1-2\rho}{4}}\,e^{-2x^{1/2}}\,{}_2F_0\{\tfrac{3}{2}-\rho;\ \rho-\tfrac{1}{2};\ -1/(4x^{1/2})\}.$$

We thus extend the range of validity of the asymptotic equality from $|\arg x| < \pi$ to $|\arg x| < 3\pi$.

Let $S(s)$ denote the function defined when $R(2s+\rho) > 1/2$ by the series

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t/2)\,\Gamma(1-\rho-s+t/2)}{\Gamma(t/2+1)\,\Gamma(t/2+1/2)}.$$

We will first shew that

$$S(s) = -\frac{\cos\pi\rho}{\pi}\,2^{2\rho+4s}\,\Gamma(-2s)\,\Gamma(2-2\rho-2s)\,\Gamma(2s+\rho-1/2).$$

If $R(2s+\rho) > 1/2$, we have

$$\begin{aligned} S(s) &= \sum_{t=0}^{\infty} \frac{\Gamma(-s+t)\,\Gamma(1-\rho-s+t)}{\Gamma(t+1)\,\Gamma(t+1/2)} + \sum_{t=0}^{\infty} \frac{\Gamma(-s+t+1/2)\,\Gamma(3/2-\rho-s+t)}{\Gamma(t+3/2)\,\Gamma(t+1)} \\ &= {}_2F_1\{-s,\ 1-\rho-s;\ 1/2;\ 1\}\,\frac{\Gamma(-s)\,\Gamma(1-\rho-s)}{\Gamma(1/2)} \\ &\qquad + {}_2F_1\{-s+1/2,\ 3/2-\rho-s;\ 3/2;\ 1\}\,\frac{\Gamma(-s+1/2)\,\Gamma(3/2-\rho-s)}{\Gamma(3/2)}. \end{aligned}$$

By Gauss' theorem we therefore have

$$\begin{aligned} S(s) &= \Gamma(\rho+2s-1/2)\left\{\frac{\Gamma(-s)\,\Gamma(1-\rho-s)}{\Gamma(1/2+s)\,\Gamma(\rho+s-1/2)} + \frac{\Gamma(-s+1/2)\,\Gamma(3/2-\rho-s)}{\Gamma(1+s)\,\Gamma(\rho+s)}\right\} \\ &= \Gamma(2s+\rho-1/2)\,\frac{\Gamma(-s)\,\Gamma(1-\rho-s)}{\Gamma(1/2+s)\,\Gamma(\rho+s-1/2)}\,\{1+\tan\pi s\tan\pi(\rho+s)\} \\ &= -\frac{\cos\pi\rho}{\pi^2}\,\Gamma(1/2-s)\,\Gamma(-s)\,\Gamma(3/2-\rho-s)\,\Gamma(1-\rho-s)\,\Gamma(2s+\rho-1/2) \\ &= -\frac{\cos\pi\rho}{\pi}\,2^{2\rho+4s}\,\Gamma(-2s)\,\Gamma(2-2\rho-2s)\,\Gamma(2s+\rho-1/2). \end{aligned}$$

Let now C be a contour embracing the positive half of the real axis and enclosing the points

$$0,\ 1,\ 2,\ \ldots,\ \text{and}\ 1-\rho,\ 2-\rho,\ 3-\rho,\ \ldots.$$

Then by Cauchy's theory of residues for all values of $|\arg x|$ we have

$$\begin{aligned} I &= -\frac{1}{2\pi\iota}\int_C \Gamma(-s)\,\Gamma(1-\rho-s)\,x^s\,ds \\ &= \Gamma(1-\rho)\,{}_0F_1\{\rho;\ x\} + x^{1-\rho}\Gamma(\rho-1)\,{}_0F_1\{2-\rho;\ x\}. \end{aligned}$$

Also we have

$$e^{2x^{1/2}}\,I/\sqrt{\pi} = -\frac{1}{2\pi\iota}\sum_{t=0}^{\infty}\int_C \frac{\Gamma(-s)\,\Gamma(1-\rho-s)}{\pi^{1/2}}\,\frac{2^t\,x^{s+t/2}}{t\,!}\,ds.$$

And hence by the methods employed either in § 6 or § 14 we have

$$e^{2x^{1/2}}\,I/\sqrt{\pi} = -\frac{1}{2\pi\iota}\int_C S(s)\,x^s\,ds,$$

if the contour C excludes the points $\frac{1}{4}-\frac{\rho}{2}-\frac{n}{2}$, $n = 0,\ 1,\ 2,\ \ldots$.

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We therefore have

$$e^{2x^{1/2}} I/\sqrt{\pi} = 2^{2\rho} \frac{\cos \pi\rho}{\pi} \frac{1}{2\pi\iota} \int_C \Gamma(-2s)\, \Gamma(2-2\rho-2s)\, \Gamma(2s+\rho-\tfrac{1}{2})\, 2^{4s}\, x^s\, ds.$$

Now if $|\arg x| < 3\pi$ the integral will vanish when taken along a contour at infinity for which $R(s)$ is greater than a finite negative quantity. It is therefore equal to minus the sum of the residues of the subject of integration at the points

$$1/4 - \rho/2 - n/2,\ n = 0,\ 1,\ 2,\ \ldots,\ k$$

together with J_k, where J_k denotes an integral along a contour parallel to the imaginary axis which passes between the points $s = 1/4 - \rho/2 - k/2$ and $s = -3/4 - \rho/2 - k/2$.

Thus
$$e^{2x^{1/2}} I/\sqrt{\pi} = -\frac{\cos \pi\rho}{\pi} x^{\frac{1-2\rho}{4}} \sum_{n=0}^{k} \frac{\Gamma(n+\rho-1/2)\, \Gamma(n+3/2-\rho)}{(-4x^{1/2})^n} + J_k$$
$$= x^{\frac{1-2\rho}{4}}\, {}_2F_0\{3/2-\rho,\ \rho-1/2;\ 1/(-4x^{1/2})\}_{k+1} + J_k,$$

where $|J_k x^{1/4-\rho/2-k/2}|$ tends to zero as $|x|$ tends to infinity.

We thus have the result stated.

§ 28. This result, since it is valid when $|\arg x| < 3\pi$, is equivalent to three different results.

In fact, if $-\pi < \arg x < \pi$, we have, in addition to the previous result,

$$\Gamma(1-\rho)\, {}_0F_1\{\rho;\ x\} + x^{1-\rho} e^{2m\pi\iota(1-\rho)}\, \Gamma(\rho-1)\, {}_0F_1\{2-\rho;\ x\}$$
$$= \sqrt{\pi}\, x^{(1-2\rho)/4} e^{m\pi\iota(1-2\rho)/2} e^{2x^{1/2}}\, {}_2F_0\{3/2-\rho;\ \rho-1/2;\ 1/(4x^{1/2})\}, \qquad \text{(A)}$$

where $m = 1$ or -1.

We thus have found two combinations of the functions ${}_0F_1\{\rho;\ x\}$ and ${}_0F_1\{2-\rho;\ x\}$ which give a solution of the equation

$$x \frac{d^2y}{dx^2} + \rho \frac{dy}{dx} - y = 0,$$

which to a first approximation behaves like $e^{2x^{1/2}}$ when $|x|$ is very large and $|\arg x| < \pi$.

From the two results (A) we can obtain anew the asymptotic equality (§ 23)

$${}_0F_1\{\rho;\ x\} = \frac{\Gamma(\rho)}{2\sqrt{\pi}} x^{1/4-\rho/2} e^{2x^{1/2}}\, {}_2F_0\{\rho-1/2,\ 3/2-\rho;\ 1/(4x^{1/2})\},$$

when $|\arg x| < \pi$.

For on subtracting one form of (A) from the other we find

$$x^{1-\rho} \sin 2\pi(1-\rho)\, \Gamma(\rho-1)\, {}_0F_1\{2-\rho;\ x\} \Big/ \sin \frac{\pi}{2}(1-2\rho)$$
$$= \sqrt{\pi}\, x^{(1-2\rho)/4} e^{2x^{1/2}}\, {}_2F_0\{3/2-\rho;\ \rho-1/2;\ 1/(4x^{1/2})\},$$

or
$${}_0F_1\{2-\rho;\ x\} = \frac{\Gamma(2-\rho)}{2\sqrt{\pi}} x^{(\rho-2)/2+1/4} e^{2x^{1/2}}\, {}_2F_0\{3/2-\rho,\ \rho-1/2;\ 1/(4x^{1/2})\}.$$

Writing ρ for $2-\rho$, we have

$${}_0F_1\{\rho;\ x\} = \frac{\Gamma(\rho)}{2\sqrt{\pi}} x^{1/4-\rho/2} e^{2x^{1/2}}\, {}_2F_0\{3/2-\rho,\ \rho-1/2;\ 1/(4x^{1/2})\},$$

the result of § 23, when $|\arg x| < \pi$.

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THE APPLICATION OF INTEGRAL EQUATIONS TO THE DETERMINATION OF EXPANSIONS IN SERIES OF OSCILLATING FUNCTIONS.

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XII. *The application of integral equations to the determination of expansions in series of oscillating functions.*

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§ 1. The theory of integral equations as developed by Fredholm* and Hilbert† has provided us with a new starting-point for a systematic study of the important classes of expansions which occur frequently in the mathematical theory of vibrating systems.

The analytical result upon which the present theory depends is that the homogeneous integral equation

$$\psi_n(s) = \lambda_n \int_a^b k(s, t)\, \psi_n(t)\, dt \quad \text{.....} \quad (1),$$

in which $k(s, t)$ is continuous for values of s and t on the path of integration, only possesses a solution $\psi_n(t)$ different from zero when the quantity λ_n is a root of a certain transcendental‡ equation $\delta(\lambda) = 0$, and that corresponding to each value of λ and each function ψ, there is another function ϕ for which the adjoint equation

$$\phi_n(t) = \lambda_n \int_a^b k(s, t)\, \phi_n(s)\, ds \quad \text{.....} \quad (2)$$

is satisfied.

If the equation (1) is multiplied by the function $\phi_m(s)$ corresponding to a different root λ_m and integrated between a and b we have

$$\int_a^b \phi_m(s)\, \psi_n(s)\, ds = \lambda_n \int_a^b \int_a^b \phi_m(s)\, k(s, t)\, \psi_n(t)\, ds\, dt$$

$$= \frac{\lambda_n}{\lambda_m} \int_a^b \phi_m(t)\, \psi_n(t)\, dt,$$

which gives

$$\int_a^b \phi_m(t)\, \psi_n(t)\, dt = 0, \quad (\lambda_m \neq \lambda_n) \quad \text{.....} \quad (3).$$

* *Acta Math.* XXVII. (1903).

† *Gött. Nachr.* Hefte 1 and 3 (1904).

‡ $\delta(\lambda)$ is in general a transcendental integral function of λ, but it may happen that in some special cases it is a rational integral function of λ.

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If λ_n is a simple root of the equation $\delta(\lambda)=0$ there is only one pair of corresponding functions $\phi_n(s)$, $\psi_n(s)$, and the arbitrary constants multiplying them are chosen so that

$$\int_a^b \phi_n(s)\,\psi_n(s)\,ds = 1 \quad\text{.....}\quad (4).$$

When however λ_n is a multiple root there may be several different pairs, but Fredholm has shown that we can find a set of linearly independent functions ψ_n^r and a corresponding set of linearly independent functions ϕ_n^r so that

$$\int_a^b \phi_n^r(s)\,\psi_n^p(s)\,ds \left.\begin{matrix}=0 & r\neq p\\ =1 & r=p\end{matrix}\right\} \quad\text{.....}\quad (5).$$

The number of linearly independent functions $\psi_n^p(s)$ cannot be greater than the multiplicity of the root λ_r: it is usually equal to it, but there is at present no evidence to show that this is always the case.

The formulae (3), (4) and (5) enable us to determine the coefficients in the two types of expansions

$$\left.\begin{aligned} f(s) &= \sum_0^\infty a_n\psi_n(s)\\ f_1(s) &= \sum_0^\infty a_n\phi_n(s)\end{aligned}\right\} \quad\text{.....}\quad (6),$$

whenever uniformly convergent expansions of this kind are known to exist and sometimes when the expansions are non-uniformly convergent. The above analysis however does not indicate how the adjoint set of functions $\phi_n(s)$ can be determined when the functions $\psi_n(s)$ are given or vice-versa. The object of this paper is to extend the analysis so that this defect may be remedied. Starting from a single integral equation

$$f(s) = \phi_\lambda(s) - \lambda\int_a^b k(s,t)\,\phi_\lambda(t)\,dt \quad\text{.....}\quad (7),$$

whose solution $\phi_\lambda(s)$ is known, we can deduce the law of the coefficients in the expansion of an arbitrary function in the form

$$\Sigma\, a_\lambda\phi_\lambda(s) \quad\text{.....}\quad (8),$$

where the quantities λ are the roots of the equation in λ obtained by imposing a linear condition of any kind upon the function $\phi_\lambda(s)$.

Since the functions $f(s)$ and $k(s,t)$ are at our disposal it is evident that the results obtained in this paper are of a very general character. The integral equation due to Hilbert, which is chosen to illustrate the method, leads to some interesting results in connection with expansions in trigonometrical series and puts in evidence the power of the new analysis.

§ 2. Many of the well known expansions in series of oscillating functions connect themselves naturally with the problem of determining the values of λ for which a linear differential equation of the form

$$Lu + \lambda u = 0 \quad\text{.....}\quad (9),$$

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where L is some differential operator, can possess a solution which satisfies certain linear conditions. It is natural then to expect that the problem of determining the values of λ for which the solution of an integral equation may satisfy given linear conditions will also lead to important results. This problem really includes the previous one because a linear differential equation can in general be replaced by an integral equation.

Let us consider then the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b k(s, t)\, \phi(t)\, dt \quad \text{.....} \quad (10),$$

concerning ourselves with the solutions which satisfy various types of linear conditions.

If λ be regarded as the quantity at our disposal only one linear condition can be imposed and then the corresponding values of λ will be in general the roots of a certain transcendental equation. If we impose the linear condition

$$\phi(x) = c, \quad (a < x < b) \quad \text{.....} \quad (11),$$

supposing λ_0 to be one of the possible values of λ we shall have

$$f(x) = c - \lambda_0 \int_a^b k(x, t)\, \phi_0(t)\, dt \quad \text{.....} \quad (12).$$

Combining this with the equation

$$f(s) = \phi_0(s) - \lambda_0 \int_a^b k(s, t)\, \phi_0(t)\, dt,$$

we obtain the homogeneous integral equation

$$\phi_0(s) = \lambda_0 \int_a^b k_1(s, t)\, \phi_0(t)\, dt \quad \text{.....} \quad (13),$$

where

$$k_1(s, t) = k(s, t) - \frac{f(s)}{f(x) - c}\, k(x, t) \quad \text{.....} \quad (14).$$

Now in order that this equation may be satisfied λ_0 must be a zero of the integral function $\delta_1(\lambda)$ connected with the equation

$$\psi(s) = \chi(s) - \lambda \int_a^b k_1(s, t)\, \chi(t)\, dt \quad \text{.....} \quad (15).$$

Accordingly, we can anticipate that the function $\delta_1(\lambda)$ contains $\phi(x) - c$ as a factor, for $\delta_1(\lambda)$ is zero whenever λ satisfies the equation $\phi(x) = c$.

The solution of equation (10) is known to be given uniquely by the formula*

$$\phi(s) = f(s) + \lambda \int_a^b K(s, t)\, f(t)\, dt \quad \text{.....} \quad (16),$$

except when λ is a zero of a certain integral function $\delta(\lambda)$ connected with this equation.

* The formulae quoted will all be found in Fredholm's paper.

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The solving function $K(s, t)$ depends of course upon λ and is infinite at each zero of $\delta(\lambda)$; moreover in the neighbourhood of a simple zero λ_n it has the form

$$\frac{\psi_n(s)\,\phi_n(t)}{\lambda_n - \lambda} + F(s, t) \quad \text{.....(17)},$$

where $F(s, t)$ remains finite and $\psi_n(s)$, $\phi_n(t)$ are the functions mentioned at the beginning of § 1.

It will be verified in § 3 that the corresponding functions $\delta_1(\lambda)$ and $K_1(s, t)$ for equation (15) are given by

$$\left.\begin{aligned} \delta_1(\lambda) &= \frac{\phi(x) - c}{f(x) - c}\,\delta(\lambda) \\ K_1(s, t) &= K(s, t) - \frac{\phi(s)}{\phi(x) - c}\,K(x, t) \end{aligned}\right\} \quad \text{.....(18)}.$$

Now it follows from equation (13) that when λ is a root of the equation $\phi(x) = c$, the corresponding function $\phi_0(s)$ is a fundamental function for equation (15). To find the adjoint function we apply formula (17) to the function $K_1(s, t)$, remembering that $\psi_n(s)$ corresponds to $\phi_0(s)$, and $\phi_n(t)$ to the function we want. Calling this function $\theta_0(t)$ we have

$$\begin{aligned} \phi_0(s)\,\theta_0(t) &= \operatorname*{Lt}_{\lambda=\lambda_0} (\lambda_0 - \lambda)\,K_1(s, t) \\ &= \frac{\phi_0(s)\,K_0(x, t)}{\dfrac{d}{d\lambda_0}\phi(x)}, \qquad K_0(x, t) = [K(x, t)]_{\lambda=\lambda_0}, \end{aligned}$$

the terms $K(s, t)$, $\phi(s)$ and $K(x, t)$ remaining finite for $\lambda = \lambda_0$.

The function $\theta_0(t)$ is thus given by the formula

$$\theta_0(t) = \frac{K_0(x, t)}{\dfrac{d}{d\lambda_0}\phi(x)} \quad \text{.....(19)},$$

and so if the function $K(x, t)$ is known the coefficients in the expansion of an arbitrary function $F(s)$ in the form

$$F(s) = \sum_{\lambda_0} A(\lambda_0)\,\phi(\lambda_0, s) \quad \text{.....(20)},$$

for the different values of λ_0 which satisfy the equation

$$\phi(\lambda_0, x) = c \quad \text{.....(21)},$$

are given by

$$A(\lambda_0) = \int_a^b \frac{K_0(x, t)}{\dfrac{d}{d\lambda_0}\phi_0(x)}\,F(t)\,dt \quad \text{.....(22)}.$$

It will be noticed that $K_1(s, t)$ is obtained from $k_1(s, t)$ by replacing the functions f and k wherever they occur by ϕ and K respectively: this rule enables us to write

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down the functions $\delta_1(\lambda)$, $K_1(s, t)$ for the integral equation obtained by imposing any other linear condition upon the function ϕ. For instance if we require $\phi(s)$ to satisfy the linear condition

$$\int_a^b \rho(s)\,\phi(s)\,ds = c \quad \text{.....} \quad (23),$$

the values of λ_0 will be such that

$$\int_a^b \rho(s)\,f(s)\,ds = c - \lambda_0 \int_a^b \int_a^b k(s, t)\,\rho(s)\,\phi_0(t)\,ds\,dt,$$

$$f(s) = \phi_0(s) - \lambda_0 \int_a^b k(s, t)\,\phi_0(t)\,dt,$$

and the equation which replaces (13) is

$$\phi_0(s) = \lambda_0 \int_a^b k_1(s, t)\,\phi_0(t)\,dt,$$

where
$$k_1(s, t) = k(s, t) + \frac{f(s)}{c - \int_a^b \rho(x)\,f(x)\,dx} \int_a^b \rho(x)\,k(x, t)\,dx \quad \text{.....} \quad (24).$$

It will be shown in § 3 that for this equation

$$K_1(s, t) = K(s, t) + \frac{\phi(s)}{c - \int_a^b \rho(x)\,\phi(x)\,dx} \int_a^b \rho(x)\,K(x, t)\,dx,$$

$$\left.\begin{aligned} \delta_1(\lambda) &= \frac{c - \int_a^b \rho(x)\,\phi(x)\,dx}{c - \int_a^b \rho(x)\,f(x)\,dx}\,\delta(\lambda) \\ \theta_0(t) &= \frac{\int_a^b \rho(x)\,K_0(x, t)\,dx}{\frac{\partial}{\partial\lambda_0}\int_a^b \rho(x)\,\phi(x)\,dx} \end{aligned}\right\} \quad \text{.....} \quad (25).$$

As an example of the application of these results we may consider Hilbert's integral equation

$$\phi(s) = f(s) - \lambda \int_0^1 k(s, t)\,\phi(t)\,dt,$$

in which
$$k(s, t) = \left.\begin{matrix} (1-s)\,t & t \leqslant s \\ (1-t)\,s & t \geqslant s \end{matrix}\right\}.$$

Here we have
$$\left.\begin{aligned} K(s, t) &= \frac{\sin\{\sqrt{\lambda}\,(1-s)\}\,\sin t\sqrt{\lambda}}{\sqrt{\lambda}\,\sin\sqrt{\lambda}} & t \leqslant s \\ &= \frac{\sin\{\sqrt{\lambda}\,(1-t)\}\,\sin s\sqrt{\lambda}}{\sqrt{\lambda}\,\sin\sqrt{\lambda}} & t \geqslant s \end{aligned}\right\} \quad \text{.....} \quad (26),$$

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and if we take the particular function $f(s) = s$, the formula

$$\phi(s) = f(s) + \lambda \int_0^1 K(s, t)\, \phi(t)\, dt$$

gives

$$\phi(s) = \frac{\sin s\sqrt{\lambda}}{\sin \sqrt{\lambda}} \quad \text{.....} \quad (27).$$

The previous work thus enables us to determine the coefficients in the expansion of an arbitrary function in the form

$$F(s) = \sum_{\lambda} A_\lambda \sin s\sqrt{\lambda} \quad \text{.....} (28),$$

where the summation extends over the values of λ for which the function $\phi(s)$ satisfies any given linear condition.

Suppose in the first place that the condition is $\phi(x) = c$ and write $\lambda_0 = m^2$, then the values of m must satisfy the equation

$$\sin mx = c \sin m \quad \text{.....} (29).$$

Now
$$\frac{d}{d\lambda_0}\phi_0(x) = \frac{1}{2\sqrt{\lambda_0}\sin^2\sqrt{\lambda_0}}\left[x\cos x\sqrt{\lambda_0}\sin\sqrt{\lambda_0} - \sin x\sqrt{\lambda_0}\cos\sqrt{\lambda_0}\right]$$
$$= \frac{x\cos mx - c\cos m}{2m\sin m}.$$

Calling the different roots of equation (29) $m_1, m_2, \ldots m_r \ldots$ the coefficients in the expansion

$$F(s) = B_m \sin ms,$$

as determined by formula (22), are given by

$$\frac{B_m}{2}\sin m\,.\,[x\cos mx - c\cos m]$$
$$= \sin m(1-x)\int_0^x F(t)\sin mt\, dt + \sin mx \int_x^1 F(t)\sin m(1-t)\, dt \quad \text{.....} (30).$$

Again, if the condition is $\int_0^1 \phi(s)\, ds = c$, the values of m are given by

$$\int_0^1 \frac{\sin ms}{\sin m}\, ds = c,$$

i.e.
$$\frac{\tan\frac{m}{2}}{m} = c \quad \text{.....} (31).$$

In this case
$$\int_0^1 K(s, t)\, ds = \frac{2\sin\frac{mt}{2}\sin\frac{m(1-t)}{2}}{m^2\cos\frac{m}{2}},$$

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$$\frac{d}{d\lambda_0}\int_0^1 \phi_0(s)\,ds = \frac{1}{4\lambda_0}\sec^2\frac{\sqrt{\lambda_0}}{2} - \frac{1}{2\lambda_0^{\frac{3}{2}}}\tan\frac{\sqrt{\lambda_0}}{2} = \frac{1}{4m^2}\left(\sec^2\frac{m}{2} - 2c\right),$$

and so the coefficients in the expansion of an arbitrary function in the form $\sum_m B_m \sin ms$, where the summation extends over the values of m for which

$$\tan\frac{m}{2} = cm,$$

are given by $$4B_m \sin\frac{m}{2}\left(1 - 2c\cos^2\frac{m}{2}\right) = \int_0^1 F(t)\sin\frac{mt}{2}\,.\,\sin\frac{m(1-t)}{2}\,dt \quad\text{.....(32).}$$

§ 3. *Verification of the formulae given in the previous section.*

In this section we require a convenient notation for the result of imposing an arbitrary linear condition upon a function $\phi(s)$. The linear conditions which we shall consider will in general be of the form

$$\int_a^b \alpha(s)\,\phi(s)\,ds + \sum_y \sum_n \rho_n(y)\frac{d^n}{dy^n}\phi(y) = \text{constant} \quad\text{..... (33),}$$

and it will be supposed that the operator which produces the left-hand side when it acts on $\phi(s)$ is such that it can be introduced under the integral sign whenever $\phi(s)$ has the form

$$\int_a^b k(s, t)\,\chi(t)\,dt.$$

If the result of this operation is expressed by replacing the variable s in the function $\phi(s)$ by means of a star, the effect of imposing the given linear condition upon $\phi(s)$ will be represented by

$$\phi(*) = c \quad\text{..... (34),}$$

and the effect of our operation upon the function represented by the above integral will be

$$\int_a^b k(*, t)\,\chi(t)\,dt.$$

Let λ_0 be a value of λ for which the function $\phi(s, \lambda)$ given by the equation

$$f(s) = \phi(s) - \lambda\int_a^b k(s, t)\,\phi(t)\,dt$$

satisfies the linear condition $\phi(*) = c$; then we have

$$f(*) = c - \lambda_0\int_a^b k(*, t)\,\phi_0(t)\,dt \quad\text{..... (35).}$$

This relation may be used to reduce the equation

$$f(s) = \phi_0(s) - \lambda_0\int_a^b k(s, t)\,\phi_0(t)\,dt$$

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to the homogeneous form

$$\phi_0(s) = \lambda_0 \int_a^b \left[k(s, t) - \frac{f(s)}{f(*) - c} k(*, t) \right] \phi_0(t)\, dt \quad \text{.....(36).}$$

Now according to the rule stated in § 2 the solving function of the integral equation

$$\psi(s) = \chi(s) - \lambda \int_a^b k_1(s, t)\, \chi(t)\, dt \quad \text{..... (37),}$$

in which
$$k_1(s, t) = k(s, t) - \frac{f(s)}{f(*) - c} k(*, t),$$

should be given by
$$K_1(s, t) = K(s, t) - \frac{\phi(s)}{\phi(*) - c} K(*, t) \quad \text{.....(38).}$$

To verify that this is the case it is simply necessary to show that the characteristic relation

$$k_1(s, t) = K_1(s, t) - \lambda \int_a^b k_1(s, x)\, K_1(x, t)\, dx \quad \text{.....(39),}$$

which is obtained by substituting in (37) the value of χ given by

$$\chi(s) = \psi(s) + \lambda \int_a^b K_1(s, t)\, \psi(t)\, dt,$$

is satisfied.

Now
$$\lambda \int_a^b k(s, x)\, K(x, t)\, dx = K(s, t) - k(s, t),$$

$$\lambda \int_a^b k(*, x)\, \phi(x)\, dx = \phi(*) - f(*),$$

$$\lambda \int_a^b k(*, x)\, K(x, t)\, dx = K(*, t) - k(*, t),$$

$$\lambda \int_a^b k(s, x)\, \phi(x)\, dx = \phi(s) - f(s);$$

accordingly, when the values of $k_1(s, x)$ and $K_1(x, t)$ are substituted, we have

$$\lambda \int_a^b k_1(s, x)\, K_1(x, t)\, dx = K(s, t) - k(s, t) - \frac{K(*, t)}{\phi(*) - c} [\phi(s) - f(s)]$$

$$- \frac{f(s)}{f(*) - c} [K(*, t) - k(*, t)] + \frac{f(s)\, K(*, t)}{[f(*) - c]\, [\phi(*) - c]} [\phi(*) - f(*)],$$

$$= K(s, t) - k(s, t) - \frac{K(*, t)}{\phi(*) - c} \phi(s) + \frac{k(*, t)}{f(*) - c} f(s),$$

$$= K_1(s, t) - k_1(s, t);$$

hence the relation is satisfied.

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To obtain the value of $\delta_1(\lambda)$ we make use of the formula

$$\int_a^b K_1(t, t)\, dt = -\frac{d}{d\lambda}[\log \delta_1(\lambda)].$$

Substituting the value of $K_1(t, t)$ we have

$$-\frac{d}{d\lambda}[\log \delta_1(\lambda)] = -\frac{d}{d\lambda}[\log \delta(\lambda)] - \frac{1}{\phi(*) - c}\int_a^b K(*, t)\,\phi(t)\, dt \quad \text{.....} (40).$$

To obtain the value of $\int_a^b K(s, t)\,\phi(t)\, dt$ we multiply the equation

$$k(s, t) = K(s, t) - \lambda \int_a^b k(s, x)\, K(x, t)\, dx$$

by $\phi(t)\, dt$ and integrate, thus

$$\int_a^b k(s, t)\,\phi(t)\, dt = \int_a^b K(s, t)\,\phi(t)\, dt - \lambda \int_a^b k(s, x)\, dx \int_a^b K(x, t)\,\phi(t)\, dt.$$

But if we differentiate the equation

$$f(s) = \phi(s) - \lambda \int_a^b k(s, x)\,\phi(x)\, dx,$$

with regard to λ, we obtain

$$0 = \frac{\partial \phi(s)}{\partial \lambda} - \int_a^b k(s, x)\,\phi(x)\, dx - \lambda \int_a^b k(s, x)\frac{\partial \phi(x)}{\partial \lambda}\, dx.$$

Comparing this with the previous equation we see that

$$\int_a^b K(s, t)\,\phi(t)\, dt = \frac{\partial}{\partial \lambda}\phi(s) \quad \text{.....} (41),$$

and therefore

$$\int_a^b K(*, t)\,\phi(t)\, dt = \frac{\partial}{\partial \lambda}\phi(*) \quad \text{.....} (42).$$

Substituting this in equation (40) we get

$$-\frac{d}{d\lambda}[\log \delta_1(\lambda)] = -\frac{d}{d\lambda}[\log \delta(\lambda)] - \frac{d}{d\lambda}[\log \{\phi(*) - c\}].$$

Now when $\lambda = 0$, $\delta_1(\lambda) = \delta(\lambda) = 1$, $\phi(s) = f(s)$; therefore on integrating the above equation we get

$$\delta_1(\lambda) = \frac{\phi(*) - c}{f(*) - c}\,\delta(\lambda) \quad \text{.....} (43),$$

which is the other formula that was used in § 2.

To determine the function $\theta_0(t)$ adjoint to $\phi_0(s)$ we must find the limit of $(\lambda_0 - \lambda)\, K_1(s, t)$ when $\lambda = \lambda_0$. Substituting the value

$$K_1(s, t) = K(s, t) - \frac{\phi(s)}{\phi(*) - c}\, K(*, t),$$

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and remembering that $K(s, t)$, $\phi(s)$ and $K(*, t)$ remain finite for $\lambda = \lambda_0$, we have

$$\underset{\lambda=\lambda_0}{\mathrm{Lt}}\ (\lambda_0 - \lambda) K_1(s, t) = \frac{\phi_0(s) K_0(*, t)}{\frac{d}{d\lambda_0}\phi(*)},$$

accordingly

$$\theta_0(t) = \frac{K_0(*, t)}{\frac{d}{d\lambda_0}\phi(*)} \quad \text{.....} \quad (44).$$

Also since

$$\int_a^b K(s, t)\, \phi(t)\, dt = \frac{d}{d\lambda}\phi(s),$$

it is evident that the equation

$$\int_a^b \theta_0(t)\, \phi_0(t)\, dt = 1$$

is satisfied.

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THE VARIATION OF THE ABSORPTION BANDS OF A CRYSTAL IN A MAGNETIC FIELD.

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XIII. *The Variation of the Absorption Bands of a Crystal in a Magnetic Field.*

By W. M. Page, B.A., King's College.

[Communicated by Mr H. W. Richmond.]

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In March and April, 1906, M. Jean Becquerel published an account of some observations on the effect of a magnetic field on the absorption bands of xenotime, a uniaxal substance giving sharp bands.

Two regions, one in the green, and one in the red, were observed, and it was found that a diffusion and doubling of the bands occurred, analogous to the Zéeman Effect for metallic vapours.

The first series of observations* deals with light rays perpendicular to the magnetic field, and the variations of the bands are given when the optic axis of the crystal is

(1) in the direction of the light rays,

(2) perpendicular to the light rays and to the field,

(3) parallel to the field.

The most important results obtained were

(1) Certain bands are displaced to a greater extent than would be expected from a knowledge of the Zéeman Effect in metallic vapours.

(2) The bands of the ordinary spectrum behave very differently for different directions of the axis of the crystal, although the orientation of the ordinary vibration remains the same with respect to the magnetic field.

Thus when the axis is normal to the field, and the vibration is also normal to the field, the green band No. 18 in Becquerel's Table (wave-length 522·14 $\mu\mu$) only gives one component, slightly displaced towards the violet; but when the axis is parallel to the field, the same band, for the same direction of the vibrations, doubles, and the separation of the components is 0·53 $\mu\mu$ in a field of 31,800 C.G.S. units.

* Jean Becquerel, *C. R.* March 26, 1906.

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On the other hand, the green band No. 22 (524·57 $\mu\mu$) undergoes a large displacement in the first case, and hardly varies in the second.

Again, there is a remarkable difference in intensity in the components of the doublets separated by the field.

For example, in red band No. 6 (653·71 $\mu\mu$), and green band No. 22, the component displaced towards the red can only be seen at the instant of the field establishing itself progressively. This component disappears as soon as the field becomes intense.

In a later paper* the case of light rays parallel to the field is considered. If the optic axis is perpendicular to the field, the modifications of the bands for the ordinary vibration normal to the field and to the axis are identical with those given in the first paper. The same is true for the extraordinary vibration normal to the field.

When the optic axis is parallel to the field the ordinary spectrum alone is visible. As soon as the magnetic field is established the edges of all the bands diffuse, and the bands double.

The components are found to be circularly polarised in opposite senses.

If, however, the axis of the crystal is inclined at a few degrees to the lines of force, it is found that for two symmetric positions A, B, independent of the strength of the field, but depending on the thickness of the crystal, the components become plane polarised in two perpendicular directions, independent of the field strength, and turning through a right angle when the sense of the field is changed.

A very slight displacement from these positions A, B makes this state of polarisation cease, and the bands again become circularly polarised.

However, in the immediate neighbourhood of these positions a partial polarisation of the two components is observed.

These phenomena are produced for all the bands at almost the same instant.

In observations on the Zéeman Effect previously made it had always been found that the components of the doublets which exhibited right-handed circular polarisation were displaced all in the same direction, and that the left-handed components were displaced in the other direction.

M. Becquerel, however, finds that for a large number of bands this is not the case, but that the components are displaced in a sense opposite to that previously observed.

We shall suppose the crystal to be made up of a series of electrons moving about centres under forces varying as the distance. For any one electron the intensity of this force along the direction of the optic axis will differ from its intensity in any direction perpendicular to the axis. In addition to these forces there will be forces of so-called 'frictional' type, due to the action of neighbouring electrons, causing absorption.

We will first consider the case of a single electron moving under the action of the central forces and external magnetic field only.

* Jean Becquerel, *C. R.* April 9, 1906.

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Suppose the magnetic field, H, to be parallel to the z axis. If the axis of the system is also along Oz, the equations of motion are

$$m\ddot{x} = -kx + e\dot{y}H,$$
$$m\ddot{y} = -ky - e\dot{x}H,$$
$$m\ddot{z} = -k_1 z,$$

e, m being the charge and mass of the electron, and k, k_1 constants.

If we suppose x, y, z to vary as e^{ipt}, we obtain from these equations

$$(-mp^2 + k)\, x_0 = \iota epHy_0,$$
$$(-mp^2 + k)\, y_0 = -\iota epHx_0,$$
$$(-mp^2 + k_1)\, z_0 = 0.$$

Hence the period of the z vibration is unaltered by the field.

The periods of the x, y vibrations are given by

$$(mp^2 - k)^2 = e^2H^2p^2,$$

or

$$mp^2 - k = \pm\, eHp.$$

If p_0 denote the value of p before the field is established, and $p_0 + \delta p_0$ the disturbed value, we have

$$2mp_0\delta p_0 = \pm\, eHp_0.$$

Thus we now get the two periods

$$p_0 + \frac{eH}{2m}, \quad p_0 - \frac{eH}{2m}.$$

Hence an emission line of the system polarised in a plane perpendicular to the xy plane will be split up into two, owing to the action of the magnetic field, if the axis of the system is parallel to the field.

The line polarised in the xy plane will be unaltered.

If, however, the axis of the system is perpendicular to the field, and lies along Oy, the period of the z vibration is given by

$$mp^2 = k,$$

while the periods of the xy vibrations are given by

$$(mp^2 - k)(mp^2 - k_1) = e^2H^2p^2 \qquad (1).$$

In the absence of the field, the values of p are given by

$$mp_0^2 = k, \quad \text{or} \quad mp_0^2 = k_1.$$

The disturbance in the first period is found from (1) by substituting

$$p = p_0 + \delta p_0, \quad \text{when } mp_0^2 = k.$$

Provided that $k - k_1$ is not small, we thus get

$$2mp_0\delta p_0\,(k - k_1) = e^2H^2p_0^2,$$

or

$$\delta p_0 = \frac{e^2H^2p_0}{2m\,(k - k_1)}.$$

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Thus the band corresponding to this value of p_0 is not doubled but is displaced by an amount depending on H^2. A similar statement holds for the band corresponding to the other value of p_0.

Hence we get a doubling effect due to the field when the plane containing the directions of equal period is perpendicular to the field, but this doubling does not occur when the plane perpendicular to the field contains directions of unequal period.

The above is to be regarded merely as an illustration giving us some idea of the nature of the effects that are to be expected in a crystalline system in a magnetic field. It can have no real validity since the ideal system contemplated is a powerful radiator, and its vibrations could not persist for more than an infinitesimal time.

We now return to the general case.

If ξ, η, ζ be the x, y, z components of the displacement of the electron from the position of equilibrium the equations of motion will be of the form

$$m\frac{\partial^2\xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta}\xi - re^2\frac{\partial\xi}{\partial t},$$

with similar equations in η, ζ.

Here e, m are the charge and mass of the electron; θ and r are positive constants; X is the x component of the external electric field.

The last terms in this equation are respectively the x components of the central and 'frictional' forces.

The quantity e^2 is introduced into the coefficients of these terms to shew that the corresponding forces are independent of the sign of the charge.

For a crystalline body θ will be supposed different in different directions; but we shall assume, for simplicity, that r is independent of direction, although this is in all probability not the case when the asymmetric structure of the crystal is taken into consideration.

The components of electric current at any point will consist of two parts, viz. (considering the x component only)

(1) The current of aethereal displacement produced by the electric force X. This is equal to $\frac{1}{4\pi}\frac{\partial X}{\partial t}$.

(2) The current due to the motion of the charges carried by the electrons. This is equal to $\Sigma eN\frac{\partial\xi}{\partial t}$, where N denotes the number of electrons of any particular type in unit volume.

Thus the components of electric current are given by

$$\gamma_x = \frac{1}{4\pi}\frac{\partial X}{\partial t} + \Sigma eN\frac{\partial\xi}{\partial t},$$

$$\gamma_y = \frac{1}{4\pi}\frac{\partial Y}{\partial t} + \Sigma eN\frac{\partial\eta}{\partial t},$$

$$\gamma_z = \frac{1}{4\pi}\frac{\partial Z}{\partial t} + \Sigma eN\frac{\partial\zeta}{\partial t}.$$

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Now suppose that a magnetic field H is acting. In consequence of this each electron will be affected by a force $\frac{1}{c}.e[v.H]$ where v is the velocity of the electron and c the velocity of light in aether.

Hence if θ_1, θ_2, θ_3 denote the values of θ in the directions of Ox, Oy, Oz the equations of motion of the electron become

$$m\frac{\partial^2\xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta_1}\xi - re^2\frac{\partial\xi}{\partial t} + \frac{e}{c}\left(\frac{\partial\eta}{\partial t}H_z - \frac{\partial\zeta}{\partial t}H_y\right),$$

$$m\frac{\partial^2\eta}{\partial t^2} = eY - \frac{4\pi e^2}{\theta_2}\eta - re^2\frac{\partial\eta}{\partial t} + \frac{e}{c}\left(\frac{\partial\zeta}{\partial t}H_x - \frac{\partial\xi}{\partial t}H_z\right),$$

$$m\frac{\partial^2\zeta}{\partial t^2} = eZ - \frac{4\pi e^2}{\theta_3}\zeta - re^2\frac{\partial\zeta}{\partial t} + \frac{e}{c}\left(\frac{\partial\xi}{\partial t}H_y - \frac{\partial\eta}{\partial t}H_x\right).$$

Now let us suppose that light waves of period $2\pi\tau$ are traversing the crystal. Then in the steady state which supervenes ξ, η, ζ will vary as $e^{\iota\frac{t}{\tau}}$.

Hence from the equations of motion we get

$$e\xi\lambda_1 - \iota\frac{e\mu_1}{H}(\eta H_z - \zeta H_y) = \frac{\theta_1}{4\pi}X,$$

$$e\eta\lambda_2 - \iota\frac{e\mu_2}{H}(\zeta H_x - \xi H_z) = \frac{\theta_2}{4\pi}Y,$$

$$e\zeta\lambda_3 - \iota.\frac{e\mu_3}{H}(\xi H_y - \eta H_x) = \frac{\theta_3}{4\pi}Z,$$

where
$$\lambda_1 = 1 + \iota.\frac{a_1}{\tau} - \frac{b_1}{\tau^2}, \quad \mu_1 = \frac{\theta_1 H}{4\pi c\tau e},$$

$$a_1 = \frac{r\theta_1}{4\pi}, \quad b_1 = \frac{m\theta_1}{4\pi e^2},$$

with similar meanings for $\lambda_2\mu_2$, etc.

We have assumed the magnetic force in the light wave to have no appreciable effect. If this were not the case the equations of the field which we shall derive would no longer be linear as regards the electric and magnetic forces of the light wave. This would mean that the optical qualities of the body would depend on the intensity of the light. An effect of this kind, however, has never been observed *.

Taking the magnetic field parallel to Oz the above equations become

$$e\xi\lambda_1 - \iota e\mu_1\eta = \frac{\theta_1}{4\pi}X,$$

$$e\eta\lambda_2 + \iota e\mu_2\xi = \frac{\theta_2}{4\pi}Y,$$

$$e\zeta\lambda_3 \qquad = \frac{\theta_3}{4\pi}Z.$$

* Drude, *Optics*, pp. 452 *et seq.*

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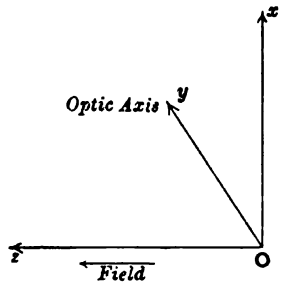
Optic Axis normal to the Field.

Suppose the axis to lie along Oy.

We have



$$\theta_1 = \theta_3 = \theta \text{ say.}$$

$$\theta_2 = \theta' \text{ say.}$$

$$a_1 = a_3 = a = \frac{r\theta}{4\pi},$$

$$a_2 = a' = \frac{r\theta'}{4\pi},$$

$$b_1 = b_3 = b = \frac{m\theta}{4\pi e^2},$$

$$b_2 = b' = \frac{m\theta'}{4\pi e^2},$$

$$\lambda_1 = \lambda_3 = \lambda = 1 + \iota\frac{a}{\tau} - \frac{b}{\tau^2},$$

$$\lambda_2 = \lambda' = 1 + \iota\,.\,\frac{a'}{\tau} - \frac{b'}{\tau^2},$$

$$\mu_1 = \mu_3 = \mu = \frac{\theta H}{4\pi c\tau e},$$

$$\mu_2 = \mu' = \frac{\theta' H}{4\pi c\tau e};$$

$$\therefore\ e\xi\lambda - \iota e\mu\eta = \frac{\theta}{4\pi} X,$$

$$e\eta\lambda' + \iota e\mu'\xi = \frac{\theta'}{4\pi} Y,$$

$$e\zeta\lambda = \frac{\theta}{4\pi} Z.$$

Solving for ξ, η, ζ we get

$$4\pi e\xi\,(\lambda\lambda' - \mu\mu') = \lambda'\theta X + \iota\mu\theta' Y,$$

$$4\pi e\eta\,(\lambda\lambda' - \mu\mu') = \lambda\theta' Y - \iota\mu'\theta X,$$

$$4\pi e\zeta\lambda = \theta\,.\,Z.$$

Therefore the current components are given by

$$4\pi\gamma_x = \frac{\partial X}{\partial t}\left(1 + \Sigma\frac{\theta N\lambda'}{\lambda\lambda' - \mu\mu'}\right) + \iota\frac{\partial Y}{\partial t}\Sigma\frac{\theta' N\mu}{\lambda\lambda' - \mu\mu'}.$$

$$4\pi\gamma_y = \frac{\partial Y}{\partial t}\left(1 + \Sigma\frac{\theta' N\lambda}{\lambda\lambda' - \mu\mu'}\right) - \iota\frac{\partial X}{\partial t}\Sigma\frac{\theta N\mu'}{\lambda\lambda' - \mu\mu'},$$

$$4\pi\gamma_z = \frac{\partial Z}{\partial t}\left(1 + \Sigma\frac{\theta N}{\lambda}\right).$$

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If we put
$$\epsilon_1 = 1 + \Sigma \frac{\theta N \lambda'}{\lambda\lambda' - \mu\mu'}, \qquad \nu_1 = \Sigma \frac{\theta' N \mu}{\lambda\lambda' - \mu\mu'},$$
$$\epsilon_2 = 1 + \Sigma \frac{\theta' N \lambda}{\lambda\lambda' - \mu\mu'}, \qquad \nu_2 = \Sigma \frac{\theta N \mu'}{\lambda\lambda' - \mu\mu'},$$
$$\epsilon_3 = 1 + \Sigma \frac{\theta N}{\lambda},$$

we obtain
$$4\pi\gamma_x = \epsilon_1 \frac{\partial X}{\partial t} + \iota\nu_1 \frac{\partial Y}{\partial t},$$
$$4\pi\gamma_y = \epsilon_2 \frac{\partial Y}{\partial t} - \iota\nu_2 \frac{\partial X}{\partial t},$$
$$4\pi\gamma_z = \epsilon_3 \frac{\partial Z}{\partial t}.$$

Optic Axis parallel to the Field.

The axis lies along Oz and we have
$$\theta_1 = \theta_2 = \theta, \quad \theta_3 = \theta'.$$
With the above meanings for λ, λ', μ, μ' we get
$$e\xi\lambda - \iota e\mu\eta = \frac{\theta}{4\pi} X,$$
$$e\eta\lambda + \iota e\mu\xi = \frac{\theta}{4\pi} Y,$$
$$e\zeta\lambda' = \frac{\theta'}{4\pi} Z.$$

Hence
$$4\pi\gamma_x = \frac{\partial X}{\partial t}\left(1 + \Sigma \frac{N\lambda\theta}{\lambda^2 - \mu^2}\right) + \iota \frac{\partial Y}{\partial t} \Sigma \frac{N\mu\theta}{\lambda^2 - \mu^2},$$
$$4\pi\gamma_y = \frac{\partial Y}{\partial t}\left(1 + \Sigma \frac{N\lambda\theta}{\lambda^2 - \mu^2}\right) - \iota \frac{\partial X}{\partial t} \Sigma \frac{N\mu\theta}{\lambda^2 - \mu^2},$$
$$4\pi\gamma_z = \frac{\partial Z}{\partial t}\left(1 + \Sigma \frac{N\theta'}{\lambda'}\right).$$

Writing
$$\epsilon_1 = 1 + \Sigma \frac{N\lambda\theta}{\lambda^2 - \mu^2}, \qquad \nu_1 = \Sigma \frac{N\mu\theta}{\lambda^2 - \mu^2};$$
$$\epsilon_2 = 1 + \Sigma \frac{N\theta'}{\lambda'},$$

we have
$$4\pi\gamma_x = \epsilon_1 \frac{\partial X}{\partial t} + \iota\nu_1 \frac{\partial Y}{\partial t},$$
$$4\pi\gamma_y = \epsilon_1 \frac{\partial Y}{\partial t} - \iota\nu_1 \frac{\partial X}{\partial t},$$
$$4\pi\gamma_z = \epsilon_2 \frac{\partial Z}{\partial t}.$$

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Light Rays perpendicular to the magnetic field.

If α, β, γ are the components of magnetic force in the light wave, Maxwell's Equations may be written

$$\frac{1}{c}\frac{\partial}{\partial t}(\alpha, \beta, \gamma) = -\operatorname{curl}(X, Y, Z),$$

$$\frac{4\pi}{c}(\gamma_x, \gamma_y, \gamma_z) = \operatorname{curl}(\alpha, \beta, \gamma).$$

We shall consider light waves travelling along Oy, so that the quantities involved are functions of y only.

Then Maxwell's Equations give

$$\beta = 0, \quad \gamma_y = 0,$$

$$\frac{1}{c}\frac{\partial \alpha}{\partial t} = -\frac{\partial Z}{\partial y}, \qquad \frac{1}{c}\frac{\partial \gamma}{\partial t} = \frac{\partial X}{\partial y},$$

$$\frac{4\pi}{c}\gamma_x = \frac{\partial \gamma}{\partial y}, \qquad \frac{4\pi}{c}\gamma_z = -\frac{\partial \alpha}{\partial y}.$$

Hence, eliminating α, γ we get

$$\gamma_y = 0,$$

$$\frac{4\pi}{c^2}\frac{\partial}{\partial t}\gamma_z = \frac{\partial^2 Z}{\partial y^2}, \qquad \frac{4\pi}{c^2}\frac{\partial}{\partial t}\gamma_x = \frac{\partial^2 X}{\partial y^2}.$$

Optic Axis perpendicular to the Field, and parallel to the Rays.

Substituting in these equations the values of $\gamma_x, \gamma_y, \gamma_z$ given on p. 297 we get

$$\epsilon_2 Y - \iota\nu_2 X = 0,$$

$$\frac{1}{c^2}\left(\epsilon_1 \frac{\partial^2 X}{\partial t^2} + \iota\nu_1 \frac{\partial^2 Y}{\partial t^2}\right) = \frac{\partial^2 X}{\partial y^2},$$

$$\frac{\epsilon_3}{c^2}\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial y^2}.$$

Eliminating Y between the first two equations, we obtain

$$\frac{1}{c^2}\left(\epsilon_1 - \frac{\nu_1\nu_2}{\epsilon_2}\right)\frac{\partial^2 X}{\partial t^2} = \frac{\partial^2 X}{\partial y^2}.$$

Put $$X = Le^{\frac{\iota}{\tau}(t-py)}, \quad Y = Me^{\frac{\iota}{\tau}(t-py)}, \quad Z = Ne^{\frac{\iota}{\tau}(t-p_1 y)}.$$

Then $$M = \frac{\iota\nu_2}{\epsilon_2}L,$$

$$p^2c^2 = \epsilon_1 - \frac{\nu_1\nu_2}{\epsilon_2},$$

$$p_1^2c^2 = \epsilon_3.$$

Since $\epsilon_3 = 1 + \Sigma\frac{\theta N}{\lambda}$, and so is independent of H, it follows that the z vibrations are unaffected by the field.

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To find the effect of the field on the X and Y vibrations put

$$pc = n(1 - \iota\kappa),$$

so that n is the index of refraction, and κ the coefficient of absorption.

Then $$n^2(1-\iota\kappa)^2 = \epsilon_1 - \frac{\nu_1\nu_2}{\epsilon_2} = 1 + \Sigma\frac{\theta N\lambda'}{\lambda\lambda' - \mu\mu'} - \frac{\Sigma\dfrac{\theta' N\mu}{\lambda\lambda' - \mu\mu'} \cdot \Sigma\dfrac{\theta N\mu'}{\lambda\lambda' - \mu\mu'}}{1 + \Sigma\dfrac{\theta' N\lambda}{\lambda\lambda' - \mu\mu'}},$$

from p. 297.

The quantities λ, λ' are complex, and it is due to the imaginary parts of these terms that we get an absorption κ.

We have $$\lambda = 1 + \iota\,.\,\frac{a}{\tau} - \frac{b}{\tau^2}, \quad \lambda' = 1 + \iota\,.\,\frac{a'}{\tau} - \frac{b'}{\tau^2},$$

$$\mu = \frac{\theta H}{4\pi c\tau e}, \qquad \mu' = \frac{\theta' H}{4\pi c\tau e}.$$

We shall suppose the frictional terms $\frac{a}{\tau}$, $\frac{a'}{\tau}$ in λ, λ' to be negligible unless the impressed period τ is almost coincident with a 'natural' period of vibration of the electron.

Returning to the equation of motion of an ion given on p. 294 and supposing the frictional and impressed forces to be absent, we have

$$m\frac{\partial^2\xi}{\partial t^2} + \frac{4\pi e^2}{\theta}\xi = 0,$$

or $$\frac{\partial^2\xi}{\partial t^2} + \frac{1}{b}\xi = 0.$$

Thus the 'natural' period of a vibration of the electron along the x or z axis is given by

$$\tau_1 = \sqrt{b}.$$

Suppose that the impressed period is given by

$$\tau = \sqrt{b}(1+g) = \tau_1(1+g), \text{ where } g \text{ is small.}$$

Then for ions of the type we are considering

$$\lambda = 1 + \iota\,.\,\frac{a}{\tau} - \frac{1}{(1+g)^2} = 2g + \iota\,.\,\frac{a}{\tau} \text{ neglecting } g^2,$$

and so is small.

$\lambda' = 1 - \frac{b'}{b}$ neglecting g compared with the other terms.

For electrons of other types we have

$$\lambda = 1 - \frac{b}{\tau_1^2}, \qquad \lambda' = 1 - \frac{b'}{\tau_1^2}.$$

If Σ' refer to these other types only, we have

$$\epsilon_1 = 1 + \Sigma'\frac{\theta N\lambda'}{\lambda\lambda' - \mu\mu'} + \frac{\theta N\left(1 - \dfrac{b'}{b}\right)}{\left(1 - \dfrac{b'}{b}\right)\left(2g + \iota\,.\,\dfrac{a}{\tau_1}\right) - \theta\theta'\left(\dfrac{H}{4\pi c\tau_1 e}\right)^2}.$$

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Neglecting terms which depend on H^2 we get

$$\epsilon_1 = 1 + \Sigma' \frac{\theta N}{\lambda} + \frac{\theta N}{2g + \iota . \frac{a}{\tau_1}}.$$

Again $\frac{\nu_1 \nu_2}{\epsilon_2}$ is of the second order in H, and so will be neglected.

Therefore
$$p^2c^2 = n^2(1 - \iota\kappa)^2 = 1 + \Sigma' \frac{\theta N}{\lambda} + \frac{\theta N}{2g + \iota . \frac{a}{\tau_1}}.$$

Since $\Sigma' \frac{\theta N}{\lambda}$ is real, it follows that the imaginary part of the right-hand side, on which κ depends, will be a maximum when $g = 0$.

Thus the maximum of κ, which gives the position of the absorption band, is unaltered to the first order by the presence of the magnetic field.

Hence, to the first order, there is no displacement of the bands of the ordinary spectrum when the axis is perpendicular to the field.

Optic Axis parallel to the Field.

We substitute the values of γ_x, γ_y, γ_z given on p. 297 in the equations on p. 298. Supposing that

$$X = Le^{\frac{\iota}{\tau}(t - py)}, \quad Y = Me^{\frac{\iota}{\tau}(t - py)}, \quad Z = Ne^{\frac{\iota}{\tau}(t - p_1 y)},$$

we obtain by the same process as above

$$M = \frac{\iota\nu_1}{\epsilon_1} L,$$

$$\frac{1}{c^2}\left(\epsilon_1 - \frac{\nu_1^2}{\epsilon_1}\right) = p^2,$$

$$\frac{\epsilon_2}{c^2} = p_1^2.$$

Since from p. 297

$$\epsilon_2 = 1 + \Sigma \frac{N\theta'}{\lambda'},$$

and so is independent of H, it follows that the z vibrations are unaltered by the field.

Again, putting $pc = n(1 - \iota\kappa)$ we get

$$n^2(1 - \iota\kappa)^2 = \frac{\epsilon_1^2 - \nu_1^2}{\epsilon_1} = \frac{\left\{1 + \Sigma \frac{N\lambda\theta}{\lambda^2 - \mu^2}\right\}^2 - \left\{\Sigma \frac{N\mu\theta}{\lambda^2 - \mu^2}\right\}^2}{1 + \Sigma \frac{N\lambda\theta}{\lambda^2 - \mu^2}}$$

$$= \frac{\left(1 + \Sigma \frac{N\theta}{\lambda + \mu}\right)\left(1 + \Sigma \frac{N\theta}{\lambda - \mu}\right)}{1 + \Sigma \frac{N\lambda\theta}{\lambda^2 - \mu^2}}.$$

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As before, we suppose the impressed period nearly to coincide with a 'natural' period, and put

$$\tau = \sqrt{b}(1+g) = \tau_1(1+g).$$

Then for the particular electrons under consideration

$$\lambda = 2g + \iota . \frac{a}{\tau_1},$$

which is small, while for the other electrons

$$\lambda = 1 - \frac{b}{\tau_1^2}, \quad \lambda' = 1 - \frac{b'}{\tau_1^2},$$

and these are not small.

The electric force in the incident wave may be resolved into two components, one along Oz, the other along Ox.

It is the vibration corresponding to the latter with which we are dealing at the present moment.

For the other electrons μ is small compared with λ. Hence for these, we have

$$\frac{1}{\lambda+\mu} = \frac{1}{\lambda}\left(1 - \frac{\mu}{\lambda}\right) = \frac{1}{1-\frac{b}{\tau_1^2}}\left(1 - \frac{\mu}{1-\frac{b}{\tau_1^2}}\right).$$

Thus if Σ' refer to these other electrons only, we get

$$1 + \Sigma' \frac{N\theta}{\lambda+\mu} = 1 + \Sigma' \frac{N\theta}{\lambda} - \Sigma' \frac{N\theta\mu}{\lambda^2}$$
$$= A - A',$$

where

$$A = 1 + \Sigma' \frac{N\theta}{1-\frac{b}{\tau_1^2}},$$

$$A' = \Sigma' \frac{N\theta\mu}{\left(1-\frac{b}{\tau_1^2}\right)^2}.$$

Thus
$$n^2(1-\iota\kappa)^2 = \frac{\left(A - A' + \dfrac{N\theta}{2g + \iota . \frac{a}{\tau} + \mu}\right)\left(A + A' + \dfrac{N\theta}{2g + \iota . \frac{a}{\tau} - \mu}\right)}{A + \dfrac{N\theta\left(2g + \iota . \frac{a}{\tau}\right)}{\left(2g + \iota\frac{a}{\tau}\right)^2 - \mu^2}}$$

$$= \frac{A^2 + A . \dfrac{2N\theta\left(2g + \iota\frac{a}{\tau}\right)}{\left(2g + \iota\frac{a}{\tau}\right)^2 - \mu^2} + \dfrac{N^2\theta^2}{\left(2g + \iota\frac{a}{\tau}\right)^2 - \mu^2} - \dfrac{2A'\mu N\theta}{\left(2g + \iota\frac{a}{\tau}\right)^2 - \mu^2} - A'^2}{A + \dfrac{N\theta\left(2g + \iota . \frac{a}{\tau}\right)}{\left(2g + \iota . \frac{a}{\tau}\right)^2 - \mu^2}}.$$

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Now A', μ both contain H as a factor, and also $2g + \iota . \frac{a}{\tau}$ will be of the same order as μ.

Hence the terms in the numerator which contain A' may both be neglected in comparison with the other terms.

Therefore
$$n^2(1-\iota\kappa)^2 = A + \frac{A . \dfrac{N\theta\left(2g+\iota.\frac{a}{\tau}\right)}{\left(2g+\iota\frac{a}{\tau}\right)^2-\mu^2} + \dfrac{N^2\theta^2}{\left(2g+\iota\frac{a}{\tau}\right)^2-\mu^2}}{A + \dfrac{N\theta\left(2g+\iota.\frac{a}{\tau}\right)}{\left(2g+\iota\frac{a}{\tau}\right)^2-\mu^2}}$$

$$= A + \frac{N\theta\left\{A\left(2g+\iota\frac{a}{\tau}\right)+N\theta\right\}}{A\left\{\left(2g+\iota\frac{a}{\tau}\right)^2-\mu^2\right\}+N\theta\left(2g+\iota.\frac{a}{\tau}\right)}.$$

Now, when no magnetic field is present the dielectric constant ϵ_1 is given by

$$\epsilon_1 = 1 + \Sigma\frac{N\theta}{\lambda}$$
$$= 1 + \Sigma\frac{N\theta}{1-\frac{\tau^2}{\tau_1^2}},$$

where τ_1 gives the unpressed period, and τ the natural period of an ion. We have supposed τ_1 to be not very close to τ, so that the frictional terms may be neglected.

Thus $1-\frac{\tau^2}{\tau_1^2}$ is a quantity which will never be very large for visible impressed periods, but may be fairly small.

Hence, since ϵ_1 does not vary very much with the period, and since there must be a large number of vibrating systems in the crystal (as is seen from the large number of bands in the spectrum) it is legitimate to assume that $N\theta$ is very small.

Hence we may write

$$n^2(1-\iota\kappa)^2 = A + \frac{N\theta . A\left(2g+\iota.\frac{a}{\tau}\right)}{A\left\{\left(2g+\iota.\frac{a}{\tau}\right)^2-\mu^2\right\}}.$$

Hence if $\frac{a}{\tau}$ is small, the imaginary part of the right-hand side, determining κ, will be a maximum when $4g^2-\mu^2=0$, i.e. when $2g = \pm\mu$.

These maxima of κ determine the positions of the absorption bands.

Thus, when the axis of the crystal is parallel to the field, a line corresponding to a vibration perpendicular to the axis is split up into two.

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Remembering the result given on p. 300, it may be stated that, according to our present theory, a band of the ordinary spectrum remains unaltered when the axis of the crystal is parallel to the ray, but doubles when the axis is parallel to the field.

The bands of the extraordinary spectrum remain unaltered in the latter case, while the bands of the ordinary spectrum corresponding to vibrations parallel to the field remain unaltered in the former case.

Becquerel, however, found that a large number of bands in the ordinary spectrum varied more or less when the Optic Axis was parallel to the incident rays, and so the results which we have obtained above do not agree with observation.

They are approximately verified for Green band No. 18 (522·14$\mu\mu$), which only varies very slightly when the axis is parallel to the rays. The other bands, however, in many cases double for this position of the axis, which is quite contrary to our results.

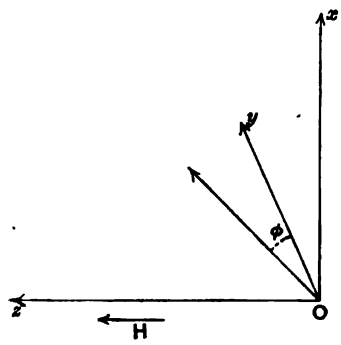
Hence we must seek some extension of our theory to bring it into closer accord with facts.

Becquerel suggests that an orientation of some of the molecules by the magnetic field might remove some of the discrepancies.

We will therefore suppose the axes of one system of electrons to be deflected from the axis of y, and to take up a position in the plane yz.

This position must be definite, otherwise the absorption bands obtained will be blurred and indistinct.

Consider a vibration parallel to Ox. We shall expect a doubling effect to occur, for we have shewn that when there is a magnetic force perpendicular to the plane containing the directions of vibrations of equal period of the electron, the absorption band is split up into two.



When the axis of the electron system is deflected through an angle ϕ, there is a magnetic field $H \sin \phi$ parallel to the axis, and so perpendicular to the plane containing the directions of equal period.

Thus we should expect a doubling, with separation proportional to $H \sin \phi$, to occur.

We will see whether this will be the case, and whether a band corresponding to vibrations along Oz is similarly split up in consequence of this magnetic orientation.

Orientation of the axes of one system of electrons by the Magnetic Field.

We suppose the axis of the crystal to be perpendicular to the field and to lie along Oy. We also suppose that when the magnetic field is established the axes corresponding to one set of electrons are orientated and lie in the plane yz making an angle ϕ with the axis of y.

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The equations of motion of these electrons will be altered. An electron of this system is subject to elastic forces of intensity represented by θ along Ox, θ' along its axis, and θ in a direction perpendicular to Ox and this axis.

Hence the components of the elastic force on this electron along the axes are

$$\frac{4\pi e^2}{\theta}\xi \qquad \text{along } Ox,$$

$$\eta\left(\frac{4\pi e^2}{\theta'}\cos^2\phi + \frac{4\pi e^2}{\theta}\sin^2\phi\right) + \zeta\sin\phi\cos\phi\left(\frac{4\pi e^2}{\theta'} - \frac{4\pi e^2}{\theta}\right) \qquad \text{,, } Oy,$$

$$\zeta\left(\frac{4\pi e^2}{\theta'}\sin^2\phi + \frac{4\pi e^2}{\theta}\cos^2\phi\right) + \eta\sin\phi\cos\phi\left(\frac{4\pi e^2}{\theta'} - \frac{4\pi e^2}{\theta}\right) \qquad \text{,, } Oz.$$

Thus the equations of motion for ions of this type are

$$m\frac{\partial^2\xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta}\xi - re^2\frac{\partial\xi}{\partial t} + \frac{eH}{c}\frac{\partial\eta}{\partial t},$$

$$m\frac{\partial^2\eta}{\partial t^2} = eY - \left(\frac{4\pi e^2}{\theta'}\cos^2\phi + \frac{4\pi e^2}{\theta}\sin^2\phi\right)\eta - \sin\phi\cos\phi\left(\frac{4\pi e^2}{\theta'} - \frac{4\pi e^2}{\theta}\right)\zeta - re^2\frac{\partial\eta}{\partial t} - \frac{eH}{c}\frac{\partial\xi}{\partial t},$$

$$m\frac{\partial^2\zeta}{\partial t^2} = eZ - \left(\frac{4\pi e^2}{\theta'}\sin^2\phi + \frac{4\pi e^2}{\theta}\cos^2\phi\right)\zeta - \sin\phi\cos\phi\left(\frac{4\pi e^2}{\theta'} - \frac{4\pi e^2}{\theta}\right)\eta - re^2\frac{\partial\zeta}{\partial t}.$$

With the same meanings for a, a', b, b' as on p. 296 these equations become

$$b\frac{\partial^2\xi}{\partial t^2} = \frac{\theta}{4\pi e}X - \xi - a\frac{\partial\xi}{\partial t} + \frac{eH\theta}{4\pi ce^2}\frac{\partial\eta}{\partial t},$$

$$b'\frac{\partial^2\eta}{\partial t^2} = \frac{\theta'}{4\pi e}Y - (\cos^2\phi + \delta\sin^2\phi)\eta - (1-\delta)\sin\phi\cos\phi\zeta - a'\frac{\partial\eta}{\partial t} - \frac{eH\theta'}{4\pi ce^2}\frac{\partial\xi}{\partial t},$$

$$b\frac{\partial^2\zeta}{\partial t^2} = \frac{\theta}{4\pi e}Z - \left(\cos^2\phi + \frac{1}{\delta}\sin^2\phi\right)\zeta - \left(\frac{1}{\delta} - 1\right)\sin\phi\cos\phi\eta - a\frac{\partial\zeta}{\partial t},$$

where $\dfrac{\theta'}{\theta} = \delta.$

If we suppose everything to vary as $e^{\iota\frac{t}{\tau}}$ and use the values of λ, λ', μ, μ' on p. 296, we obtain

$$e\xi\lambda - \iota e\mu\eta = \frac{\theta}{4\pi}X,$$

$$e\eta(\lambda' + \sigma\sin^2\phi) - e\sigma\sin\phi\cos\phi\zeta + \iota e\mu'\xi = \frac{\theta'}{4\pi}Y,$$

$$e\zeta(\lambda + \sigma'\sin^2\phi) + e\sigma'\sin\phi\cos\phi\eta \qquad = \frac{\theta}{4\pi}Z,$$

where $$\delta - 1 = \sigma, \quad \frac{1}{\delta} - 1 = \sigma'.$$

Solving these equations for ξ, η, ζ and writing, for brevity,

$$\rho = \lambda\lambda' - \mu\mu' + \lambda\sigma\sin^2\phi + \left(\lambda' - \frac{\mu\mu'}{\lambda}\right)\sigma'\sin^2\phi + \sigma\sigma'\sin^2\phi,$$

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we obtain, for the components of electric current

$$4\pi\gamma_x = \frac{\partial}{\partial t}\left[X + \Sigma N \,.\, \frac{\theta X\{\lambda\lambda' + (\lambda\sigma + \lambda'\sigma' + \sigma\sigma')\sin^2\phi\} + \iota\mu\theta' Y(\lambda + \sigma'\sin^2\phi) + \iota\mu\theta Z\sigma\sin\phi\cos\phi}{\lambda\rho}\right],$$

$$4\pi\gamma_y = \frac{\partial}{\partial t}\left[Y + \Sigma N \,.\, \frac{-\iota\mu\theta' X(\lambda + \sigma'\sin^2\phi) + \lambda\theta' Y(\lambda + \sigma'\sin^2\phi) + \lambda\theta Z\sigma\sin\phi\cos\phi}{\lambda\rho}\right],$$

$$4\pi\gamma_z = \frac{\partial}{\partial t}\left[Z + \Sigma N \,.\, \frac{\iota\mu'\theta X\sigma'\sin\phi\cos\phi - \lambda\theta' Y\sigma'\sin\phi\cos\phi + \theta \,.\, Z(\lambda\lambda' - \mu\mu' + \lambda\sigma\sin^2\phi)}{\lambda\rho}\right].$$

Of course in all terms under the summation denoted by Σ which do not correspond to the ions which have had their axes rotated, we put $\phi = 0$.

Put
$$\epsilon_1 = 1 + \Sigma\frac{\theta N\lambda'}{\lambda\lambda' - \mu\mu'}, \qquad \nu_1 = \Sigma\frac{\theta' N\mu}{\lambda\lambda' - \mu\mu'},$$

$$\epsilon_2 = 1 + \Sigma\frac{\theta' N\lambda}{\lambda\lambda' - \mu\mu'}, \qquad \nu_2 = \Sigma\frac{\theta N\mu'}{\lambda\lambda' - \mu\mu'},$$

$$\epsilon_3 = 1 + \Sigma\frac{\theta N}{\lambda}.$$

Then we have

$$4\pi\gamma_x = \left(\epsilon_1 - \frac{N\theta\sigma\mu\mu'}{\lambda\rho} \cdot \frac{\lambda + \sigma'}{\lambda\lambda' - \mu\mu'}\sin^2\phi\right)\frac{\partial X}{\partial t} + \iota\left(\nu_1 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda + \sigma'}{\lambda\lambda' - \mu\mu'} \,.\, \sin^2\phi\right)\frac{\partial Y}{\partial t}$$
$$+ \iota \,.\, \frac{N\mu\theta\sigma}{\lambda\rho}\sin\phi\cos\phi\frac{\partial Z}{\partial t},$$

$$4\pi\gamma_y = \left(\epsilon_2 - \frac{N\theta'\sigma\lambda}{\rho} \cdot \frac{\lambda + \sigma'}{\lambda\lambda' - \mu\mu'}\sin^2\phi\right)\frac{\partial Y}{\partial t} - \iota\left(\nu_2 - \frac{N\theta\mu'\sigma}{\rho} \cdot \frac{\lambda + \sigma'}{\lambda\lambda' - \mu\mu'}\sin^2\phi\right)\frac{\partial X}{\partial t}$$
$$+ \frac{N\theta\sigma\sin\phi\cos\phi}{\rho} \cdot \frac{\partial Z}{\partial t},$$

$$4\pi\gamma_z = \left(\epsilon_3 - \frac{N\theta\sigma'}{\rho} \cdot \frac{\lambda' + \sigma - \frac{\mu\mu'}{\lambda}\sin^2\phi}{\lambda}\right)\frac{\partial Z}{\partial t} + \iota \,.\, \frac{N\mu'\theta\sigma'}{\lambda\rho}\sin\phi\cos\phi\frac{\partial X}{\partial t} - \frac{N\theta'\sigma'\sin\phi\cos\phi}{\rho}\frac{\partial Y}{\partial t}.$$

Consider a wave travelling along the axis of y. If we put

$$(X,\ Y,\ Z) = (M_1,\ N_1,\ P_1)\, e^{\frac{\iota}{\tau}(t - py)},$$

we obtain from Maxwell's Equations as on p. 298,

$$\left(\epsilon_1 - \frac{N\theta\sigma\mu\mu'}{\lambda\rho} \cdot \frac{\lambda + \sigma'}{\lambda\lambda' - \mu\mu'}\sin^2\phi\right)M_1 + \iota\left(\nu_1 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda + \sigma'}{\lambda\lambda' - \mu\mu'}\sin^2\phi\right)N_1 + \iota \,.\, \frac{N\mu\theta\sigma}{\lambda\rho}\sin\phi\cos\phi P_1$$
$$= M_1 p^2 c^2,$$

$$\left(\epsilon_2 - \frac{N\theta'\sigma\lambda}{\rho} \cdot \frac{\lambda + \sigma'}{\lambda\lambda' - \mu\mu'}\sin^2\phi\right)N_1 - \iota\left(\nu_2 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda + \sigma'}{\lambda\lambda' - \mu\mu'}\sin^2\phi\right)M_1 + \frac{N\theta\sigma\sin\phi\cos\phi}{\rho} \,.\, P_1 = 0,$$

$$\left(\epsilon_3 - \frac{N\theta\sigma'}{\rho} \cdot \frac{\lambda' + \sigma - \frac{\mu\mu'}{\lambda}}{\lambda}\sin^2\phi\right)P_1 + \iota \,.\, \frac{N\mu'\theta\sigma'}{\lambda\rho}\sin\phi\cos\phi M_1 - \frac{N\theta'\sigma'\sin\phi\cos\phi}{\rho}N_1 = P_1 p^2 c^2.$$

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Eliminating M_1, N_1, P_1 between these equations, we get for the period equation

$$\left(\epsilon_2 - \frac{\theta' N\lambda\sigma \sin^2\phi}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'}\right)\left(\epsilon_1 - \frac{\theta N\sigma}{\rho} \cdot \frac{\mu\mu'}{\lambda} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi - p^2c^2\right)$$

$$\left(\epsilon_3 - \frac{N\theta\sigma'}{\lambda} \cdot \frac{\lambda' - \frac{\mu\mu'}{\lambda} + \sigma}{\rho} \sin^2\phi - p^2c^2\right)$$

$$+ \frac{N^2\theta\theta'\sigma\sigma'}{\rho^2} \sin^2\phi\cos^2\phi\left(\epsilon_1 - \frac{N\theta\sigma}{\rho} \cdot \frac{\mu\mu'}{\lambda} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi - p^2c^2\right)$$

$$-\left(\nu_1 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right)\left(\nu_2 - \frac{N\theta\mu'\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right)$$

$$\left(\epsilon_3 - \frac{N\theta\sigma'}{\lambda} \cdot \frac{\lambda' - \frac{\mu\mu'}{\lambda} + \sigma}{\rho} \sin^2\phi - p^2c^2\right)$$

$$-\frac{N^2\theta\sigma\sigma'}{\lambda\rho^2} \sin^2\phi\cos^2\phi\left\{\mu'\theta\left(\nu_1 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right) + \mu\left(\theta'\nu_2 - \frac{\mu'\theta}{\lambda}\epsilon_2\right)\right\} = 0.$$

Vibrations parallel to Oz.

If there were no orientation the period of these vibrations would be given by

$$p_0^2c^2 = \epsilon_3.$$

Suppose that when orientation occurs, the period is given by

$$p = p_0 + \delta p_0.$$

Substituting this value in the period equation above, we get

$$-\left(\epsilon_2 - \frac{\theta' N\lambda\sigma \sin^2\phi}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'}\right)\left(\epsilon_1 - \epsilon_3 - \frac{\theta N\sigma}{\rho} \cdot \frac{\mu\mu'}{\lambda} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi - 2p_0c^2\delta p_0\right)$$

$$\left(\frac{N\theta\sigma'}{\lambda} \cdot \frac{\lambda' - \frac{\mu\mu'}{\lambda} + \sigma}{\rho} \sin^2\phi + 2p_0c^2\delta p_0\right)$$

$$+ \frac{N^2\theta\theta'\sigma\sigma'}{\rho^2} \sin^2\phi\cos^2\phi\left(\epsilon_1 - \epsilon_3 - \frac{N\theta\sigma}{\rho} \cdot \frac{\mu\mu'}{\lambda} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi - 2p_0c^2\delta p_0\right)$$

$$+\left(\nu_1 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right)\left(\nu_2 - \frac{N\theta\mu'\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right)$$

$$\left(\frac{N\theta\sigma'}{\lambda} \cdot \frac{\lambda' - \frac{\mu\mu'}{\lambda} + \sigma}{\rho} \sin^2\phi + 2p_0c^2\delta p_0\right)$$

$$-\frac{N^2\theta\sigma\sigma'}{\lambda\rho^2} \sin^2\phi\cos^2\phi\left\{\mu'\theta\left(\nu_1 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right) + \mu\left(\theta'\nu_2 - \frac{\mu'\theta}{\lambda}\epsilon_2\right)\right\} = 0.$$

We cannot neglect the term $2p_0c^2\delta p_0$ in the middle factor of the first line or the second factor of the second line, since $\epsilon_1 - \epsilon_3$ is of the second order in H.

For $$\epsilon_1 - \epsilon_3 = \Sigma\frac{\theta N\lambda'}{\lambda\lambda'-\mu\mu'} - \Sigma\frac{\theta N}{\lambda} = \Sigma\frac{\theta N}{\lambda} \cdot \frac{\mu\mu'}{\lambda\lambda'-\mu\mu'}.$$

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Vibrations parallel to Ox.

When there is no orientation, the period of these is given by

$$p_0^2c^2 = \epsilon_1 - \frac{\nu_1\nu_2}{\epsilon_2}.$$

When orientation is present suppose the value of p to become

$$p = p_0 + \delta p_0.$$

Substituting this value in the period equation on p. 306, we get

$$\left(\epsilon_2 - \frac{\theta' N\lambda\sigma \sin^2\phi}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'}\right)\left(\frac{\nu_1\nu_2}{\epsilon_2} - \frac{\theta N\sigma}{\rho} \cdot \frac{\mu\mu'}{\lambda} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi - 2p_0c^2\delta p_0\right)$$
$$\left(\epsilon_3 - \epsilon_1 + \frac{\nu_1\nu_2}{\epsilon_2} - \frac{N\theta\sigma'}{\lambda} \cdot \frac{\lambda' - \frac{\mu\mu'}{\lambda} + \sigma}{\rho} \sin^2\phi - 2p_0c^2\delta p_0\right)$$
$$+ N^2\theta\theta' \cdot \frac{\sigma\sigma'}{\rho^2} \sin^2\phi\cos^2\phi \left(\frac{\nu_1\nu_2}{\epsilon_2} - \frac{\theta N\sigma}{\rho} \cdot \frac{\mu\mu'}{\lambda} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi - 2p_0c^2\delta p_0\right)$$
$$-\left(\nu_1 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right)\left(\nu_2 - \frac{N\theta\mu'\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right)$$
$$\left(\epsilon_3 - \epsilon_1 + \frac{\nu_1\nu_2}{\epsilon_2} - \frac{N\theta\sigma'}{\lambda} \cdot \frac{\lambda' - \frac{\mu\mu'}{\lambda} + \sigma}{\rho} \sin^2\phi - 2p_0c^2\delta p_0\right)$$
$$-\frac{N^2\theta\sigma\sigma'}{\lambda\rho^2} \sin^2\phi\cos^2\phi \left\{\mu'\theta\left(\nu_1 - \frac{N\theta'\mu\sigma}{\rho} \cdot \frac{\lambda+\sigma'}{\lambda\lambda'-\mu\mu'} \sin^2\phi\right) + \mu\left(\theta'\nu_2 - \frac{\mu'\theta}{\lambda}\epsilon_2\right)\right\} = 0.$$

We shall return to this equation later.

Thus whether the vibration be parallel to Ox or Oz we get an equation of the second degree in δp_0. Corresponding to each value of δp_0 we get an absorption band.

Hence we conclude that if one system of electrons undergoes orientation, the corresponding bands may become doubled, whether the vibration be perpendicular to the Optic Axis and the Field, or parallel to the Field.

This conclusion removes many of the discrepancies which existed between our former results and Becquerel's observations. In his table, doubling occurs in bands Nos. 6, 16 when the vibration is parallel to the field, and in bands Nos. 4, 10, 22 when the vibration is perpendicular to the field.

From the equations for δp_0 above it is clear that the expressions giving the index and absorption when orientation occurs are not of the same type as the quantities determining the displacements in the ordinary theory of the Zéeman Effect. These latter displacements are determined from the equation $2g = \pm\mu$.

This may account for the fact that some of the displacements observed by Becquerel were greater than those found in previous observations on the Zéeman Effect.

These large displacements may be due to the corresponding electrons being somewhat loosely held in their orbits.

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For if the Optic Axis is parallel to the field the separation of the components is given by $g = \mu$.

Now
$$\mu = \frac{\theta H}{4\pi c \tau e}.$$

Hence if θ is large the separation is large.

But a large value of θ means a small central force, as is seen from the equations of motion on p. 294.

Thus if the electron is easily displaced from its position of equilibrium we should expect a large separation of the components of the doublet resulting from the action of a magnetic field.

Again, it will be noticed that in some cases a band is not doubled, but is merely displaced or enlarged.

If the 'friction,' determined by the quantity r in the equations of motion, is small, the maximum absorption deduced from the equation

$$n^2(1-\iota\kappa)^2 = 1 + \Sigma' \frac{\theta N}{\lambda} + \frac{\theta N}{2g + \iota \frac{a}{\tau_1}},$$

on p. 300, will be large, and it may easily be shewn that κ decreases very rapidly from its maximum value as g increases.

Hence, if the friction is small, the bands will be very dark and narrow, while if the friction is large, the bands will be wide and not very intense,—i.e. they shade off very gradually from the point of maximum intensity. Thus if the friction is large, the band has much the same intensity throughout a large portion of its breadth, and so, when the crystal is acted on by a magnetic field the components of the resulting doublet will run into one another, and the separation will not be observed. The band, however, on the whole, will increase in breadth. This explains how, in some cases, the bands are observed merely to broaden, without being doubled.

The fact that some of the bands observed broaden for some directions of the vibrations and separate for others indicates that the friction is different in different directions. If the friction were the same in all directions we should either get doubling for both directions of the vibration or enlargement without separation for both directions.

We have now to explain the fact that in the case of bands Nos. 6, 16, 22 one of the components becomes fainter as the field increases and finally is almost invisible.

The systems giving these bands have to be supposed orientated by the field in order to explain the doubling.

To find the intensities of the bands we must find the absorptions corresponding to the values $p_0 + \delta p_0$ of p, where δp_0 is given by the equations on pp. 306 and 307.

Since the values of δp_0 in each of these equations are quite distinct, the increments of absorption corresponding to these two bands will be different.

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Also these increments will be quite as important as the absorptions deduced from the equation for p_0, since these latter mainly depend on the one system with which we are dealing.

Hence the two components will be of different intensities. To get a fuller explanation we will return to the equation for δp_0 on p. 307.

We must first state one or two results.

We have
$$\begin{aligned}\lambda' &= 1 + \iota\frac{a'}{\tau} - \frac{b'}{\tau^2}\\ &= 1 - \frac{\theta'}{\theta} + \frac{\theta'}{\theta}\left(1 + \iota\frac{a}{\tau} - \frac{b}{\tau^2}\right)\\ &= -\sigma + \frac{\theta'}{\theta}\lambda.\end{aligned}$$

Similarly
$$\lambda + \sigma' = \frac{\theta}{\theta'}\lambda'.$$

For the 'particular' electrons λ and μ are of the same order, and λ' is large compared with λ.

Hence $\lambda\lambda'$ is large compared with $\mu\mu'$, and we may therefore write $\lambda\lambda'$ instead of $\lambda\lambda' - \mu\mu'$.

$$\begin{aligned}\therefore\ \rho &= \lambda\lambda' - \mu\mu' + \lambda\sigma\sin^2\phi + \frac{\lambda\lambda' - \mu\mu'}{\lambda}.\sigma'\sin^2\phi + \sigma\sigma'\sin^2\phi\\ &= \lambda\lambda' + \lambda\sigma\sin^2\phi + \lambda'\sigma'\sin^2\phi + \sigma\sigma'\sin^2\phi\\ &= \lambda\lambda'\cos^2\phi + (\lambda + \sigma')(\lambda' + \sigma)\sin^2\phi\\ &= \lambda\lambda'\cos^2\phi + \frac{\theta}{\theta'}.\lambda'.\frac{\theta'}{\theta}\lambda\sin^2\phi\\ &= \lambda\lambda'.\end{aligned}$$

Therefore if we write K for $2p_0c^2\delta p_0$, the equation for δp_0 becomes

$$\begin{aligned}&K^2\left(\epsilon_2 - \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\right)\\ &\quad - K\left[\left(\epsilon_2 - \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\right)\left(\epsilon_3 - \epsilon_1 + \frac{2\nu_1\nu_2}{\epsilon_2} + \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'} + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}.\frac{\mu\mu'}{\lambda^2}\right)\right.\\ &\quad \left. + \frac{N^2\theta\theta'\sigma\sigma'}{\lambda^2\lambda'^2}\sin^2\phi\cos^2\phi - \left(\nu_1 - \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}.\frac{\mu}{\lambda}\right)\left(\nu_2 + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}.\frac{\mu'}{\lambda}\right)\right]\\ &\quad + \left(\epsilon_2 - \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\right)\left(\frac{\nu_1\nu_2}{\epsilon_2} + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}.\frac{\mu\mu'}{\lambda^2}\right)\left(\epsilon_3 - \epsilon_1 + \frac{\nu_1\nu_2}{\epsilon_2} + \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\right)\\ &\quad + \frac{N^2\theta\theta'\sigma\sigma'}{\lambda^2\lambda'^2}.\sin^2\phi\cos^2\phi\left(\frac{\nu_1\nu_2}{\epsilon_2} + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}.\frac{\mu\mu'}{\lambda^2}\right)\\ &\quad - \left(\nu_1 - \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}.\frac{\mu}{\lambda}\right)\left(\nu_2 + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}.\frac{\mu'}{\lambda}\right)\left(\epsilon_3 - \epsilon_1 + \frac{\nu_1\nu_2}{\epsilon_2} + \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\right)\\ &\quad - \frac{N^2\theta\sigma\sigma'}{\lambda^2\lambda'^2}\sin^2\phi\cos^2\phi\left\{\mu'\theta\left(\nu_1 - \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}.\frac{\mu}{\lambda}\right) + \mu\left(\theta'\nu_2 - \frac{\mu'\theta}{\lambda}\epsilon_2\right)\right\} = 0.\end{aligned}$$

Consider the second bracket in the second line.

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We may neglect terms in $\mu\mu'$ compared with the last term but one.

The last term will be retained since μ, μ' are of the same order as λ.

Again $$\epsilon_3 - \epsilon_1 = \Sigma\left(\frac{N\theta}{\lambda} - \frac{N\theta\lambda'}{\lambda\lambda' - \mu\mu'}\right) = -\Sigma\frac{N\theta}{\lambda}\cdot\frac{\mu\mu'}{\lambda\lambda' - \mu\mu'}.$$

All the terms in this summation except the last are of the second order in μ, μ'.

The last term is approximately equal to $-\frac{N\theta}{\lambda'}\cdot\frac{\mu\mu'}{\lambda^2}$ since $\lambda\lambda' - \mu\mu'$ is approximately equal to $\lambda\lambda'$.

Thus we may replace the second bracket by

$$-\frac{N\theta}{\lambda'}\cdot\frac{\mu\mu'}{\lambda^2} + \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'} + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\cdot\frac{\mu\mu'}{\lambda^2}.$$

Now μ, μ' are of the same order as λ so that $\frac{\mu\mu'}{\lambda^2}$ is of the same order as unity. Also λ is small, so that $\frac{N\theta\sigma}{\lambda\lambda'}$ is large compared with $\frac{N\theta}{\lambda'}$.

Therefore we may neglect the first term and replace the second bracket by

$$\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'} + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\cdot\frac{\mu\mu'}{\lambda^2}.$$

Again in the term $\nu_1 - \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\cdot\frac{\mu}{\lambda}$, ν_1 is equal to $\Sigma\frac{N\mu\theta'}{\lambda\lambda'}$. Hence since λ is large compared with λ' we may neglect ν_1 in comparison with the other term.

Similarly in the expression $\nu_2 + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'(1+\cos^2\phi)}\cdot\frac{\mu'}{\lambda}$ we may neglect ν_2.

Thus the coefficient of $-K$ may be written

$$\left(\epsilon_2 - \frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\right)\left(-\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'} + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\cdot\frac{\mu\mu'}{\lambda^2}\right)$$
$$+ \frac{N^2\theta\theta'\sigma\sigma'}{\lambda^2\lambda'^2}\sin^2\phi\cos^2\phi + \frac{N^2\theta^2\sigma\sigma'\sin^4\phi}{\lambda^2\lambda'^2}\cdot\frac{\mu\mu'}{\lambda^2}.$$

Now $\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}$ may be neglected in comparison with ϵ_2.

Hence the coefficient of $-K$ becomes

$$\epsilon_2\left(\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'} + \frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\cdot\frac{\mu\mu'}{\lambda^2}\right).$$

Next consider the absolute term.

The first line of it is approximately equal to

$$\epsilon_2\cdot\frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\cdot\frac{\mu\mu'}{\lambda^2}\cdot\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}.$$

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The second line may be replaced by

$$\frac{N^2\theta\theta'\sigma\sigma'}{\lambda^2\lambda'^2}\sin^2\phi\cos^2\phi\,.\,\frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\,.\,\frac{\mu\mu'}{\lambda^2}\,.$$

The next line reduces to

$$\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\,.\,\frac{\mu}{\lambda}\,.\,\frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\,.\,\frac{\mu'}{\lambda}\,.\,\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\,.$$

The last line is approximately

$$\frac{N^2\theta\sigma\sigma'}{\lambda^2\lambda'^2}\sin^2\phi\cos^2\phi\left\{\mu'\theta\,.\,\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\,.\,\frac{\mu}{\lambda}+\frac{\mu\mu'}{\lambda}\,\theta\epsilon_2\right\},$$

or

$$\frac{N^2\theta^2\sigma\sigma'}{\lambda^2\lambda'^2}\sin^2\phi\cos^2\phi\,\frac{\mu\mu'}{\lambda^2}\left(\epsilon_2+\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}\right),$$

or

$$\epsilon_2\,.\,\frac{N^2\theta^2\sigma\sigma'}{\lambda^2\lambda'^2}\,.\,\sin^2\phi\cos^2\phi\,.\,\frac{\mu\mu'}{\lambda^2}\,.$$

Since ϵ_2 is large compared with $\dfrac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}$, the second and third lines may be neglected, and the absolute term becomes

$$\epsilon_2\,.\,\frac{\mu\mu'}{\lambda^2}\,.\,\frac{N^2\theta^2\sigma\sigma'}{\lambda^2\lambda'^2}\sin^2\phi.$$

Hence the equation for K becomes

$$\epsilon_2K^2-K\epsilon_2\left(\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'}+\frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\,.\,\frac{\mu\mu'}{\lambda^2}\right)+\epsilon_2\,.\,\frac{\mu\mu'}{\lambda^2}\,.\,\frac{N^2\theta^2\sigma\sigma'}{\lambda^2\lambda'^2}\sin^2\phi=0,$$

which may be written

$$K^2-K(\alpha+\beta)\sin^2\phi+\alpha\beta\sin^2\phi=0,$$

where

$$\alpha=\frac{N\theta\sigma\sin^2\phi}{\lambda\lambda'},$$

$$\beta=\frac{N\theta\sigma'\sin^2\phi}{\lambda\lambda'}\,.\,\frac{\mu\mu'}{\lambda^2}\,.$$

If n_1, κ_1 denote the index of refraction and the coefficient of absorption, we have

$$\begin{aligned}n^2(1-\iota\kappa_1)^2&=(p_0+\delta p_0)^2c^2\\&=p_0^2c^2+K\\&=\epsilon_1-\frac{\nu_1\nu_2}{\epsilon_2}+K.\end{aligned}$$

Neglecting the term $\dfrac{\nu_1\nu_2}{\epsilon_2}$, we have

$$\begin{aligned}n_1^2(1-\iota\kappa_1)^2&=1+\Sigma\frac{N\theta}{\lambda}+K\\&=1+\Sigma\frac{N\theta}{\lambda}+\frac{(\alpha+\beta)\sin^2\phi\pm\sin^2\phi\sqrt{(\alpha+\beta)^2\sin^2\phi-4\alpha\beta}}{2}\\&=1+\Sigma\frac{N\theta}{\lambda}+\frac{N\theta}{\lambda\lambda'}\,.\,\frac{\left(\sigma+\sigma'\dfrac{\mu\mu'}{\lambda^2}\right)\sin^2\phi\pm\sin\phi\sqrt{\left(\sigma+\sigma'\,.\,\dfrac{\mu\mu'}{\lambda^2}\right)^2\sin^2\phi-4\sigma\sigma'\,.\,\dfrac{\mu\mu'}{\lambda^2}}}{2}.\end{aligned}$$

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The imaginary part, determining K, is derived from the terms

$$\frac{N\theta}{\lambda}+\frac{N\theta\sin\phi}{\lambda\lambda'}\cdot\frac{\left(\sigma+\sigma'\cdot\frac{\mu\mu'}{\lambda^2}\right)\sin\phi\pm\sqrt{\left(\sigma+\sigma'\cdot\frac{\mu\mu'}{\lambda^2}\right)^2\sin^2\phi-4\sigma\sigma'\cdot\frac{\mu\mu'}{\lambda^2}}}{2}.$$

From p. 309 we may put $\lambda'=-\sigma$, since λ is small.

Also
$$\frac{\sigma'}{\sigma}=\frac{\frac{1}{\delta}-1}{\delta-1}=-\frac{1}{\delta},$$

and
$$\frac{1}{\delta}\cdot\mu\mu'=\mu^2.$$

Therefore the above expression becomes

$$\frac{N\theta}{\lambda}-\frac{N\theta}{\lambda}\sin\phi\cdot\frac{\left(1-\frac{\mu^2}{\lambda^2}\right)\sin\phi\pm\sqrt{\left(1-\frac{\mu^2}{\lambda^2}\right)^2\sin^2\phi+4\frac{\mu^2}{\lambda^2}}}{2}.$$

Also
$$\lambda=2g+\iota h \text{ when } h=\frac{a}{\tau_1}.$$

We shall retain only the first power of h.

Thus we get

$$\frac{N\theta}{2g+\iota h}-\frac{N\theta\sin\phi}{8g^2+12\iota g^2h}\cdot\frac{(4g^2-\mu^2+4\iota gh)\sin\phi\pm\sqrt{(4g^2-\mu^2)^2\sin^2\phi+16\mu^2g^2+8\iota gh\{(4g^2-\mu^2)\sin^2\phi+2\mu^2\}}}{2}.$$

We shall suppose g to be large compared with h so that terms like $\frac{1}{2g+\iota h}$ may be expanded in powers of h.

We shall further suppose $(4g^2-\mu^2)^2\sin^2\phi+16\mu^2g^2$ not to become very small.

Under these suppositions it is easy to shew that the imaginary part of the above expression is equal to

$$-\iota\cdot\frac{N\theta h}{\gamma}\left\{1+\sin\phi\left[\sin\phi\pm\frac{(\gamma-m)\sin^2\phi+2m}{\sqrt{(\gamma-m)^2\sin^2\phi+4\gamma m}}-\frac{3}{2\gamma}\{(\gamma-m)\sin\phi\pm\sqrt{(\gamma-m)^2\sin^2\phi+4\gamma m}\}\right]\right\},$$

where $\gamma=4g^2$, $m=\mu^2$.

This expression determines the two values of κ.

We shall now see whether one of these coefficients of absorption can vanish for a particular value of ϕ.

If so, ϕ is determined by the equation

$$\frac{1}{\sin\phi}+\sin\phi-\frac{3}{2\gamma}(\gamma-m)\sin\phi\pm\left\{\frac{(\gamma-m)\sin^2\phi+2m}{\sqrt{(\gamma-m)^2\sin^2\phi+4\gamma m}}-\frac{3}{2\gamma}\sqrt{(\gamma-m)^2\sin^2\phi+4\gamma m}\right\}=0,$$

or
$$\frac{1}{\sin^2\phi}+\sin^2\phi+\frac{9}{4\gamma^2}(\gamma-m)^2\sin^2\phi+2-\frac{3}{\gamma}(\gamma-m)-\frac{3}{\gamma}(\gamma-m)\sin^2\phi$$
$$=\frac{\{(\gamma-m)\sin^2\phi+2m\}^2}{(\gamma-m)^2\sin^2\phi+4\gamma m}+\frac{9}{4\gamma^2}\{(\gamma-m)^2\sin^2\phi+4\gamma m\}-\frac{3}{\gamma}\{(\gamma-m)\sin^2\phi+2m\},$$

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or
$$\frac{1}{\sin^2\phi} - 1 = \frac{4m^2\cos^2\phi}{(\gamma - m)^2\sin^2\phi + 4\gamma m}.$$
$$\therefore \frac{1}{\sin^2\phi} = \frac{4m^2}{(\gamma - m)^2\sin^2\phi + 4\gamma m}.$$
$$\therefore \sin^2\phi = \frac{4\gamma m}{(m+\gamma)(3m-\gamma)}.$$

The value of ϕ deduced from this equation will be real provided that $\gamma < m$,

i.e. $2g < \mu$.

Hence if the point of maximum absorption is given by a value of g less than this limiting value it is possible for the one component of the doublet to become very faint as the corresponding electron system becomes more and more orientated by the growing magnetic field.

Thus as the magnetic field is progressively established, we may get the orientation approaching the above critical value, and as this occurs, one of the components of the doublet will become weaker and weaker until it may be almost invisible if the orientation gets near enough to the value of ϕ deduced above.

Thus the hypothesis of magnetic orientation explains the doubling effects when the optic axis is perpendicular to the field, and also explains how one component of a doublet may be much weaker than the other.

It is thus in full agreement with Becquerel's observations.

Light Rays parallel to the Magnetic Field.

It is clear that if the Optic Axis is perpendicular to the field we get phenomena identical with those considered above.

Hence we need only consider the case of

Optic Axis Parallel to the Field.

The light waves are travelling along the z axis, so that the quantities involved are functions of z, t only.

Thus using the values of γ_x, γ_y, γ_z given on p. 297 we get
$$\frac{1}{c}\frac{\partial\alpha}{\partial t} = \frac{\partial Y}{\partial z}, \quad \frac{1}{c}\frac{\partial\beta}{\partial t} = -\frac{\partial X}{\partial z}, \quad \gamma = 0,$$
$$\frac{1}{c}\left(\epsilon_1\frac{\partial X}{\partial t} + \iota\nu_1\frac{\partial Y}{\partial t}\right) = -\frac{\partial\beta}{\partial z}, \quad \frac{1}{c}\left(\epsilon_1\frac{\partial Y}{\partial t} - \iota\nu_1\frac{\partial X}{\partial t}\right) = \frac{\partial\alpha}{\partial z},$$
$$Z = 0.$$

Eliminating α, β and putting
$$X = Me^{\frac{\iota}{\tau}(t-pz)} \quad Y = Ne^{\frac{\iota}{\tau}(t-pz)},$$
we get
$$\epsilon_1 M + \iota\nu_1 N = Mp^2c^2,$$
$$\epsilon_1 N - \iota\nu_1 M = Np^2c^2$$

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Eliminating p we find $\quad M^2 + N^2 = 0$ or $M = \pm \iota N$.

Therefore we have $\quad M = \iota N$ and $p^2c^2 = \epsilon_1 + \nu_1$,

or $\quad M = -\iota N$ and $p^2c^2 = \epsilon_1 - \nu_1$.

$M = \iota N$ corresponds to a left-handed circularly polarised wave, and $M = -\iota N$ corresponds to a right-handed wave.

Thus a plane polarised wave is split up by the magnetic field into two circularly polarised waves of opposite senses.

For the left-handed wave the index and absorption are given by

$$p^2c^2 = n'^2(1 - \iota\kappa')^2 = \epsilon_1 + \nu_1,$$

and for the right-handed wave by

$$p^2c^2 = n''^2(1 - \iota\kappa'')^2 = \epsilon_1 - \nu_1.$$

Remembering the values of ϵ_1 and ν_1, we have

$$\epsilon_1 + \nu_1 = 1 + \Sigma \frac{N\theta}{\lambda - \mu},$$

$$\epsilon_1 - \nu_1 = 1 + \Sigma \frac{N\theta}{\lambda + \mu}.$$

If τ lies close to a 'natural' period τ_1 of a system of ions we put

$$\tau = \sqrt{b_1}(1 + g) = \tau_1(1 + g),$$

and for this system we have $\lambda = 2g + \iota . \frac{a}{\tau}$, while for the other systems $\lambda = 1 - \frac{b}{b_1}$.

Therefore if Σ' refer to these other systems, we have

$$n'^2(1 - \iota\kappa')^2 = \epsilon_1 + \nu_1 = 1 + \Sigma' \frac{N\theta}{1 - \frac{b}{b_1}} + \frac{N\theta}{2g + 1 . \frac{a}{\tau} - \mu},$$

$$n''^2(1 - \iota\kappa'')^2 = \epsilon_1 - \nu_1 = 1 + \Sigma' \frac{N\theta}{1 - \frac{b}{b_1}} + \frac{N\theta}{2g + \iota . \frac{a}{\tau} + \mu}.$$

Hence the absorption for the left-handed wave is a maximum when $2g = \mu$, and for the right-handed wave when $2g = -\mu$.

Thus any absorption band in the spectrum is split up into two components by the action of the magnetic field.

These bands are circularly polarised in opposite senses.

Thus far the development of the theory of the variation of the bands is identical with that for isotropic bodies, the crystalline quality of the medium not having entered into the equations.

We shall now suppose the Optic Axis to be slightly inclined to the direction of the field and shall find the states of polarisation of the bands composing the doublets.

Suppose the Optic Axis to make an angle ϕ with the field. We obtain the equations of motion of an ion from those on p. 304 by writing $\frac{\pi}{2} - \phi$ for ϕ.

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Thus the equations of motion are

$$m\frac{\partial^2\xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta}\xi - re^2\frac{\partial\xi}{\partial t} + \frac{eH}{c}\frac{\partial\eta}{\partial t},$$

$$m\frac{\partial^2\eta}{\partial t^2} = eY - \left(\frac{4\pi e^2}{\theta'}\sin^2\phi + \frac{4\pi e^2}{\theta}\cos^2\phi\right)\eta - re^2\frac{\partial\eta}{\partial t} - \frac{eH}{c}\frac{\partial\xi}{\partial t} - \sin\phi\cos\phi\zeta\left(\frac{4\pi e^2}{\theta'} - \frac{4\pi e^2}{\theta}\right),$$

$$m\frac{\partial^2\zeta}{\partial t^2} = eZ - \left(\frac{4\pi e^2}{\theta'}\cos^2\phi + \frac{4\pi e^2}{\theta}\sin^2\phi\right)\zeta - \sin\phi\cos\phi\eta\left(\frac{4\pi e^2}{\theta'} - \frac{4\pi e^2}{\theta}\right) - re^2\frac{\partial\zeta}{\partial t}.$$

If everything be supposed to vary as $e^{\iota\frac{t}{\tau}}$, we derive from these equations exactly as on p. 304, the equations

$$e\xi\lambda - \iota e\mu\eta = \frac{\theta}{4\pi}X,$$

$$e\eta(\lambda + \sigma'\sin^2\phi) + e\sigma'\sin\phi\cos\phi\zeta + \iota e\mu\xi = \frac{\theta}{4\pi}Y,$$

$$e\zeta(\lambda' - \sigma\sin^2\phi) - e\sigma\sin\phi\cos\phi\eta = \frac{\theta'}{4\pi}Z,$$

where the quantities involved have the same meanings as on p. 304.

Correct to ϕ^2 these may be written

$$e\xi\lambda - \iota e\mu\eta = \frac{\theta}{4\pi}X,$$

$$e\eta(\lambda + \sigma'\phi^2) + e\sigma'\phi\zeta + \iota e\mu\xi = \frac{\theta}{4\pi}Y,$$

$$e\zeta(\lambda' - \sigma\phi^2) - e\sigma\phi\eta = \frac{\theta'}{4\pi}Z.$$

Solving these equations for ξ, η, ζ, and neglecting powers of ϕ above the second, we get for the components of current

$$4\pi\gamma_x = \epsilon_1'\frac{\partial X}{\partial t} + \iota\nu_1'\frac{\partial Y}{\partial t} + \iota\,.\,\omega\phi\frac{\partial Z}{\partial t},$$

$$4\pi\gamma_y = -\iota\nu_1'\frac{\partial X}{\partial t} + \epsilon_1''\frac{\partial Y}{\partial t} + \chi\,.\,\phi\frac{\partial Z}{\partial t},$$

$$4\pi\gamma_z = -\iota\omega\phi\frac{\partial X}{\partial t} + \chi\,.\,\phi\frac{\partial Y}{\partial t} + \epsilon_2'\frac{\partial Z}{\partial t},$$

where

$$\epsilon_1' = 1 + \Sigma N\theta\,.\,\frac{\lambda\lambda' + (\lambda'\sigma' - \lambda\sigma + \sigma\sigma')\phi^2}{\rho},$$

$$\epsilon_1'' = 1 + \Sigma N\lambda\theta\,.\,\frac{\lambda' - \sigma\phi^2}{\rho},$$

$$\epsilon_2' = 1 + \Sigma N\theta'\,.\,\frac{\lambda^2 - \mu^2 + \lambda\sigma'\phi^2}{\rho},$$

$$\nu_1' = \Sigma N\mu\theta\,.\,\frac{\lambda' - \sigma\phi^2}{\rho},$$

$$\chi = \Sigma\frac{N\lambda\theta\sigma}{\rho} = -\Sigma\frac{N\lambda\theta'\sigma'}{\rho}, \text{ for } \theta\sigma = \theta' - \theta = -\theta'\sigma',$$

$$\omega = \Sigma\frac{N\mu\theta\sigma}{\rho} = -\Sigma\frac{N\mu\theta'\sigma'}{\rho},$$

$$\rho = \lambda'(\lambda^2 - \mu^2) + (\lambda\lambda'\sigma' - \sigma\lambda^2 + \sigma\mu^2 + \lambda\sigma\sigma')\phi^2.$$

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Consider waves travelling along the axis of z. Then the quantities involved are functions of z, t only.

Putting $(X, Y, Z) = (A, B, C)e^{\frac{\iota}{\tau}(t-pz)}$ and substituting in Maxwell's Equations as before, we get

$$\epsilon_1' A + \iota\nu_1' B + \iota\omega\phi C = A \,.\, p^2c^2$$
$$-\iota\nu_1' A + \epsilon_1'' B + \chi \,.\, \phi \,.\, C = B \,.\, p^2c^2$$
$$-\iota\omega\phi A + \chi\phi B + \epsilon_2' C = 0.$$

Eliminating p and C between these equations, we find

$$A^2 - \iota \,.\, \frac{\epsilon_2'(\epsilon_1' - \epsilon_1'') + (\chi^2 - \omega^2)\phi^2}{\nu_1'\epsilon_2' - \omega\chi\phi^2} AB + B^2 = 0.$$

From this equation we derive two values for the ratio of A to B; complex in general. If either of these values be substituted in the immediately preceding equations we get a value of p^2 to correspond, giving us the index and absorption for each component.

The states of polarisation of the components are determined by the values of the ratio A to B.

Thus when the Optic Axis is parallel to the field, we have $\phi = 0$ and $A^2 + B^2 = 0$, or $A = \pm\iota B$, indicating that the components are circularly polarised.

We must evaluate $$\frac{\epsilon_2'(\epsilon_1' - \epsilon_1'') + (\chi^2 - \omega^2)\phi^2}{\nu_1'\epsilon_2' - \omega\chi\phi^2}.$$

We have $$\epsilon_1' - \epsilon_1'' = \Sigma N\theta \,.\, \frac{\lambda'\sigma' + \sigma\sigma'}{\rho} \,.\, \phi^2.$$

Now ϕ is small, and for all systems of ions except that under immediate consideration λ, λ' etc. are not small.

Hence for these systems we may put

$$\rho = \lambda'(\lambda^2 - \mu^2)$$
$$= \lambda'\lambda^2 \text{ approximately};$$

μ being small compared with λ for these systems.

Thus if Σ' refer to all the other systems, we have

$$\epsilon_1' - \epsilon_1'' = \phi^2 \Sigma' N\theta\sigma' \,.\, \frac{\lambda' + \sigma}{\lambda'\lambda^2} + N\theta \,.\, \frac{\lambda'\sigma' + \sigma\sigma'}{\rho}\phi^2.$$

Suppose that the impressed period is almost equal to a natural period of the system we are considering, and put $\tau = \tau_1(1+g) = \sqrt{b_1}(1+g)$ where g is small.

Then for the particular group of ions

$$\lambda = 2g + \iota \,.\, \frac{a}{\tau_1}.$$

From p. 309 we have

$$\lambda' + \sigma = \frac{\theta'}{\theta}\lambda.$$

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For the particular system of electrons considered λ is small, and we may sometimes use $\lambda' = -\sigma$ as an approximation.

For this group of electrons

$$\rho = (\lambda^2 - \mu^2)(\lambda' - \sigma\phi^2) + \lambda\sigma'(\lambda' + \sigma)\phi^2$$

$$= -\sigma(\lambda^2 - \mu^2)(1 + \phi^2) + \frac{\lambda^2\theta'\sigma'}{\theta}\phi^2 \text{ approximately}$$

$$= -\sigma(\lambda^2 - \mu^2) - \lambda^2\sigma\phi^2 \text{ approximately}$$

$$= -\sigma(\lambda^2 - \mu^2) \text{ approximately};$$

$$\therefore\ \epsilon_1' - \epsilon_1'' = \phi^2\Sigma' N\theta\sigma' \cdot \frac{\lambda' + \sigma}{\lambda'\lambda^2} - N\theta\sigma' \cdot \frac{\dfrac{\lambda\theta'}{\theta}}{\sigma(\lambda^2 - \mu^2)}\phi^2$$

$$= \phi^2\Sigma' N\theta\sigma' \cdot \frac{\lambda' + \sigma}{\lambda'\lambda^2} + N\theta \cdot \frac{\lambda}{\lambda^2 - \mu^2}\phi^2,$$

since

$$\theta'\sigma' = -\theta\sigma.$$

Again

$$\epsilon_2' = 1 + \Sigma N\theta' \cdot \frac{\lambda^2 - \mu^2 + \lambda\sigma'\phi^2}{\rho}$$

$$= \epsilon_2 - \Sigma N\theta'\left(\frac{1}{\lambda'} - \frac{\lambda^2 - \mu^2 + \lambda\sigma'\phi^2}{\rho}\right),$$

where

$$\epsilon_2 = 1 + \Sigma\frac{N\theta'}{\lambda'};$$

$$\therefore\ \epsilon_2' = \epsilon_2 - \Sigma N\theta'\sigma \cdot \frac{\lambda\sigma - \lambda^2 + \mu^2}{\lambda'\rho} \cdot \phi^2$$

$$= \epsilon_2 - \phi^2 \cdot \Sigma' \frac{N\theta'\sigma}{\lambda'^2} \cdot \frac{\lambda\sigma' - \lambda^2 + \mu^2}{\lambda^2 - \mu^2} + \frac{N\theta'}{\lambda'} \cdot \frac{\lambda\sigma'}{\lambda^2 - \mu^2} \cdot \phi^2$$

$$= \epsilon_2 - \phi^2 \cdot \Sigma' \frac{N\theta'\sigma}{\lambda'^2} \cdot \frac{\sigma' - \lambda}{\lambda} + N\theta \cdot \frac{\lambda}{\lambda^2 - \mu^2}\phi^2.$$

Also

$$\nu_1' = \Sigma N\mu\theta \cdot \frac{\lambda' - \sigma\phi^2}{\rho}$$

$$= \nu_1 - \Sigma N\mu\theta\left(\frac{1}{\lambda^2 - \mu^2} - \frac{\lambda' - \sigma\phi^2}{\rho}\right)$$

$$= \nu_1 - \Sigma\frac{N\mu\theta}{\lambda^2 - \mu^2} \cdot \frac{\lambda\sigma'(\lambda' + \sigma)}{\rho}\phi^2$$

$$= \nu_1 - \phi^2\Sigma' \frac{N\mu\theta\lambda\sigma'}{\lambda'} \cdot \frac{\lambda' + \sigma}{(\lambda^2 - \mu^2)^2} + \frac{N\mu\theta\lambda\sigma'}{\sigma} \cdot \frac{\lambda \cdot \dfrac{\theta'}{\theta}}{(\lambda^2 - \mu^2)^2}\phi^2$$

$$= \nu_1 - \Sigma' \frac{N\mu\theta\lambda\sigma'}{\lambda'} \cdot \frac{\lambda' + \sigma}{(\lambda^2 - \mu^2)^2} \cdot \phi^2 - N\mu\theta \cdot \frac{\lambda^2}{(\lambda^2 - \mu^2)^2} \cdot \phi^2$$

$$= \Sigma' \frac{N\mu\theta}{\lambda^2} + \frac{N\mu\theta}{\lambda^2 - \mu^2} - \Sigma' \frac{N\mu\theta\sigma'}{\lambda'} \cdot \frac{\lambda' + \sigma}{\lambda^3}\phi^2 - N\mu\theta \cdot \frac{\lambda^2}{(\lambda^2 - \mu^2)^2}\phi^2,$$

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$$\omega = \Sigma \frac{N\mu\theta\sigma}{\rho} = \Sigma' \frac{N\mu\theta\sigma}{\lambda'(\lambda^2-\mu^2)} - \frac{N\mu\theta}{\lambda^2-\mu^2}$$
$$= \Sigma' \frac{N\mu\theta\sigma}{\lambda'\lambda^2} - \frac{N\mu\theta}{\lambda^2-\mu^2},$$
$$\chi = \Sigma \frac{N\lambda\theta\sigma}{\rho}$$
$$= \Sigma' \frac{N\theta\sigma}{\lambda\lambda'} - \frac{N\lambda\theta}{\lambda^2-\mu^2};$$
$$\therefore\ \nu_1'\epsilon_2' - \omega\chi\phi^2 = \left(\epsilon_2 - \phi^2 . \Sigma' \frac{N\theta'\sigma}{\lambda'^2} . \frac{\sigma'-\lambda}{\lambda} + \frac{N\theta\lambda}{\lambda^2-\mu^2}\phi^2\right)$$
$$\left\{\Sigma' \frac{N\mu\theta}{\lambda^2} + \frac{N\mu\theta}{\lambda^2-\mu^2} - \phi^2\Sigma' \frac{N\mu\theta\sigma'}{\lambda'} . \frac{\lambda'+\sigma}{\lambda^3} - N\mu\theta\phi^2 . \frac{\lambda^2}{(\lambda^2-\mu^2)^2}\right\}$$
$$-\left(\Sigma' . \frac{N\mu\theta\sigma}{\lambda'\lambda^2} - \frac{N\mu\theta}{\lambda^2-\mu^2}\right)\left(\Sigma' . \frac{N\theta\sigma}{\lambda\lambda'} - \frac{N\lambda\theta}{\lambda^2-\mu^2}\right)\phi^2.$$

The terms in the expressions for ν_1', ϵ_2', etc., which depend on the particular system of electrons, become complex when the impressed period approaches the free period. We may suppose the real part of these terms negligible in comparison with the remainder of the expression for ν_1', etc.

Thus we may regard the quantities ν_1', etc., as made up of a real part, and an imaginary part which is small compared with the real part.

Hence we may write

$$\nu_1'\epsilon_2' - \omega\chi\phi^2 = \epsilon_2\Sigma' \frac{N\mu\theta}{\lambda^2} - \left(\epsilon_2\Sigma' \frac{N\mu\theta\sigma'}{\lambda'} . \frac{\lambda'+\sigma}{\lambda^3} + \Sigma' \frac{N\mu\theta}{\lambda^2} . \Sigma' \frac{N\theta'\sigma}{\lambda'^2} . \frac{\sigma'-\lambda}{\lambda}\right.$$
$$\left. + \Sigma' \frac{N\mu\theta\sigma}{\lambda'\lambda^2}\Sigma' . \frac{N\theta\sigma}{\lambda\lambda'}\right)\phi^2$$
$$+ \iota . G,$$

where G is small compared with $\epsilon_2\Sigma' \frac{N\mu\theta}{\lambda^2}$.

If it be remembered that

$$\sigma = \frac{\theta'}{\theta} - 1 = \frac{b'}{b} - 1,$$
$$\sigma' = \frac{\theta}{\theta'} - 1 = \frac{b}{b'} - 1,$$
$$\lambda = 1 - \frac{b}{\tau_1^2} \qquad \lambda' = 1 - \frac{b'}{\tau_1^2}$$
$$= 1 - \frac{b}{b_1}, \qquad = 1 - \frac{b'}{b_1},$$

so that σ, σ', λ, λ' are all fractions of much the same order of magnitude, it will be seen that each of the three terms

$$\epsilon_2\Sigma' \frac{N\mu\theta\sigma'}{\lambda'} . \frac{\lambda'+\sigma}{\lambda^3}, \text{ etc.,}$$

composing the coefficient of ϕ^2 in $(\nu_1'\epsilon_2' - \omega\chi\phi^2)$ is of the same character as $\epsilon_2\Sigma' \frac{N\mu\theta}{\lambda^2}$.

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Hence we may suppose this coefficient of ϕ^2 to be fairly large compared with

$$\epsilon_2 \Sigma' \frac{N\mu\theta}{\lambda^2}.$$

Thus we may write

$$\nu_1' \epsilon_2' - \omega\chi\phi^2 = E - F\phi^2 + \iota G,$$

where F is large compared with E and G is small compared with E.

Again, owing to the presence of the factor μ, ω is small compared with χ, and will be neglected.

Neglecting the small imaginary part arising from the particular system considered, we therefore have

$$\epsilon_2' (\epsilon_1' - \epsilon_1'') + (\chi^2 - \omega^2)\phi^2 = \phi^2 \left\{ \epsilon_2 \Sigma' \frac{N\theta\sigma'}{\lambda'} \cdot \frac{\lambda' + \sigma}{\lambda^2} + \left(\Sigma' \cdot \frac{N\theta\sigma}{\lambda\lambda'} \right)^2 \right\}$$

$$= D\phi^2, \text{ say.}$$

Therefore we have

$$\frac{\epsilon_2' (\epsilon_1' - \epsilon_1'') + (\chi^2 - \omega^2)\phi^2}{\nu_1' \epsilon_2' - \omega\chi\phi^2} = \frac{D}{E - F\phi^2 + \iota G}\phi^2,$$

where D is large compared with E, since E contains the small factor μ.

When $\phi = 0$ the state of polarisation of the bands is given by $A^2 + B^2 = 0$.

Thus the bands are circularly polarised in opposite senses. As ϕ increases, the coefficient of AB also increases, remaining almost purely imaginary, since G is small compared with E.

But now suppose that ϕ takes one of the values given by $E - F\phi^2 = 0$.

Then the coefficient of AB becomes equal to $-\frac{D}{G}\phi^2$, which is large and real.

Thus the equation giving $\frac{A}{B}$ is

$$A^2 - \frac{D}{G}\phi^2 . AB + B^2 = 0.$$

Since $\frac{D}{G}\phi^2$ is large and real, it follows that the two values of $\frac{A}{B}$ are real.

Hence for this value of ϕ the two components are *plane polarised.*

When ϕ changes from the value $\pm\sqrt{\frac{E}{F}}$ this state of polarisation abruptly alters from plane to elliptic; for the term ιG may now be again neglected compared with $E - F\phi^2$, and thus the coefficient of AB again becomes imaginary.

The ratio $\frac{A}{B}$ is now given by

$$A^2 - \iota k . AB + B^2 = 0, \qquad k \text{ real.}$$

$$\therefore \frac{A}{B} = \frac{\iota k \pm \sqrt{-k^2 - 4}}{2},$$

$$= \iota . \frac{k \pm \sqrt{k^2 + 4}}{2}.$$

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This shews that the two bands are now elliptically polarised in opposite senses, since $k+\sqrt{k^2+4}$ is positive and $k-\sqrt{k^2+4}$ is negative.

The value of ϕ for which we get plane polarisation is given by $\phi^2=\frac{E}{F}$,

where $$E=\epsilon_2\Sigma'\frac{N\mu\theta}{\lambda^2}$$

$$F=\epsilon_2\Sigma'\frac{N\mu\theta\sigma'}{\lambda'}\cdot\frac{\lambda'+\sigma}{\lambda^2}+\Sigma'\frac{N\mu\theta}{\lambda^2}\cdot\Sigma'\frac{N\theta'\sigma}{\lambda'^2}\cdot\frac{\sigma'-\lambda}{\lambda}+\Sigma'\frac{N\mu\theta\sigma}{\lambda'\lambda^2}\cdot\Sigma'\frac{N\theta\sigma}{\lambda\lambda'}.$$

Since E, F both contain throughout the factor μ, which depends on the field, it follows that ϕ is independent of the strength of the field.

Moreover, ϕ will be practically the same for all the bands, since the summations which occur in E, F refer to all the systems except the one giving the band.

According to Becquerel the direction of polarisation of the bands is independent of the strength of the field. We will now deduce this from our theory. We shall also shew that the directions of polarisation of the components are perpendicular.

The direction of vibration depends upon the term G.

The imaginary part of $\nu_1'\epsilon_2'-\omega\chi\phi^2$ is contained in

$$\frac{N\mu\theta}{\lambda^2-\mu^2}\left(\epsilon_2-\phi^2\Sigma'\frac{N\theta'\sigma}{\lambda'^2}\cdot\frac{\sigma'-\lambda}{\lambda}\right)+\frac{N\theta\lambda}{\lambda^2-\mu^2}\phi^2\cdot\Sigma'\frac{N\mu\theta}{\lambda^2}-\epsilon_2N\mu\theta\phi^2\frac{\lambda^2}{(\lambda^2-\mu^2)^2}$$
$$+\frac{N^2\theta^2\lambda\mu}{(\lambda^2-\mu^2)^2}\phi^2+\frac{N\lambda\theta}{\lambda^2-\mu^2}\phi^2\Sigma'\frac{N\mu\theta\sigma}{\lambda'\lambda^2}+\frac{N\mu\theta}{\lambda^2-\mu^2}\phi^2\Sigma'\frac{N\theta\sigma}{\lambda\lambda'}-\frac{N^2\theta^2\lambda\mu}{(\lambda^2-\mu^2)^2}\phi^2.$$

The terms $\pm\frac{N^2\theta^2\lambda\mu}{(\lambda^2-\mu^2)^2}\phi^2$ cancel.

Now $\lambda=2g+\iota h$ where $h=\iota\frac{a}{\tau_1}$, and the positions of the bands are given approximately by $2g=\pm\mu$. (Their positions will be altered by an amount depending on ϕ^2.) Therefore λ is of the same order as μ for our particular electrons. Hence it appears that the term $\frac{N\mu\theta}{\lambda^2-\mu^2}\epsilon_2$ is the only one we need retain, for the other terms are all small compared with this.

Now $$\frac{N\mu\theta}{\lambda^2-\mu^2}\epsilon_2=\frac{N\mu\theta}{(2g+\iota h)^2-\mu^2}\epsilon_2.$$

If we take $2g=\mu$ this becomes $\frac{N\mu\theta}{2\iota\mu h}\epsilon_2$ or $-\iota\frac{N\theta}{2h}\epsilon_2$, and if we take $2g=-\mu$ it becomes $+\iota\frac{N\theta}{2h}\epsilon_2$.

Therefore we have $G=\mp\frac{N\theta}{2h}\epsilon_2$, and G is independent of the strength of the field.

[It is to be noticed that whether the above is the only term to be retained or not, G will be independent of the field strength, since this will cancel out in each term.]

Thus the direction of the polarisation, which depends on $\frac{D}{G}\phi^2$, is independent of the field strength.

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Again, suppose that $\frac{B_1}{A_1}$, $\frac{B_2}{A_2}$ are the values of $\frac{B}{A}$ for the two components. The positions of the components are determined very approximately by $2g = \mu$ and $2g = -\mu$. For these values of g, G takes equal and opposite values. Since the two values of G are of opposite sign it follows that $\frac{B_1}{A_1}$ and $\frac{B_2}{A_2}$ are of opposite sign.

Further, since the values of G are equal in magnitude, it follows that if we reverse the sign of one of the ratios $\frac{B_1}{A_1}$, $\frac{B_2}{A_2}$ they will both be roots of the equation

$$A^2 - \frac{N\theta}{2h}\epsilon_2 . AB + B^2 = 0.$$

Hence we have $$\left(\frac{B_1}{A_1}\right)\left(-\frac{B_2}{A_2}\right) = 1 \text{ or } \frac{B_1}{A_1} . \frac{B_2}{A_2} = -1.$$

If the normals to the planes of polarisation of the components make angles θ_1, θ_2 with the axis of x we have

$$\frac{B_1}{A_1} = \tan\theta_1, \quad \frac{B_2}{A_2} = \tan\theta_2,$$

$$\therefore \tan\theta_1 . \tan\theta_2 = -1,$$

$$\therefore \theta_1 \sim \theta_2 = \frac{\pi}{2}.$$

Hence the bands are polarised in perpendicular directions.

Again, the direction of polarisation will turn through 90° when the field is reversed.

For the displacements of the bands are given by $2g = \pm\mu$.

Thus the bands change places when the field is reversed, since μ changes sign; the component that was accelerated being now retarded, and vice versa.

Hence, since the direction of polarisation of a given band does not change with the field, it follows that the direction of polarisation of the band which occupies a *given position* turns through 90°, since the bands are polarised in perpendicular directions.

The results thus deduced for the states of polarisation of the components when the axis is slightly inclined to the field agree fully with Becquerel's observations.

He, however, observed that the angle ϕ depended on the thickness of the crystal. The above theory seems to leave no room for such a dependence.

With the Optic Axis parallel to the Rays and to the Field it was found that in a large number of the resulting doublets the direction of displacement was contrary to that observed in the Zéeman Effect.

According to our theory the displacement for the right-handed components is given by $2g = -\mu$, and for the left-handed by $2g = +\mu$.

Thus for these 'inverse' displacements the sign of μ must be different from the sign for the direct displacements.

Now $$\mu = \frac{\theta H}{4\pi\iota\tau e}.$$

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Hence μ can only change sign when either H or e changes sign.

In observations on the Zéeman Effect it has always been found that the vibrations are due to negative electrons, and vibrations due to positive electrons have never been observed.

From the separation of the components of the band whose wave length is $642{\cdot}4\,\mu\mu$ Becquerel infers that the vibrations are due to negative electrons for which $\frac{e}{m} = 1{\cdot}1 \,.\, 10^8$.

But for the band whose wave length is $522{\cdot}14\,\mu\mu$ he finds almost the same value for $\frac{e}{m}$, but considers it due to positive electrons.

The coincidence of these values for $\frac{e}{m}$ leads us to suspect that the electrons are, however, negative in both cases, for it is well known that the ratio $\frac{e}{m}$ is approximately constant for negative electrons.

Also values of $\frac{e}{m}$ for positive electrons, so far as known, are always much smaller than this.

Hence it seems preferable to conclude that the vibrations are due to negative electrons in all cases, and to suppose that in the neighbourhood of the systems which give 'inverse' displacements the magnetic field changes sign, i.e. that these inverse effects are diamagnetic in character.

M. Becquerel * and Herr Voigt † have recently given explanations of the observed phenomena, starting from a hypothesis rather different from that used above.

* J. Becquerel, *C. R.*, 19 Nov., 3, 10, 24 Dec. 1906.

† W. Voigt, *Gött. Nach.* 1906, Heft 5.

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ON THE ASYMPTOTIC APPROXIMATION TO FUNCTIONS DEFINED BY HIGHLY-CONVERGENT PRODUCT-FORMS.

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XIV. *On the Asymptotic Approximation to Functions defined by Highly-convergent Product-forms.*

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Communicated by E. W. Barnes, Sc.D.

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§ 1. In the present paper we shall develop certain methods by means of which it is possible to obtain asymptotic approximations for certain classes of functions defined by a product-form of the type $\prod_{s=1}^{\infty}\left[1+\left(\frac{z}{a_s}\right)^{f(s)}\right]$. In this form we suppose that $a_1, a_2, \ldots a_s, \ldots$ is a sequence of numbers arranged in order of increasing moduli, and that $|a_s|$ tends either to an infinite limit or to a definite finite limit when s tends to infinity. In the former case $|z|$ is supposed large, in the latter $|z|$ is supposed less than, but near, the limit to which $|a_s|$ tends. The earlier part of the paper is devoted to certain results which hold only in the former case, while in the latter part the two cases are treated simultaneously.

§ 2. Throughout the paper we shall employ the following ideas and notation.

We write $$F(z)=\prod_{s=1}^{\infty}\left[1+\left(\frac{z}{a_s}\right)^{f(s)}\right], \quad |a_s|=\alpha_s, \quad |z|=r.$$

The index $f(s)$ is supposed to be real. When $f(s)$ is an integer for all values of s, the function $F(z)$ is uniform. When $f(s)$ has non-integral values, $F(z)$ is multiform. In this case we suppose a cross-cut made from the origin along the negative real axis, extending to infinity, if $\operatorname{Lt}_{s=\infty} \alpha_s=\infty$, in which case z ranges over the whole plane, and extending to the point $-c$ if $\operatorname{Lt}_{s=\infty} \alpha_s=c$, in which case z is supposed interior to the circle $|z|=c$. Then in the cut plane $z^{f(s)}$ is interpreted to mean $\exp[f(s)\log z]$, when $\log z$ has its principal value, and the function $f(z)$ is uniform in the cut plane.

When $\operatorname{Lt}_{s=\infty} \alpha_s=\infty$, we propose to find an approximate expression for $F(z)$ when r is large, and when $\operatorname{Lt}_{s=\infty} \alpha_s=$ a finite c, to find a similar expression when r is less than but nearly equal to c. In either case there is a single integer associated with r, which we shall always denote by n, such that $\alpha_{n+1} \geqslant r \geqslant \alpha_n$.

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When r is near its limit, n is clearly large. The approximations which we shall obtain are expressed practically in terms of n, and it is the fact that n is large which enables us to deal with the case when $\operatorname*{Lt}_{s=\infty} a_s = c$.

We shall always use the following notation

$$P = \left[\sum_{s=1}^{n} f(s)\right] . \log z - \left[\sum_{s=1}^{n} f(s) \log a_s\right],$$

$$R = \sum_{s=n+2}^{\infty} \log\left[1 + \left(\frac{z}{a_s}\right)^{f(s)}\right] = \sum_{s=2}^{\infty} \log\left[1 + \left(\frac{z}{a_{n+s}}\right)^{f(n+s)}\right],$$

$$S = \sum_{s=1}^{n-1} \log\left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right] = \sum_{s=1}^{n-1} \log\left[1 + \left(\frac{a_{n-s}}{z}\right)^{f(n-s)}\right],$$

$$T_1 = \log\left[1 + \left(\frac{z}{a_{n+1}}\right)^{f(n+1)}\right],$$

$$T_2 = \log\left[1 + \left(\frac{a_n}{z}\right)^{f(n)}\right].$$

We then have, as is easily seen,

$$\log F(z) = P + R + S + T_1 + T_2 \quad \text{.....(1).}$$

We shall also denote the real parts of P, T_1, etc., by $\bar{P}$, $\bar{T}_1$, etc.

It is customary, and it is more convenient, to express such results as we propose to obtain, in terms of the function $\log F(z)$ rather than $F(z)$, but we are none the less primarily interested in $F(z)$. Now $\log F(z)$ has an infinity of values, differing by multiples of $2\pi\iota$, and when $\log F(z)$ is given, each of its values gives one and the same value of $F(z)$. For this reason we shall regard all possible values of $\log F(z)$ as equivalent, and we shall therefore not concern ourselves with the proper multiple of $2\pi\iota$ which should be added to the right-hand side of equation (1).

Again, the imaginary part of any approximation which we may have obtained for $\log F(z)$ may have a modulus which tends to infinity as r tends to its limit. But since this imaginary part may be reduced, by the subtraction of a suitable multiple of $2\pi\iota$, to an expression whose modulus is less than π, and since it corresponds to a factor of $F(z)$ whose modulus is unity, we shall not regard this imaginary part as being a *large* term in the approximation for $\log F(z)$, but as being a *finite* term.

After what has been said it is clear that we are free to choose any determination we please for the logarithms which occur in the above expressions for P, R, etc., provided only that the infinite series for R converge. We determine that all the logarithms in the expressions for R, S, T_1 and T_2 shall have their principal values. The logarithms in the expression for P we shall choose as is most convenient in each particular case*. This is clearly legitimate on account of the nature of equation (1). We shall, however, sometimes employ the two following variations of equation (1)

$$\log F(z) = \bar{P} + \bar{R} + \bar{S} + T_1 + T_2 + \iota\gamma_z \quad \text{.....(2),}$$

and

$$\log F(z) = \bar{P} + R + S + T_1 + T_2 + \iota\gamma_z',$$

where we may suppose that $|\gamma_z| \leqslant \pi$, $|\gamma_z'| \leqslant \pi$.

* For example if $a_s = se^{s\gamma\iota}$, it is clearly most convenient for purposes of calculation to take $\log a_s = \log s + s\gamma\iota$.

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§ 3. We shall first establish a general formula for $\log F(z) = \log \Pi \left[1 + \left(\frac{z}{a_s}\right)^{f(s)}\right]$, under certain general conditions for $f(s)$ and the a's. We shall then consider more special forms of $f(s)$, and shall replace the general conditions by others more precise.

Theorem. Let $a_1, a_2 \ldots$ be a set of numbers arranged in order of increasing* moduli $\alpha_1, \alpha_2 \ldots$. Let $f(s)$ be a number depending on s, such that $f(s+1) \geqslant f(s)$ for all values of s, and let the following conditions be satisfied,

(A) $\operatorname*{Lt}_{s=\infty} \left[f(s)\left\{\frac{\log \alpha_{s+1}}{\log \alpha_s} - 1\right\}\right] = \infty$.

(B) $\operatorname*{Lt}_{s=\infty} \left[\frac{f(s)}{\log s}(\log \alpha_{s+1} - \log \alpha_s)\right] = \infty$.

(C) Suppose we are given any finite integer p. Suppose r is a positive number, and let n be the integer such that $\alpha_n \leqslant r < \alpha_{n+1}$. Let $\beta_s(r) = \left(\frac{\alpha_s}{r}\right)^{f(s)}$, $(s \leqslant n)$.

Then the following properties shall obtain when r is greater than some finite value.

(1) An integer $m < n$ exists, such that

$$\beta_1(r) > \beta_2(r) > \ldots > \beta_m(r),$$

and

$$\beta_m(r) \leqslant \beta_{m+1}(r) < \beta_{m+2}(r) < \ldots < \beta_n(r).$$

(2) A positive absolute constant $c < 1$, and an integer m' depending on n, exist, such that

$$(a) \quad \beta_{s+1}(r)/\beta_s(r) < c, \text{ when } s < m',$$

and

$$(b)\dagger \quad \operatorname*{Lt}_{r=\infty} \left[n\left(\frac{\alpha_{m'}}{r}\right)^{f(m')} r^p\right] = 0.$$

Then when z is restricted to lie in a region R of the z plane, presently to be defined, we have the asymptotic expansion‡,

$$\log F(z) = P + \sum_{s=1}^{\infty} C_s/z^{\nu_s},$$

where

$$P = \left[\sum_{s=1}^{n} f(s)\right] . \log z - \sum_{s=1}^{n} [f(s) \log a_s],$$

and $\sum_1^{\infty} C_s/z^{\nu_s}$ is the formal development of $\log \prod_{s=1}^{\infty} \left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right]$, arranged in such a manner that $\nu_{s+1} > \nu_s$.

No asymptotic expansion for $\log F(z)$ of the above form can be valid throughout the whole extent of any region which contains zeros of $F(z)$. We must therefore confine z to some region R which contains no zeros. We shall proceed to define this region.

Assign any constant k between 0 and $\frac{1}{2}$, both exclusive, but which we may take as near to zero as we please.

* The condition (A) requires that ultimately $a_{s+1} > a_s$, equality being excluded.

† n is, of course, a function of r.

‡ This expansion is clearly always divergent.

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The annulus A_s, $\alpha_{s+1}^{k}\alpha_s^{1-k} \leqslant |z| \leqslant \alpha_{s+1}^{1-k}\alpha_s^{k}$ is evidently contained in the annulus $\alpha_{s-1} \leqslant |z| \leqslant \alpha_{s+1}$. We take the region R to be the sum of all the annuli A_s, and we suppose z to lie in R.

We have
$$\log F(z) = P + R + T_1 + S + T_2 \qquad (1).$$

We shall prove that $\operatorname*{Lt}_{r=\infty} |(R+T_1)r^p| = 0$, and that $S+T_2$ possesses the asymptotic expansion $\Sigma C_s/z^{\nu_s}$.

First consider $T_1 + R$.

Condition (A) may be rewritten in the form
$$\alpha_{s+1} = \alpha_s^{1+\frac{\chi(s)}{f(s)}}, \text{ where } \operatorname*{Lt}_{s=\infty} \chi(s) = \infty \qquad (2).$$

Hence, since z lies in R,
$$\left|\left(\frac{z}{\alpha_{n+1}}\right)^{f(n+1)}\right| = \left(\frac{r}{\alpha_{n+1}}\right)^{f(n+1)}$$
$$\leqslant \left(\frac{\alpha_{n+1}^{1-k}\cdot\alpha_n^{k}}{\alpha_{n+1}}\right)^{f(n+1)} \leqslant \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{kf(n+1)} < \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{kf(n)}$$
$$< \left[\alpha_n^{-\frac{\chi(n)}{f(n)}}\right]^{kf(n)} < \alpha_n^{-k\chi(n)}$$
$$< \left[\alpha_{n+1}^{\frac{f(n)}{f(n)+\chi(n)}}\right]^{-k\chi(n)} < \alpha_{n+1}^{-k\omega(n)} < r^{-k\omega(n)} \qquad (3),$$

where
$$\omega(n) = \chi(n) \Big/ \left(1 + \frac{\chi(n)}{f(n)}\right).$$

Again
$$\left(\frac{z}{\alpha_{n+s}}\right)^{f(n+s)}\Bigg| = \left(\frac{r}{\alpha_{n+1}}\right)^{f(n+s)} \cdot \left(\frac{\alpha_{n+1}}{\alpha_{n+2}}\right)^{f(n+s)} \cdots \left(\frac{\alpha_{n+s-1}}{\alpha_{n+s}}\right)^{f(n+s)}$$
$$\leqslant \left(\frac{r}{\alpha_{n+1}}\right)^{f(n+1)} \cdot \left(\frac{\alpha_{n+1}}{\alpha_{n+2}}\right)^{f(n+2)} \cdots \left(\frac{\alpha_{n+s-1}}{\alpha_{n+s}}\right)^{f(n+s)}$$
$$< r^{-k\cdot\omega(n)} \cdot \alpha_{n+1}^{-\omega(n+1)} \cdot \alpha_{n+2}^{-\omega(n+2)} \cdots \alpha_{n+s-1}^{-\omega(n+s-1)} \qquad (3)',$$

by a proof similar to the above.

Now if
$$\chi(n) > f(n),\ \omega(n) > \frac{\chi(n)}{2\dfrac{\chi(n)}{f(n)}} > \tfrac{1}{2}f(n),$$

and if
$$\chi(n) < f(n),\ \omega(n) > \frac{\chi(n)}{2}.$$

Hence in all cases $\operatorname*{Lt}_{n=\infty} \omega(n) = \infty$.

Then if p be any integer, there is a finite μ such that, when $n > \mu$, we have,
$$k\omega(n) > p+1, \text{ and also } \omega(n+s) > 1,\ (s = 0, 1, 2 \ldots).$$

Hence, when $n > \mu$, we have from (3)′
$$\left|\left(\frac{z}{\alpha_{n+s}}\right)^{f(n+s)}\right| < r^{-(p+1)} \cdot r^{-(s-1)},\ (s = 1, 2 \ldots).$$

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Choose n so large that we have $r > 2$, so that

$$|x| = \left|\left(\frac{z}{a_{n+s}}\right)^{f(n+s)}\right| < \tfrac{1}{2}.$$

Then
$$\left|\log\left[1+\left(\frac{z}{a_{n+s}}\right)^{f(n+s)}\right]\right| = |x + \tfrac{1}{2}x^2 + \dots|$$
$$\leqslant |x| + \tfrac{1}{2}|x|^2 + \dots$$
$$< |x| + |x|^2 + \dots$$
$$< \frac{|x|}{1-|x|} < 2|x|$$
$$< 2r^{-p-s} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(4).$$

Hence
$$|T_1 + R| = \left|\sum_{s=1}^{\infty} \log(1+x)\right|$$
$$\leqslant \sum_{s=1}^{\infty} |\log(1+x)|$$
$$< 2r^{-p-1} . \sum_{s=1}^{\infty} r^{-s-1}$$
$$< \frac{2r^{-p-1}}{1-r^{-1}}$$
$$< 4r^{-p-1}, \text{ (since } r > 2) \dots\dots\dots\dots\dots\dots\dots\dots(4)'.$$

We shall now consider the term $(T_2 + S)$, and shall determine for it an asymptotic expression in descending powers* of z with an error of order less than z^{-p}, p being any assigned integer.

Let q be the greatest integer such that $f(q) \leqslant p$.

For a value of $s < q$, let σ be the greatest integer such that $\sigma f(s) < p$.

When p has been assigned we suppose r so large that $\left(\frac{a_q}{r}\right)^{f(1)} < \frac{1}{2}$.

We have
$$T_2 + S = \sum_{s=1}^{q} \log\left[1+\left(\frac{a_s}{z}\right)^{f(s)}\right] + \sum_{s=q+1}^{n} \log\left[1+\left(\frac{a_s}{z}\right)^{f(s)}\right] \dots\dots\dots\dots\dots\dots(5).$$

Now when $s < q$,
$$\left|\left(\frac{a_s}{z}\right)^{f(s)}\right| < \left(\frac{a_s}{r}\right)^{f(1)} < \left(\frac{a_q}{r}\right)^{f(1)} < \tfrac{1}{2}.$$

Then
$$S_1 = \sum_{s=1}^{q} \log\left[1+\left(\frac{a_s}{z}\right)^{f(s)}\right]$$
$$= \sum_{s=1}^{q} \left[\left\{\left(\frac{a_s}{z}\right)^{f(s)} - \frac{1}{2}\left(\frac{a_s}{z}\right)^{2f(s)} + \dots \pm \frac{1}{\sigma}\left(\frac{a_s}{z}\right)^{\sigma f(s)}\right\}\right.$$
$$\left. + \left(\frac{a_s}{z}\right)^{(\sigma+1)f(s)} . \left\{\frac{1}{\sigma+1} + \frac{1}{\sigma+2}\left(\frac{a_s}{z}\right)^{f(s)} + \dots\right\}\right]$$
$$= \sum_{s=1}^{q} \left\{\left(\frac{a_s}{z}\right)^{f(s)} - \frac{1}{2}\left(\frac{a_s}{z}\right)^{2f(s)} + \dots \pm \frac{1}{\sigma}\left(\frac{a_s}{z}\right)^{\sigma f(s)}\right\} + \sum_{s=1}^{q} \theta_s \left(\frac{a_s}{z}\right)^{(\sigma+1)f(s)} \quad \dots\dots\dots(6),$$

* These powers are of course not integral, except in the case when $f(s)$ is always an integer.

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where
$$|\theta_s| < \frac{1}{\sigma+1} + \frac{1}{\sigma+2}\left|\left(\frac{a_s}{z}\right)^{f(s)}\right| + \dots$$
$$< \frac{1}{\sigma+1} + \frac{1}{\sigma+2}\cdot\frac{1}{2} + \frac{1}{\sigma+3} + \frac{1}{2^2} + \dots$$
$$< \frac{2}{\sigma+1} < 2.$$

Then $\sum_1^q \theta_s . a_s^{(\sigma+1)f(s)}$ is finite and is less than a finite K, and $\left|\sum_{s=1}^q \theta_s \left(\frac{a_s}{z}\right)^{(\sigma+1)f(s)}\right| < \frac{K}{r^{-\kappa}}$, where κ is the minimum value of $(\sigma+1)f(s)$.

But by the definition of σ, $(\sigma+1)f(s) > p$ for all values of s considered. Hence $\kappa > p$.

Then from (6),
$$S_1 = \sum_{s=1}^q \left\{\left(\frac{a_s}{z}\right)^{f(s)} - \frac{1}{2}\left(\frac{a_s}{z}\right)^{2f(s)} + \dots\right\} + \lambda . r^{-\kappa} \quad \dots\dots\dots\dots\dots\dots\dots\dots (6)',$$
where $|\lambda| < K$.

We may arrange the finite sum on the left-hand side of (6)′ as $\sum_{s=1}^{\varpi} C_s/z^{\nu_s}$, where $\nu_{s+1} > \nu_s$, and it is clear that this expression consists of all the terms of the formal development of $\log \prod_{s=1}^{\infty}\left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right]$, arranged so that the indices of z^{-1} increase, which are such that $\nu_s \leqslant p$.

Hence
$$S_1 = \sum_{s=1}^{\varpi} C_s/z^{\nu_s} + \lambda r^{-\kappa} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6)'',$$
where ν_{ϖ} is the greatest ν less than or equal to p, and $\kappa > p$.

Call the second term on the right-hand side of (5), S_2. We divide the sum S_2 into the two parts,
$$\sigma_1 = \sum_{s=q+1}^{m'} \log\left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right], \text{ and } \sigma_2 = \sum_{s=m'+1}^{n} \log\left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right],$$
where m' is the integer provided by condition (C) (2). From the condition (C) (2) (a), we have, when $m' > s > q$,
$$\left|\left(\frac{a_s}{z}\right)^{f(s)}\right| = \beta_s(r) < c^{s-q-1} . \beta_{q+1}(r) < c^{s-q-1} . \left(\frac{\alpha_{q+1}}{r}\right)^{f(q+1)} \quad \dots\dots\dots\dots\dots\dots (7)$$
$$< c^{s-q}\left(\frac{\alpha_q}{r}\right)^{f(q)} < c^{s-q} . \frac{\alpha_q}{r} < \tfrac{1}{2}c^{s-q} < \tfrac{1}{2} \quad \dots\dots\dots\dots\dots\dots (8).$$

From (8),
$$\left|\log\left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right]\right| < 2 . \left|\left(\frac{a_s}{z}\right)^{f(s)}\right|$$
$$< 2 . c^{s-q-1}\left(\frac{\alpha_{q+1}}{r}\right)^{f(q+1)}, \text{ (from (7))}.$$

Hence
$$|\sigma_1| \leqslant \sum_{q+1}^{m'} \left|\log\left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right]\right|$$
$$< 2 . \left(\frac{\alpha_{q+1}}{r}\right)^{f(q+1)} . \sum_{s=q+1}^{m'} c^{s-q-1}$$
$$< \frac{2}{1-c}(\alpha_{q+1})^{f(q+1)} . r^{-f(q+1)} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (9),$$
where, from the definition of q, we have $f(q+1) > p$.

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We have finally to consider σ_2.

It is evident from the inequalities by which m, m', are determined in (C), that $m \geqslant m'$.

Now by (C)(1), $\beta_n(r) > \beta_{n-1}(r) > \ldots > \beta_m(r)$; $\beta_m(r) < \beta_{m-1}(r) < \ldots < \beta_1(r)$.

Then when $n \geqslant s > m'$,

$$\left|\left(\frac{a_s}{z}\right)^{f(s)}\right| = \beta_s(r) < \text{the greater of } \left(\frac{a_{m'}}{r}\right)^{f(m')} \text{ and } \left(\frac{a_n}{r}\right)^{f(n)} \quad \ldots\ldots\ldots\ldots\ldots(10).$$

First suppose $$\left(\frac{a_n}{r}\right)^{f(n)} > \left(\frac{a_{m'}}{r}\right)^{f(m')}.$$

We have $$\left(\frac{a_n}{r}\right)^{f(n)} \leqslant \left(\frac{a_n}{a_{n+1}^{k}\, a_n^{1-k}}\right)^{f(n)} \leqslant \left(\frac{a_n}{a_{n+1}}\right)^{kf(n)}$$

$$\leqslant a_{n+1}^{-k\chi(n)} \Big/ \left[1 + \frac{\chi(n)}{f(n)}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$\leqslant a_{n+1}^{-\omega(n)}$$

$$< \tfrac{1}{2}. \text{ when } n \text{ is sufficiently large} \quad \ldots\ldots\ldots\ldots\ldots(12).$$

Hence, when n is sufficiently large,

$$|\sigma_2| \leqslant \sum_{s=m'+1}^{n} \left|\log\left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right]\right|$$

$$\leqslant (n - m' - 1).\left|\log\left[1 - \left(\frac{a_n}{r}\right)^{f(n)}\right]\right|, \text{ since } \left(\frac{a_s}{z}\right)^{f(s)} < \left(\frac{a_n}{r}\right)^{f(n)},$$

$$< 2(n - m' - 1).\left(\frac{a_n}{r}\right)^{f(n)}, \text{ from (12)},$$

$$< 2n.\, a_{n+1}^{-k\chi(n)} \Big/ \left(1 + \frac{\chi(n)}{f(n)}\right), \text{ from (11)}.$$

Therefore $$\operatorname*{Lt}_{r=\infty} \log\left[r^{-p}/|\sigma_2|\right] \geqslant \operatorname*{Lt}_{n=\infty} \log\left[\frac{1}{2n}.\, a_{n+1}^{-p+k\chi(n)/\{1+\chi(n)/f(n)\}}\right]$$

$$\geqslant \operatorname*{Lt}_{n=\infty}\left[\tfrac{1}{2}k\frac{\chi(n)}{1 + \dfrac{\chi(n)}{f(n)}} \log a_{n+1} - \log 2n\right],$$

since $\dfrac{k\chi(n)}{1 + \dfrac{\chi(n)}{f(n)}}$ tends to infinity and is therefore ultimately greater than $2p$,

$$\geqslant \tfrac{1}{2}k \operatorname*{Lt}_{n=\infty}\left[f(n)\log a_{n+1}.\left(1 - \frac{1}{1 + \dfrac{\chi(n)}{f(n)}}\right) - \frac{2}{k}\log 2n\right]$$

$$\geqslant \tfrac{1}{2}k \operatorname*{Lt}_{n=\infty}\left[\log n\left\{\frac{f(n)(\log a_{n+1} - \log a_n)}{\log n} - \frac{2}{k} - \frac{2}{k}\frac{\log 2}{\log n}\right\}\right]$$

$$= +\infty, \text{ by condition (B)}.$$

Hence $$|\sigma_2|.\, r^p = \epsilon(n), \text{ where } \operatorname*{Lt}_{n=\infty} \epsilon(n) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Next suppose $$\left(\frac{a_{m'}}{r}\right)^{f(m')} > \left(\frac{a_n}{r}\right)^{f(n)}.$$

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By condition (C) (2) (*b*),

$$\left(\frac{a_{m'}}{r}\right)^{f(m')} = \frac{\epsilon'(n)}{nr^p} \,.\, \underset{n=\infty}{\mathrm{Lt}}\ \epsilon'(n) = 0.$$

Choose n so large that $$\left(\frac{a_{m'}}{r}\right)^{f(m')} < \frac{1}{2}.$$

Then as before $$|\sigma_2| < 2n\left(\frac{a_{m'}}{r}\right)^{f(m')}$$

and therefore $$\underset{r=\infty}{\mathrm{Lt}}\ [r^p\,|\sigma_2|] \leqslant 2\ \underset{n=\infty}{\mathrm{Lt}}\left[n\left(\frac{a_{m'}}{r}\right)^{f(m')} . r^p\right] = 0$$

by (C), (2), (*b*).

Hence, in any case, $$|\sigma_2| . r^p = \epsilon(n),\ \underset{n=\infty}{\mathrm{Lt}}\ \epsilon(n) = 0 \qquad\text{.....(13)}'.$$

Then remembering that $S_2 = \sigma_1 + \sigma_2$, we have, from (4)′, (6)′, (9) and (13)′,

$$r^p . \left| T_1 + R + T_2 + S - \sum_{s=1}^{\varpi} C_s/z^{\nu_s} \right| < 4r^{-1} + K . r^{p-\kappa} + \frac{2}{1-c}(a_{q+1})^{-f(q+1)} . r^{p-f(q+1)} + \epsilon(n)$$
$$= \epsilon(r), \text{ where } \underset{r=\infty}{\mathrm{Lt}}\ \epsilon(r) = 0,$$

since $(\kappa - p)$, $[f(q+1) - p]$, are positive and do not depend on n.

Since $p \geqslant \nu_\varpi$, we have

$$\underset{r=\infty}{\mathrm{Lt}} \left| \left[\log F(z) - P - \sum_{s=1}^{\varpi} C_s/z^{\nu_s}\right] . z^{\nu_\varpi} \right| \leqslant \underset{r=\infty}{\mathrm{Lt}} \left| \left[\log F(z) - P - \sum_{s=1}^{\varpi} C_s/z^{\nu_s}\right] z^p \right| = 0,$$

and the theorem is proved.

§ 4. The formula of the last article supposes z confined to the region R.

It is easy, however, to see that when z is unrestricted, the expressions R, and S, satisfy the inequalities that we found in § 3 for $T_1 + R$, and $T_2 + S$.

Hence when z is unrestricted we have the asymptotic formula,

$$\log F(z) = P + T_1 + T_2 + \sum_{s=1}^{\infty} C_s/z^s,$$

where $$T_1 = \log\left[1 + \left(\frac{z}{a_{n+1}}\right)^{f(n+1)}\right], \qquad T_2 = \log\left[1 + \left(\frac{a_n}{z}\right)^{f(n)}\right].$$

Moreover at least one of the two expressions T_1, T_2, is asymptotically represented by zero. For if z lies in the annulus A_n, T_1 and T_2* are both asymptotically represented by zero. If z lies outside the annulus A_n, in such a manner that A_n passes between z and the origin, $|T_2|$ is less than the corresponding value when z is inside A_n and has the same argument, and hence T_2 is asymptotically represented by zero. Similarly when z lies on the other side of the annulus A_n, T_1 is asymptotically represented by zero.

The above extended formula enables us to study the behaviour of $F(z)$ when z is near the circle $|z| = |a_n|$.

* In the course of the proof in § 3 we proved that $\underset{n=\infty}{\mathrm{Lt}}\left[n . \left(\frac{a_n}{r}\right)^{f(n)} . r^p\right] = 0$, whence follows the above result for T_2.

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We may remark, in connection with the conditions (A), (B), (C), that (C), in spite of its complicated appearance, is usually satisfied, in the cases which arise most naturally, when (A) and (B) are satisfied. Moreover in the case of any particular function, it is usually fairly easy to determine whether or no (C) is satisfied. It is clear that when no further assumptions are made as to $f(s)$ than that $f(s+1) > f(s)$, we must expect complicated conditions. We shall now assume special forms for $f(s)$, and shall replace in each case the conditions (A), (B), (C), by other and simpler conditions.

§ 5. Suppose that $f(s) = s$. The function $F(z)$ is the integral function

$$\prod_{s=1}^{\infty}\left[1+\left(\frac{z}{a_s}\right)^s\right].$$

We proceed to show that the general conditions of § 3 are satisfied, if the two following conditions are satisfied:

I. $$\operatorname*{Lt}_{s=\infty} \chi(s) = \operatorname*{Lt}_{s=\infty} s\left[\frac{\log a_{s+1}}{\log a_s} - 1\right] = \infty.$$

II. $$\omega(s) \equiv \log a_{s+1} + s(\log a_{s+1} - \log a_s)$$

is an increasing function of s after some finite value of s.

Consequently, under conditions I, II, when z is confined to the region R, we have the asymptotic expansion,

$$\log F(z) = P + \sum_{s=1} C_s/z^s, \text{ where } P = \tfrac{1}{2}n(n+1)\log z - \sum_{s=1}^{n} s\log a_s,$$

and $\sum_1 C_s/z^s$ is the formal development of $\log \prod_1^{\infty}\left[1+\left(\frac{a_s}{z}\right)^s\right]$, arranged in descending powers of z.

The condition (A) is identical with I; we have then to consider (B) and (C).

We can deduce from I that when s is sufficiently large, $a_s > \exp(s^\lambda)$, for any assigned value of λ.

Let μ be the least integer such that $\chi(s) \equiv s\left[\frac{\log a_{s+1}}{\log a_s} - 1\right] > \lambda + 1$, and also $a_s > 1$, for all values of $s \geqslant \mu$.

Then

$$\log a_{s+1} = \left[1+\frac{\chi(s)}{s}\right].\log a_s = \ldots = \log a_\mu.\left(1+\frac{\chi(\mu)}{\mu}\right)\left(1+\frac{\chi(\mu+1)}{\mu+1}\right)\ldots\left(1+\frac{\chi(s)}{s}\right)$$

$$> \log a_\mu.\left(1+\frac{\lambda+1}{\mu}\right)\left(1+\frac{\lambda+1}{\mu+1}\right)\ldots\left(1+\frac{\lambda+1}{s}\right)$$

$$> \log a_\mu.\exp\left[(\lambda+1)\left(\frac{1}{\mu}+\frac{1}{\mu+1}+\ldots+\frac{1}{s}\right)\right].\left\{\prod_{t=\mu}^{n}\left[\left(1+\frac{\lambda+1}{t}\right)\exp\left(-\frac{\lambda+1}{t}\right)\right]\right\} \quad \ldots\ldots(1).$$

Now the product in crooked brackets, when $\prod_{t=\mu}^{n}$ is replaced by $\prod_{t=\mu}^{\infty}$, is convergent. Hence the modulus of the product in crooked brackets lies between finite non-zero limits, and has a lower limit K, say.

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Again $$\sum_{t=\mu}^{s} \frac{1}{s} > \int_{\mu+1}^{s+1} \frac{dx}{x} > \log \frac{s+1}{\mu+1}.$$

Then from (1), $$\log \alpha_{s+1} > K . \log \alpha_\mu . e^{-\frac{\lambda+1}{\mu+1}} . (s+1)^{\lambda+1}.$$

Choose s so large that $$(s+1) . K \log \alpha_\mu . e^{-\frac{\lambda+1}{\mu+1}} > 1.$$

Then $$\log \alpha_{s+1} > (s+1)^\lambda.$$

We therefore have

$$\operatorname*{Lt}_{s=\infty} \left[\frac{s(\log \alpha_{s+1} - \log \alpha_s)}{\log s}\right] \geqslant \operatorname*{Lt}_{s=\infty} \left[\frac{\log \alpha_s}{\log s} \chi(s)\right] \geqslant \operatorname*{Lt}_{s=\infty} \left[\frac{s^\lambda}{\log s} \chi(s)\right] = \infty,$$

since $\operatorname*{Lt}_{s=\infty} \chi(s) = \infty$.

Hence condition (B) is satisfied.

We have finally, then, to consider (C).

We have $$\beta_s(r) = \exp[s(\log \alpha_s - \log r)].$$

Then $$\gamma_s(r) = \frac{\beta_{s+1}(r)}{\beta_s(r)} = \exp[(\log \alpha_{s+1} - \log r) + s(\log \alpha_{s+1} - \log \alpha_s)]$$
$$= \exp[\omega(s) - \log r].$$

Now by condition II, $\omega(s)$ is an increasing function of s after some finite value σ of s.

When r and n are large $\omega(s) - \log r$ is negative when $s \leqslant \sigma$, and it is clearly positive when $s = n$. Hence since $\omega(s)$ is an increasing function, there is an integer m, such that $\omega(s) - \log r < 0$ when $s < m$, and $\omega(s) - \log r > 0$ when $s > m$; that is, such that $\frac{\beta_{s+1}(r)}{\beta_s(r)} < 1$, when $s < m$, and $\frac{\beta_{s+1}(r)}{\beta_s(r)} > 1$, when $s > m$; or again, such that $\beta_1(r) > \beta_2(r) > \ldots > \beta_m(r)$, and $\beta_m(r) \leqslant \beta_{m+1}(r) < \beta_{m+2}(r) < \ldots < \beta_n(r)$.

Thus (C) (1) is satisfied.

Again, assign any constant $c < 1$.

Let m' be the greatest value of s for which

$$\omega(s) - \log r < -\log\left(\frac{1}{c}\right).$$

Then for $s < m'$, we have $$\frac{\beta_{s+1}(r)}{\beta_s(r)} < c.$$

If p be assigned we clearly have $m > m' > p+1$, when n, or r, is sufficiently large.

Also we have $n < \log r$, when n is sufficiently large, since we ultimately have $\alpha_n > e^n$.

Hence $$\operatorname*{Lt}_{r=\infty} \left[n . \left(\frac{\alpha_{m'}}{r}\right)^{m'} . r^p\right] \leqslant \operatorname*{Lt}_{r=\infty} \left[n . \left(\frac{\alpha_{p+1}}{r}\right)^{p+1} . c^{m'-p-1} . r^p\right]$$
$$\leqslant \alpha_{p+1}^{p+1} \operatorname*{Lt}_{r=\infty} (\log r . r^{-1}) = 0.$$

Hence (C) (2) is satisfied, and our theorem is proved.

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§ 6. We saw that the condition I involves that $\alpha_n = \exp[n^{\omega(n)}]$, where $\underset{n=\infty}{\mathrm{Lt}}\ \omega(n) = \infty$.

When α_n is any function of this form which increases with n in a regular manner, the conditions I and II will be satisfied. It is evident that the function $F(z)$ is an integral function of zero order.

Let us take as an example $F(z) = \prod_{s=1}^{\infty}\left[1 + \left(\frac{z}{e^{e^{s\omega}}}\right)^s\right]$, where ω is real and positive.

n is determined as a function of r by the relation

$$n + 1 > \frac{1}{\omega}\log\log r > n.$$

We determine a number α by the relations,

$$0 < \alpha < e - 1, \quad \frac{1}{\omega}[\log_2 r + \log(1+\alpha)] = \text{ an integer } = n + 1.$$

The region R is the sum of all the annuli

$$\exp[e^{s\omega}\{1 + k(e^{\omega} - 1)\}] < |z| < \exp[e^{(s+1)\omega}.\{1 - k(e^{-\omega} + 1)\}].$$

When z is confined to R, we have

$$\log F(z) = \frac{1}{2\omega}[\log_2 r + \log\overline{1+\alpha}]^2.\log z - [\log_2 r + \log\overline{1+\alpha}].\left[\frac{1+\alpha}{e^{\omega}-1}.\log r + \frac{1}{2\omega}\log z\right]$$
$$+ \frac{(1+\alpha)e^{\omega}}{(e^{\omega}-1)^2}.\log r - \frac{e^{\omega}}{(e^{\omega}-1)^2} + \sum_1 C_s/z^s,$$

where $\quad C_1 = e^{e^{\omega}}, \quad C_2 = e^{e^{2\omega}} - \frac{1}{2}e^{2e^{\omega}}, \quad C_3 = e^{e^{3\omega}} - \frac{1}{3}e^{3e^{\omega}}, \quad C_4 = e^{e^{4\omega}} - \frac{1}{2}e^{2e^{2\omega}} + \frac{1}{4}e^{4e^{\omega}}$, etc.

When $z = e^{e^{\overline{n+1}\omega}}$, we have the simple expression

$$\log F(z) = \frac{1}{2\omega}.(\log_2 z)^2.\log z - \left[\frac{1}{2\omega} + \frac{1}{e^{\omega}-1}\right]\log_2 z.\log z$$
$$+ \frac{e^{\omega}}{(e^{\omega}-1)^2}.\log z - \frac{e^{\omega}}{(e^{\omega}-1)^2} + \log 2 + \sum_1 C_s/z^s.$$

§ 7. Next consider the case $f(n) = n^q$, $q > 1$.

The theorem for this case is the following.

The general conditions (A), (B), (C) are satisfied provided the following conditions are fulfilled:—

(α) $\underset{s=\infty}{\mathrm{Lt}}\ \chi(s) \equiv \underset{s=\infty}{\mathrm{Lt}}\ s^q\left[\frac{\log\alpha_{s+1}}{\log\alpha_s} - 1\right] = \infty$;

(β) $\underset{s=\infty}{\mathrm{Lt}}\ \frac{s^q}{\log s}[\log\alpha_{s+1} - \log\alpha_s] = \infty$;

(γ) $\psi(s) \equiv [(s+1)^q\log\alpha_{s+1} - s^q\log\alpha_s]/[(s+1)^q - s^q]$ is an increasing function of s when s is greater than a finite σ.

(δ) One or both of the following conditions are satisfied:—

(a) A positive non-zero constant ρ exists such that when s is sufficiently large, $\alpha_s > s^{\rho}$.

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(*b*) A positive non-zero constant $k_1 < 1$, and a positive non-zero h, exist, such that if s be sufficiently large, and if s' be any integer less than $k_1 s$, then

$$[(s'+1)^q - s'^q] \log \alpha_s - (s'+1)^q . \log \alpha_{s'+1} + s'^q \log \alpha_{s'} > h.$$

The conditions (A) and (B) are here identical with (α) and (β). It remains to consider (C).

We have

$$\frac{\beta_{s+1}(r)}{\beta_s(r)} = \left(\frac{\alpha_{s+1}}{r}\right)^{(s+1)^q} \Big/ \left(\frac{\alpha_s}{r}\right)^{s^q} = \exp\left[\{(s+1)^q - s^q\}(\psi(s) - \log r)\right] \text{.....} (1).$$

The expression in square brackets is negative when $s \leqslant \sigma$, when r is sufficiently large, and it is easily seen to be positive when $s = n$.

Moreover $[(s+1)^q - s^q]$ and $\psi(s)$ are increasing functions of s when $s > \sigma$.

Then it follows, as in the last article, that there is an integer m such that

$$\beta_1(r) > \beta_2(r) > \ldots > \beta_m(r),$$

and

$$\beta_m(r) \leqslant \beta_{m+1}(r) < \ldots < \beta_n(r).$$

Clearly also m, which differs by less than unity from the single positive root of $\psi(s) = \log r$, becomes infinite with n and r.

We see, then, that (C) (1) is satisfied.

Now suppose, in the first place, that δ (a) is satisfied. The integer p being assigned, we choose a constant $m' \geqslant q\sqrt{\left(p + \frac{2}{\rho}\right)}$, where ρ is a number provided by δ (a). When r is large we have $m > m'$.

From (1) we see that when r is sufficiently large, then, when $s < m'$, $\frac{\beta_{s+1}(r)}{\beta_s(r)} < c$, where c is any assigned constant less than unity.

Again

$$\operatorname*{Lt}_{r=\infty}\left[n\left(\frac{\alpha_{m'}}{r}\right)^{m'^q} . r^p\right] \leqslant (\alpha_{m'})^{m'^q} . \operatorname*{Lt}_{r=\infty} \frac{1}{r^{p+\frac{2}{\rho}}} . nr^p \leqslant (\alpha_{m'})^{m'^q} . \operatorname*{Lt}_{n=\infty} \frac{n}{\alpha_n^{2/\rho}} = 0, \quad \text{since } \alpha_n^{1/\rho} > n.$$

Then (C) (2) is satisfied.

Next suppose δ (b) holds.

Choose m' to be the greatest integer contained in $k_1 n$.

We have from equation (1),

$$\frac{\beta_{s+1}(r)}{\beta_s(r)} < \exp\left[-\{\log \alpha_n [(s+1)^q - s^q] - (s+1)^q \log \alpha_{s+1} + s^q \log \alpha_s\}\right].$$

Then from the condition δ (b), we have, when $s < m'$,

$$\frac{\beta_{s+1}(r)}{\beta_s(r)} < e^{-h} < 1.$$

Again $$m' = [k_1 n] \geqslant k_1 n - 1 > \frac{k_1}{2} n, \quad \text{when } n > \frac{2}{k_1}.$$

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Hence
$$\left(\frac{\alpha_{m'}}{r}\right)^{m'^q} < \left(\frac{\alpha_{m'}}{r}\right)^{\left(\frac{k_1}{2}\right)^q . n^q}, \quad \text{since } \alpha_{m'} < r$$
$$< \left(\frac{\alpha_n}{r}\right)^{\left(\frac{k_1}{2}\right)^q . n^q}$$
$$< \left[\frac{\alpha_n}{\alpha_n^{1-k} . \alpha_{n+1}^k}\right]^{\left(\frac{k_1}{2}\right) . n^q}, \quad \text{since } z \text{ lies in } R$$
$$< \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{k_2 . n^q}, \quad \text{where } k_2 = k\left(\frac{k_1}{2}\right)^q.$$

Then
$$L \equiv \underset{r=\infty}{\mathrm{Lt}} \log\left[1/\left\{n\left(\frac{\alpha_{m'}}{r}\right)^{m'^q} . r^p\right\}\right] \geqslant \underset{n=\infty}{\mathrm{Lt}} \log\left[1/\left\{n\left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{k_2 n^q} . \alpha_{n+1}^p\right\}\right]$$
$$\geqslant \underset{n=\infty}{\mathrm{Lt}} \left[k_2 n^q . (\log \alpha_{n+1} - \log \alpha_n) - p \log \alpha_{n+1} - \log n\right]$$
$$\geqslant \underset{n=\infty}{\mathrm{Lt}} \left[k_2 . \frac{\chi(n)}{1 + \frac{\chi(n)}{n^q}} \log \alpha_{n+1} - p \log \alpha_{n+1} - \log n\right].$$

Now
$$\underset{n=\infty}{\mathrm{Lt}} \frac{\chi(n)}{1 + \frac{\chi(n)}{n^q}} = \infty, \quad \text{since } \underset{n=\infty}{\mathrm{Lt}}\, \chi(n) = \infty.$$

Hence when n is sufficiently large, $p < \frac{1}{2} . \frac{k_2 \chi(n)}{1 + \frac{\chi(n)}{n^q}}$.

Hence
$$L \geqslant \underset{n=\infty}{\mathrm{Lt}} \left[\frac{1}{2} k_2 \frac{\chi(n)}{1 + \frac{\chi(n)}{n^q}} \log \alpha_{n+1} - \log n\right]$$
$$\geqslant \underset{n=\infty}{\mathrm{Lt}} \left[\tfrac{1}{2} k_2 . n^q (\log \alpha_{n+1} - \log \alpha_n) - \log n\right]$$
$$\geqslant \underset{n=\infty}{\mathrm{Lt}} \left[\left\{\frac{1}{2} k_2 \frac{n^q}{\log n} (\log \alpha_{n+1} - \log \alpha_n) - 1\right\} . \log n\right] = +\infty,$$

since, by condition (β), $\underset{n=\infty}{\mathrm{Lt}} \frac{n^q}{\log n} (\log \alpha_{n+1} - \log \alpha_n) = +\infty$.

Hence in this case again (C) (2) is satisfied.

Thus (A), (B), (C) are satisfied when (α), (β), (γ), (δ) are satisfied.

§ 8. The functions considered in the last article are integral functions if q is an integer greater than unity, but not otherwise. We obtain integral functions, however, if we take $f(n) = [n^q]$, $q > 1$, where $[x]$ is the greatest integer contained in x. If the conditions (α) to (δ) be modified by replacing s^q, wherever it occurs, by $[s^q]$ it can be proved, by slightly modifying the analysis of § 7, that the general conditions are satisfied, so that we may still apply the general formula.

With regard to the conditions (α) to (δ), we may remark that they are satisfied when $\alpha_1, \alpha_2, \ldots$ is any sequence whatever whose terms increase to infinity in a regular manner.

In support of this statement we shall make the following observations.

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In the first place if $\alpha_1, \alpha_2, \ldots$ increase in a regular* manner to infinity, we have

$$\log \alpha_{s+1} - \log \alpha_s > \frac{1}{s^{1+\epsilon}}.$$

For if not we should have $\quad \log \alpha_{s+1} - \log \alpha_s < \frac{1}{s^{1+\epsilon}},$

for some value of ϵ.

Then, adding, we have

$$\log \alpha_s < \frac{1}{s^{1+\epsilon}} + \frac{1}{\overline{s-1}^{1+\epsilon}} + \ldots,$$

$<$ a finite number, which is contrary to our suppositions.

It follows at once, since $q > 1$, that conditions (α), (β) are satisfied. It is, moreover, evident that (γ) is satisfied when $\alpha_1, \alpha_2, \ldots$ increase in a regular manner.

Concerning (δ) we observe that with our suppositions, either $\alpha_s > s^\rho$ for some ρ and all values of s sufficiently large, or else $\alpha_s < s^\rho$ for every value of ρ, when s is sufficiently large. In the first case (δ) (a) is satisfied.

Suppose then, in the second case, that $\alpha_s = s^{1/\chi(s)}$, where $\chi(s)$ increases in a regular manner, and let us consider the condition (δ) (b).

We wish to show that, for some constant k_1, when $s' < ks$, and s is large, we have

$$\psi(s', s) \equiv [(s'+1)^q - s'^q] \,.\, \log \alpha_s - (s'+1)^q \log \alpha_{s'} + s'^q \log \alpha_{s'} > h > 0.$$

If s' is supposed less than any particular finite limit, then, by choosing s sufficiently large, we can ensure that the above inequality is satisfied. We shall suppose, then, that s' is large.

Let $$\theta(s') = \frac{d}{ds'}(\log \alpha_{s'}).$$

Since $\log \alpha_{s'} = \frac{1}{\chi(s')} \log s'$ is of less order than $\log s'$, $\theta(s')$ is of less order than $\frac{1}{s'}$.

Let $\theta(s') = \frac{1}{s'\phi(s')}$, and put $s' = e^t$. Let $\phi(s') = \psi(t)$, and let $\int^t \frac{dt}{\psi(t)} = \Omega(t)$. We clearly have $\underset{t=\infty}{\mathrm{Lt}}\ \psi(t) = \infty$.

Then $$\log \alpha_{s'} = \int^{s'} \theta(x')\,dx' = \int^t \frac{dy}{\psi(y)} = \Omega(t).$$

Since $\underset{s'=\infty}{\mathrm{Lt}} \log \alpha_{s'} = \infty$, we have $\underset{t=\infty}{\mathrm{Lt}}\ \Omega(t) = \infty$.

Then $\log \alpha_s - \log \alpha_{s'} > \log \alpha_{k's'} - \log \alpha_{s'}, \quad (k' = 1/k_1)$

$$> \Omega(t + \log k') - \Omega(t)$$

$$> \log k' \Omega'(t) + \tfrac{1}{2}(\log k')^2 \Omega''(t + \lambda \log k'), \quad \lambda < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now $\Omega'(t) = \frac{1}{\psi(t)}$ is positive and decreasing, and hence $\Omega''(t)$ is negative, and decreasing in absolute magnitude. Also $\underset{t=\infty}{\mathrm{Lt}} \frac{\Omega''(t)}{\Omega'(t)} = 0$.

* The word "regular" is used above in a rough sense, and the analysis following is only rough.

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We have then from (1)

$$[\log \alpha_s - \log \alpha_{s'}] > (1-\epsilon) \log k' \,.\, \Omega'(t) > (1-\epsilon) \log k' \,.\, \frac{1}{\psi(t)} \quad \ldots\ldots\ldots\ldots (2)$$

$$> (1-\epsilon) \log k' \,.\, s'\theta(s') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)'.$$

Again $\quad \log \alpha_{s'+1} - \log \alpha_{s'} = [1 + \epsilon(s)]\, \theta(s'), \quad$ since $\theta(s') = \dfrac{d}{ds'} (\log \alpha_{s'})$,

and tends to zero with $1/s'$.

Hence $\quad \dfrac{\log \alpha_{s'+1} - \log \alpha_{s'}}{\log \alpha_s - \log \alpha_{s'}} < \dfrac{(1+\epsilon)}{(1-\epsilon) \log k' \,.\, s'} < \dfrac{2}{\log k' \,.\, s'}, \quad$ when $\epsilon < \dfrac{1}{3}$.

Then $\quad \psi(s', s) = [(s'+1)^q - s'^q] (\log \alpha_s - \log \alpha_{s'}) - (s'+1)^q (\log \alpha_{s'+1} - \log \alpha_{s'})$

$$> (\log \alpha_s - \log \alpha_{s'}) \left[q s'^{q-1} - (s'+1)^q \,.\, \frac{2}{\log k' \,.\, s'} \right]$$

$$> (\log \alpha_s - \log \alpha_{s'}) \left[q s'^{q-1} - 2^q \,.\, s'^q \,.\, \frac{2}{\log k' \,.\, s'} \right].$$

Choose k_1 such that $\quad \dfrac{2^{q+1}}{\log k'} < q/2.$

Then $\quad \psi(s', s) > (\log \alpha_s - \log \alpha_{s'}) \,.\, \dfrac{q}{2} s'^{q-1}$

$$> (\log \alpha_{k's'} - \log \alpha_{s'}) \,.\, \frac{q}{2} \,.\, s'^{q-1}$$

$$> (1-\epsilon) \log k' \,.\, \frac{q}{2} \,.\, s'^{q-1} \,.\, \frac{1}{\psi(t)} \text{(from (2))}.$$

Now $\Omega(t) = \displaystyle\int \frac{dt}{\psi(t)}$ tends to infinity with t.

Hence we must have $\psi(t) < t^2 < (\log s')^2$.

Then $\quad \psi(s', s) > \dfrac{1}{2} (1-\epsilon) \log k' \,.\, q \,.\, \dfrac{s'^{q-1}}{(\log s')^2}.$

But since $q > 1$, $\operatorname{Lt}\limits_{s'=\infty} \dfrac{s'^{q-1}}{(\log s')^2} = \infty$, and hence the inequality which we wished to prove is seen to hold.

§ 9. As an example, let us consider the case $a_n = n^\rho$, where $R\rho > 0$, and n^ρ means $\exp(\rho \log n)$.

Let $\rho = \rho_1 + \iota\rho_2$; then $a_n = n^{\rho_1}$.

It is easily seen that (α), (β), (γ) and (δ) (a) are satisfied. We may, then, apply our general formula.

The region R is the sum of all the annuli

$$(s+1)^{k\rho_1} \,.\, s^{(1-k)\rho_1} < |z| < (s+1)^{(1-k)\rho_1} \,.\, s^{k\rho_1}.$$

It may be taken to be the sum of all the annuli

$$s^{\rho_1} \,.\, \left(1 + \frac{k\rho_1}{s}\right) < |z| < s^{\rho_1} \left(1 + \frac{(1-k)\rho_1}{s}\right).$$

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The number n is determined by

$$(n+1)^{\rho_1} > r \geqslant n^{\rho_1}, \quad \text{or } n+1 > r^{1/\rho_1} \geqslant n.$$

We take $$n = r^{1/\rho_1} - \alpha, \quad \text{where } 0 < \alpha \leqslant 1 \quad \text{.....} \quad (1).$$

The restriction that z should lie in R is practically equivalent to the restriction

$$k < \alpha < 1 - k.$$

We have
$$\begin{aligned} P &= \left[\sum_{s=1}^{n} s^q .\right] \log z - \rho . \sum_{s=1}^{n} s^q \log s \\ &= \left[\sum_{s=1}^{n} s^q .\right] (\log z - \rho \log n) + \rho \left[\log n . \sum_{s=1}^{n} s^q - \sum_{s=1}^{n} s^q . \log s\right] \\ &= \left[\sum_{s=1}^{n} s^q\right] (\log z - \rho \log n) + \rho \left[\log n \sum_{s=1}^{n} s^q - \frac{\partial}{\partial q} \sum_{s=1}^{n} s^q\right] \text{.....} (2). \end{aligned}$$

Now $\sum_{s=1}^{n} s^q$ may be asymptotically expanded* as

$$\zeta(-q) + \frac{1}{q+1} n^{q+1} + \sum_{r=1} S_r'(1|1) . \frac{{}_{q+1}C_r}{q+1} . n^{q+1-r} + J_l,$$

where $$\operatorname*{Lt}_{n=\infty} |J_l . n^{l-q-1}| = 0.$$

Moreover this expansion may be differentiated with respect to q.

Hence
$$\begin{aligned} \log n . \sum_{s=1}^{n} s^q - \frac{\partial}{\partial q} \sum_{s=1}^{n} s^q &= \zeta(-q) . \log n + \zeta'(-q) + \frac{1}{(q+1)^2} n^{q+1} \\ &\quad + \sum_{r=1}^{l} S_r'(1|1) . \frac{\partial}{\partial q} \left[\frac{{}_{q+1}C_r}{q+1}\right] . n^{q+1-r} - K_l, \end{aligned}$$

where $$\operatorname*{Lt}_{n=\infty} |K_l n^{l-q-1}| = 0.$$

Hence from (2),

$$\begin{aligned} \log F(z) = \rho &\left[\zeta(-q) \log n - \zeta'(-q) + \frac{1}{(q+1)^2} n^{q+1} + \sum_{r=1}^{l} B_{2r} . n^{q+1-2r}\right] \\ &+ [\log z - \rho \log n] . \left\{\zeta(-q) + \frac{1}{q+1} . n^{q+1} + \sum_{r=1}^{l} S_r'(1|1) . \frac{{}_{q+1}C_r}{q+1} . n^{q+1-r}\right\} \\ &+ \sum_{s=1}^{l'} C_s / z^{\nu_s} + H_l + H_{l'}' \text{ .....} (3), \end{aligned}$$

where $$\operatorname*{Lt}_{n=\infty} \left| H_l . \frac{n^{2l-q-1}}{\log n} \right| = 0, \quad \text{and} \quad \operatorname*{Lt}_{r=\infty} |H_{l'}' . z^{\nu_{l'}}| = 0,$$

and where† $$B_r = S_r'(1|1) . \frac{\partial}{\partial q} \left(\frac{{}_{q+1}C_r}{q+1}\right),$$

and $\Sigma C_s / z^{\nu_s}$ is the formal development of $\log . \prod_{s=1}^{\infty} \left[1 + \left(\frac{s^\rho}{z}\right)^{s^q}\right]$.

The formula (3) provides an asymptotic expansion for $\log F(z)$ in descending powers (possibly non-integral) of z, and in descending integral powers of n, or of $[r^{1/\rho_1} - \alpha]$.

* Barnes, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. III. Part 4, p. 262.

† We clearly have $B_1 = S_1(1|1) . \frac{\partial}{\partial q}(1) = 0$, and $B_{2s-1} = 0$ $(s > 1)$, since $S_{2s-1}(1|1) = 0$.

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The obvious objection to this expansion is that it involves the number α. This, however, is unavoidable.

If we expand the various terms in n in descending powers of r^{1/ρ_1}, and arrange the whole expression in terms in r of descending order, neglecting, to our order of approximation, pure imaginaries, which count as finite terms, we obtain, when $q > 2$,

$$\log F(z) = \frac{\rho_1}{(q+1)^2} \cdot r^{\frac{q+1}{\rho_1}} + \rho_1 \cdot \frac{6\alpha(1-\alpha)-1}{12} \cdot r^{\frac{q-1}{\rho_1}} + A \cdot r^{\frac{q-2}{\rho_1}} + (\text{lower terms});$$

when* $q = 2$

$$\log F(z) = \frac{\rho_1}{9} \cdot r^{3/\rho_1} + \rho_1 \cdot \frac{6\alpha(1-\alpha)-1}{12} \cdot r^{1/\rho_1} + (\text{finite terms});$$

and when $1 < q < 2$,

$$\log F(z) = \frac{\rho_1}{(q+1)^2} r^{\frac{q+1}{\rho_1}} + \rho_1 \cdot \frac{6\alpha(1-\alpha)-1}{12} \cdot r^{\frac{q-1}{\rho_1}} + \zeta(-q)\log r + (\text{finite terms}).$$

It is clear that these expressions cannot be expressed in terms of the ordinary analytic functions, since the function α cannot be so expressed, and since it enters inevitably into the above formulae.

When $z = n^\rho$, so that $\alpha = 0$, we can find an asymptotic expansion for $\log F(z)$ in terms of known functions.

We must add to the general formula $P + \Sigma C_s/z^{\nu_s}$, the term $\log\left[1 + \left(\frac{a_n}{z}\right)^{n^t}\right]$, whose value is $\log 2$.

We then have

$$\log F(z) = \frac{\rho z^{\frac{q+1}{\rho}}}{(q+1)^2} - \rho \sum_{s=1}^{l} B_{2s} \cdot z^{\frac{q+1-2s}{\rho}}$$
$$+ \zeta(-q)\log z + \rho\zeta'(-q) + \log 2 + \sum_{s=1}^{l'} C_s/z^{\nu_s} + H_l + H_{l'}',$$

where $$\operatorname*{Lt}_{r=\infty} \left| H_l \cdot z^{\frac{2l-q-1}{\rho}} \right| = 0, \quad \text{and} \quad \operatorname*{Lt}_{r=\infty} \left| H_{l'}' \cdot z^{\nu_{l'}} \right| = 0.$$

§ 10. By means of the function $F(z)$ of the last article we can give an example of the following possibility.

It is possible to construct two integral functions $\phi_1(z)$ and $\psi_1(z)$, whose moduli differ only by a factor of the form $[1 + A/z]$, where A is finite, on the whole circumference of circles with centre at the origin and radii as large as we please, which functions are such that every zero of $\phi_1(z)$ of large modulus is at a large distance from every zero of $\psi_1(z)$, or more precisely, given any number h, as large as we please, we can find a number k such that every zero of $\phi_1(z)$, which is outside the circle $|z| = k$, is at a distance greater than h, from every zero of $\psi_1(z)$.

Let $$F_{\rho_1 p}(z) = \prod_{s=1}^{\infty} \left[1 + \left(\frac{z^\rho}{s^{p\rho}}\right)^{s^t}\right],$$

* We have $\zeta(-2) = 0$, so that the term $\zeta(-q) \cdot \log r$, which appears in the last expression, is in this case absent.

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where q is an integer $\geqslant 2$, p is an integer $> q$, and ρ is an odd integer. $F_{\rho,p}(z)$ is evidently an integral function of finite order.

From the definition of the region R, it is evident that this region is independent of ρ. Clearly also, for a given z, the integer n is independent of ρ.

For the function $F_{\rho,p}(z)$ we have

$$P = \rho \log z \,.\, \sum_{s=1}^{n} s^q - p\rho \sum_{s=1}^{n} s^q \log s$$

$$= \rho T, \quad \text{where } T \text{ is independent of } \rho.$$

Hence if $\sum_{s=1}^{\infty} {}_\rho C_s'/z^s$ be the formal development* of $\log \prod_{s=1}^{\infty} \left[1 + \left(\frac{s^{p\rho}}{z^\rho}\right)^{s^q}\right]$,

$$\log F_{\rho,p}(z) = \rho T + \sum_{s=1} {}_\rho C_s'/z^s,$$

and $$\log |F_{\rho,p}(z)| = \rho \bar{T} + \Re \left[\sum_{s=1} {}_\rho C_s'/z^s\right], \quad \text{where } \bar{T} = \Re T \text{ .....(1).}$$

We have also the particular case

$$\log |F_{1,p}(z)| = \bar{T} + \Re \left[\sum_{s=1} {}_1 C_s'/z^s\right] \text{ .....(2).}$$

Again, $\phi(z) = F_{\rho,p}(z)/F_{1,p}(z)$ is easily seen to be an integral function.

Let $$\psi(z) = [F_{1,p}(z)]^{\rho-1}.$$

Then we have from (1) and (2), when z lies in R,

$$\log |\phi(z)| - \log |\psi(z)| = \Re \sum_{s=1} ({}_\rho C_s' - \rho \,.\, {}_1 C_s') \,.\, z^{-s}.$$

Thus $|\phi(z)|$ and $|\psi(z)|$ differ only by a factor of the form $\left[1 + \frac{A}{z}\right]$, where $|A|$ is finite.

Now the zeros of $\phi(z)$ are $n^p \,.\, \exp\left[\frac{\pi\iota}{n^q}(2t+1)\right] \,.\, e^{\frac{2s\pi\iota}{\rho}}$,

where $$t = 0,\ 1 \ldots (n^q - 1),\ s = 1,\ 2, \ldots (\rho - 1), \text{ and } n = 1,\ 2,\ 3 \ldots.$$

The zeros of $\psi(z)$ are $n^p \exp \frac{\pi\iota}{n^q}(2t+1)$, each zero being taken $(\rho - 1)$ times, where

$$t = 0,\ 1, \ldots (n^q - 1), \text{ and } n = 1,\ 2,\ 3 \ldots.$$

When n^q is not a multiple of ρ the two sets of zeros on the circle $|z| = n^p$ are completely distinct.

When n^q is a multiple of ρ the two sets have common members. Let the assemblage of these common zeros, for all values of n, be $b_1, b_2 \ldots$, arranged in order of increasing moduli, each common zero b_m being taken μ times when it is a common zero of order μ for both $\phi(z)$ and $\psi(z)$.

Let $G(z)$ be any integral function with the zeros $b_1, b_2 \ldots$, and no others.

Then $\phi_1(z) = \phi(z)/G(z)$ and $\psi_1(z) = \psi(z)/G(z)$ are integral functions, and their moduli only differ by a factor of the form $\left[1 + \frac{A}{z}\right]$.

* Since q and ρ are integers, this development proceeds in descending integral powers of z.

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Now on the circle $|z| = n^p$, the least distance $d(n)$ between a zero of $\phi_1(z)$ and a zero of $\psi_1(z)$ is greater than or equal to the least distance between two distinct points of the set $n^p \exp\left[\frac{\pi\iota}{n^q}(2t+1)\right] e^{\frac{2s\pi\iota}{\rho}}$, where $t = 0, 1 \ldots (n^q - 1)$, and $s = 0, 1, 2 \ldots (\rho - 1)$; for every zero of $\phi_1(z)$ and $\psi_1(z)$ belongs to this set.

Now the difference of the arguments of the two points of this set for which s and t have the values s_1, t_1, and s_2, t_2, is

$$\frac{2(s_1 - s_2)\pi}{\rho} + \frac{2(t_1 - t_2)\pi}{n^q} = \frac{\{(s_1 - s_2)n^q + (t_1 - t_2)\rho\}}{\rho n^q} 2\pi.$$

The expression in crooked brackets does not vanish since the two points are distinct, hence this expression, which must be an integer, is equal to or greater than unity.

Hence the difference of the arguments $\geqslant \frac{2\pi}{\rho n^q}$.

Then the distance between the two points $\geqslant 2n^p . \sin \frac{2\pi}{2\rho n^q}$, and becomes infinite with n when $p > q$.

Again the shortest distance between the circles $|z| = n^p$, and $|z| = (n-1)^p$, is greater than $p(n-1)^{p-1}$, and becomes infinite with n.

Hence when n is large, any zero of $\phi_1(z)$ on the circle $|z| = n^p$, is at a large distance from every zero of $\psi_1(z)$. Thus the functions $\phi_1(z)$ and $\psi_1(z)$ have the properties mentioned at the beginning of the article.

§ 11. The functions considered in the last two articles are of finite order. As an example of a function of infinite order consider

$$F(z) = \prod_{s=3}^{\infty} \left[1 + \left\{\frac{z}{(\log s)^\rho}\right\}^{s^q}\right], \text{ where } q > 1, \ \Re\rho > 0,$$

which is an integral function when q is an integer $\geqslant 2$.

Let $\qquad \rho = \rho_1 + \iota\rho_2$, so that $\alpha_n = (\log n)^{\rho_1}$.

It is easily seen that the conditions (α), (β), (γ), of § 7 are satisfied. Moreover, following the lines of § 8, we can prove rigorously that (δ) (b) is satisfied; thus we can apply the general formula.

The region R is practically* equivalent to the region consisting of the sum of all the annuli

$$(\log s)^{\rho_1}\left(1 + \frac{k'}{s \log s}\right) < |z| < (\log s)^{\rho_1}\left(1 + \frac{1-k'}{s \log s}\right), \text{ where } k' < \tfrac{1}{2}.$$

The integer n is determined by $\log n < r^{1/\rho_1} < \log(n+1)$.

We have then $n = \exp(r^{1/\rho_1}) - \alpha$, where $0 < \alpha < 1$,

and $$\log F(z) = \left[\log z . [\sum_{s=3}^{n} s^q] - \rho . \sum_{s=3}^{n} s^q . \log_2 s\right] + \sum_{s=1} C_s / z^{\rho_s}.$$

* i.e., by choosing k' sufficiently small we can ensure that the region R' defined above is contained in R, while on the other hand, by choosing k (which occurs in the general definition of R) sufficiently small we can ensure that R is contained in R'.

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Now $\sum^{n} s^q \log_2 s$, qua function of n, cannot be completely expressed, asymptotically or otherwise, in terms of known functions of n. Hence we cannot obtain a complete expansion for $\log F(z)$ in terms of z. We can, however, find an asymptotic expression of the form $e^{Ar^\sigma}.[\Sigma B_s/r^s]$.

We shall neglect in P finite terms, and pure imaginaries which count as finite, for we propose to obtain an approximation for P in which we only retain the large terms above a certain order.

We may then replace P by its real part $\bar{P}$, i.e. replace $\log z$ by $\log r$, and ρ by ρ_1.

We have $$|\log r - \rho_1 \log_2 n| = |\rho_1 (\log_2 \overline{n+\alpha} - \log_2 n)| < \frac{\rho_1 \alpha}{n \log n}.$$

Again $\sum^{n} s^q = \frac{1}{q+1}.n^{q+1} + K.n^q$, where K is finite for all values of n.

Then $$\bar{P} = \left(\frac{1}{q+1} n^{q+1} + K.n^q\right)\left(\rho_1 \log_2 n + \frac{K_1}{n \log n}\right) - \rho_1 \sum^{n} s^q \log_2 s$$

$$= \frac{\rho_1}{q+1}.n^{q+1}.\log_2 n - \rho_1 \sum^{n} s^q.\log_2 s + K_2 n^q \log_2 n \quad \text{.....(1)},$$

where the K's denote numbers which are finite for all values of n.

Let $$u(x) = \int_3^x s^q \log_2 s\, ds.$$

Let $t = \exp(r^{1/\rho_1})$, so that $n+1 > t > n$, and $u(n+1) > u(t) > u(n)$.

We have, $$u_{n+1} - u_n < u_{n+1} < (n+1)^q \log_2 (n+1).$$

Also $$u(n+1) > \sum_3^n s^q \log_2 s > u(n).$$

Hence $|\sum_3^n s^q \log_2 s - u(t)| < u(n+1) - u(n)$, and hence is equal to $K_3.n^q \log_2 n$.....(2).

Integrating by parts, we have

$$u(t) = \left[\frac{s^{q+1}\log_2 s}{q+1}\right]_3^t - \frac{1}{q+1} v(t) \quad \text{.....(3)},$$

where $$v(t) = \int_3^t (s^q/\log s)\, ds.$$

From (1), (2) and (3) we have

$$\bar{P} = \frac{1}{q+1} v(t) + K_4.n^q \log_2 n \quad \text{.....(4)}.$$

Now $$v(t) = \int_{\log 3}^{\log t} e^{\overline{q+1}x}.\frac{dx}{x} \text{ (on putting } e^t = x).$$

This integral, as is well known, possesses the asymptotic expansion*,

$$t^{1+q}\left[\frac{1}{\overline{q+1}\log t} + \frac{1}{\overline{q+1}^2 (\log t)^2} + \ldots + \frac{(l-1)!}{\overline{q+1}^l (\log t)^l} + \frac{J_l}{(\log t)^l}\right],$$

* The expansion follows at once by integrating by parts $(l+1)$ times.

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where $$\operatorname*{Lt}_{t=\infty} J_l = 0,$$

or, replacing t by $\exp(r^{1/\rho_1})$,

$$e^{(q+1)r^{1/\rho_1}}\left[\frac{r^{-1/\rho_1}}{q+1}+\frac{r^{-2/\rho_1}}{(q+1)^2}+\ldots+(l-1)!\,\frac{r^{-l/\rho_1}}{(q+1)^l}+J_l r^{-l/\rho_1}\right] \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where $$\operatorname*{Lt}_{r=\infty} J_l = 0.$$

Now $$\log F(z) = P + (\text{small terms})$$
$$= \bar{P} + (\text{a finite expression})$$
$$= \frac{1}{q+1}\,v(t) + K_5 n^q \log_2 n, \text{ from (4).}$$

Also $$\operatorname*{Lt}_{r=\infty}\left[e^{-(q+1)r^{1/\rho_1}}\,.\,r^2 n^q \log_2 n\right] \leqslant \operatorname*{Lt}_{r=\infty} e^{-r^{1/\rho_1}} \log(r^{1/\rho_1})\, r^l = 0.$$

Hence changing l into $\overline{l+1}$ in (5),

$$\log F(z) = \frac{\exp[(q+1)r^{1/\rho_1}]}{(q+1)\,.\,r}\left[1+\frac{r^{-1/\rho_1}}{q+1}+\ldots+l!\,\frac{r^{-l/\rho_1}}{(q+1)^l}+J_l r^{-l/\rho_1}\right].$$

§ 12. We shall not work out the conditions, analogous to (α), (β), (γ), (δ), of § 7, by which the general condition (A), (B), (C), of § 3 may be replaced, in the cases of other special forms of $f(s)$, e.g. e^s, $[s^2/\log s]$. When $f(s)$ increases regularly and is of order greater than n^{1+k}, where k is any number >0, and when the terms a_1, $a_2 \ldots$ increase to infinity in a regular manner, however slowly, the general conditions will be satisfied. This will be the case, for example, when $f(s)$ is of higher order than s^q for all values of q. In this latter case it is always easy to find the dominant term* of P, and hence that of $\log F(z)$, as a function of r, but it is not always possible to find *all* the large terms of P, even qua functions of n.

For example if $f(s) = e^{e^s}$, $a_s = s^\rho$, $\rho = \rho_1 + \iota\rho_2$,

$$P = \sum_{s=1}^{n} e^{e^s} \log z - \sum_{s=1}^{n} \rho \log s\,.\,e^{e^s},$$

and we easily find that the dominant term of P and $\log F(z)$ is

$$\frac{\rho_1 \alpha}{r^{1/\rho_1}}\,.\,\exp[\exp(r^{1/\rho_1} - \alpha)],$$

where α is the fractional part of r^{1/ρ_1}.

To find P as a function of n, however, we should have to sum or asymptotically expand the series $\sum^{n} e^{e^s}$ and $\sum^{n} \log s\,.\,e^{e^s}$, which is impossible by any known analysis.

§ 13. In §§ 6, 9, 11, we gave examples of integral functions of zero, finite and infinite order respectively, for which the general formula of § 3 held. We can also obtain

* By a 'dominant term' I mean a term P_1, such that $P = (1+\epsilon_n)\,P_1$, where $\operatorname*{Lt}_{n=\infty} |\epsilon_n| = 0$.

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integral functions of transfinite order for which this formula holds. This can be done in two ways.

We may take $f(s)=s^q$ (q an integer $\geqslant 2$), and for a_s a function which increases very slowly with s, e.g. the inverse function $\vartheta^{-1}(s)$, where $\vartheta(x)=e_1^{e_2^{\cdot^{\cdot^{\cdot^{e_\nu}}}}}$, where $e_1=e_2=\ldots>1$, and ν is the integral part of x.

Again we may take $a_s=s^\rho$, and for $f(s)$ a function which increases very rapidly with s, e.g. $f(s)=e_1^{e_2^{\cdot^{\cdot^{\cdot^{e_\nu}}}}}$, where $e_1=e_2=\ldots=$ an integer $\geqslant 2$.

In each of these cases the function $\Pi\left[1+\left(\frac{z}{a_s}\right)^{f(s)}\right]$ is of transfinite order.

§ 14. The functions for which the general formula holds have a certain common property.

We have, when z is confined to the region R,

$$\log|F(z)|=\bar{P}+\Re\Sigma C_s/z^{\nu_s}.$$

Now
$$\bar{P}=\left[\sum_{s=1}^{n} f(s)\right]\log r-\left[\sum_{s=1}^{n} f(s)\log a_s\right],$$

and is clearly independent of the arguments of z and of the a's.

Hence on circles lying in the region R, whose centres are at the origin, and whose radii are large, $|F(z)|$ varies only by a factor of the form $[1+\epsilon(z)]$, where $\operatorname*{Lt}_{r=\infty}|\epsilon(z)|=0$, on the whole circumference of each circle.

Hence, in particular, integral functions exist, of every type of order, zero, finite, infinite, and transfinite, which have this property. The property is characteristic of functions of zero order*, but quite exceptional in the case of finite order, and probably also in the cases of infinite and transfinite order.

§ 15. In cases where the general formula of § 3 does not apply, it may happen that the error committed by taking P as an approximation for $\log F(z)$, is finite. We shall give a theorem for this case for the particular form s^q of $f(s)$.

This theorem applies to a more extended class of functions than we have hitherto considered: it applies to cases when the a's do not tend to infinity, but tend to a definite finite limit.

Suppose then that $a_1, a_2\ldots$ is a sequence such that the moduli $\alpha_1, \alpha_2\ldots$ increase, and such that $\operatorname*{Lt}_{s=\infty}\alpha_s=$ a finite limit, which we may suppose without loss of generality to be unity.

When $\operatorname*{Lt}_{s=\infty}\alpha_s=1$, and $q>1$, it is possible for the product form

$$F(z)=\prod_{s=1}^{\infty}\left[1+\left(\frac{z}{a_s}\right)^{s^q}\right]$$

to converge, and if q is in addition an integer, $F(z)$ will be an uniform function.

* Cf. a paper by myself, *Proc. Lond. Math. Soc.* Oct. 1907, §§ 4, 5.

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Since $F(z)$ has s^q zeros on the circle $|z| = a_s$, every point of the circle $|z| = 1$ is a limiting point of zeros of $F(z)$. Hence the circle $|z| = 1$ is a ring of essential singularities of $F(z)$, the function $F(z)$ only exists inside the circle, and the radius of convergence of the Taylor series which represents $F(z)$ is unity.

Let n be the integer such that $a_n < |z| < a_{n+1}$.

Then when $|z|$ is near its limit 1, n is large, and hence, as we shall see, $|P|$ is large. This fact enables us to apply, with a single reservation, the methods which we have so far employed, to find an approximation for $\log F(z)$ when $|z|$ is nearly unity. The sole modification which appears arises from the fact that since $|z|$ is finite, the formal development of $\log \Pi \left[1 + \left(\frac{a_s}{z}\right)^{f(s)}\right]$ cannot be an asymptotic expansion. This is the reason that we have not so far considered functions $F(z)$ of the above form.

In what follows we shall suppose that $a_1, a_2 \ldots$ is a sequence such that a_s is an increasing function of s, and such that $\operatorname*{Lt}_{s=\infty} a_s$ is either infinity or unity. In the first case $|z|$ is supposed large, and in the second case it is supposed less than unity, but near unity. In any case n is large, and the two cases can be considered together.

We shall continue to use the following notation:

$$P = \left[\sum_{s=1}^{n} f(s)\right] . \log z - \left[\sum_{s=1}^{n} f(s) \log a_s\right],$$

$$R = \sum_{s=2}^{\infty} \log \left[1 + \left(\frac{z}{a_{n+s}}\right)^{f(n+s)}\right],$$

$$S = \sum_{s=1}^{n-1} \log \left[1 + \left(\frac{a_{n-s}}{z}\right)^{f(n-s)}\right],$$

$$T_1 = \log \left[1 + \left(\frac{z}{a_{n+1}}\right)^{f(n+1)}\right],$$

$$T_2 = \log \left[1 + \left(\frac{a_n}{2}\right)^{f(n)}\right],$$

and $\qquad \bar{P} = \Re P$, etc.

§ 16. Theorem.

Let
$$F(z) = \prod_{s=1}^{\infty} \left[1 + \left(\frac{z}{a_s}\right)^{s^q}\right],$$

where $q > 0$, and where the a's are subject to the following condition:—

$$\operatorname*{Lt}_{s=\infty} s^q [\log a_{s+1} - \log a_s] \geqslant \beta > 0 \text{ *} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let R' be the region of the z plane defined as follows. Choose a positive number k which may be as small as we please and which we take in any case to be less than

* This formula does not imply that the limit on the left-hand side is definite, but only that $s^q [\log a_{s+1} - \log a_s] > \beta - \epsilon$ when s is sufficiently large.

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both $\frac{1}{2}\beta$ and π. When q is not an integer, R' is defined to be the sum of all the annuli* $\alpha_s\left(1+\frac{k}{s^q}\right)<|z|<\alpha_{s+1}\left(1-\frac{k}{s^q}\right)$.

When q is an integer, $F(z)$ is uniform and has s^q zeros on the circle $|z|=\alpha_s$.

Let these zeros be ${}_1b_s, {}_2b_s \dots {}_{sq}b_s$.

With centre ${}_tb_s$ describe a circle ${}_tA_s$ of radius $\frac{k\alpha_s}{s^q}$.

Then, in this case, R' is defined to be the region of the z plane or of the interior of the unit circle (according as $\operatorname{Lt}_{s=\infty} \alpha_s=\infty$ or 1), which is exterior to all the circles ${}_tA_s$, for all possible values of s and t.

Then when z is confined to R', and n has its usual meaning we have

$$\log F(z)=\bar{P}+Q,$$

where
$$\bar{P}=\left[\sum_{s=1}^{n} s^q \log r-\sum_{s=1}^{n} s^q \log \alpha_s\right],$$
and where $|Q|<$ a finite K.

Choose a positive number $\beta'<\beta$. Then a finite m exists, such that when $s>m$,

$$s^q(\log \alpha_{s+1}-\log \alpha_s)>\beta',$$

and hence
$$\left(\frac{\alpha_{s+1}}{\alpha_s}\right)^{s^q}>e^{\beta'} \quad \dots\dots\dots\dots(2).$$

We have, when $n>m$, $s\geqslant 2$,

$$\left|\left(\frac{z}{\alpha_{n+s}}\right)^{(n+s)^q}\right|\leqslant\left(\frac{\alpha_{n+1}}{\alpha_{n+2}}\right)^{(n+s)^q}\cdot\left(\frac{\alpha_{n+2}}{\alpha_{n+3}}\right)^{(n+s)^q}\dots\left(\frac{\alpha_{n+s-1}}{\alpha_{n+s}}\right)^{(n+s)^q}$$
$$<\left(\frac{\alpha_{n+1}}{\alpha_{n+2}}\right)^{(n+1)^q}\dots\left(\frac{\alpha_{n+s-1}}{\alpha_{n+s}}\right)^{(n+s-1)^q}$$
$$<e^{-\beta'(s-1)} \text{ from (2)} \quad \dots\dots\dots\dots(3).$$
$$<e^{-\beta'}.$$

Now when $|x|<e^{-\beta'}$, $\left|\frac{\log(1+x)}{x}\right|$ is finite, and less than a finite K'.

Hence
$$\left|\log\left[1+\left(\frac{z}{\alpha_{n+s}}\right)^{(n+s)^q}\right]\right|<K'.\left|\left(\frac{z}{\alpha_{n+s}}\right)^{(n+s)^q}\right|<K'.e^{-\beta'(s-1)},$$

and hence
$$|R|\leqslant\sum_{s=2}^{\infty}\left|\log\left[1+\left(\frac{z}{\alpha_{n+s}}\right)^{(n+s)^q}\right]\right|<K'\sum_{s=2}^{\infty}e^{-\beta'(s-1)}$$
$$<\frac{K'}{1-e^{-\beta'}} \quad \dots\dots\dots\dots(4).$$

Next consider the term S.

Assign a number θ between 0 and 1, both exclusive.

Let $u_s=\left|\log\left[1+\left(\frac{\alpha_s}{z}\right)^{s^q}\right]\right|$, $N=[\theta n]$ (i.e. the greatest integer contained in θn), and let

$$\sigma_1=\sum_{s=1}^{m}u_s,\quad \sigma_2=\sum_{s=m+1}^{N},\quad \sigma_2=\sum_{s=N+1}^{n-1}u_s.$$

Then $|S|\leqslant\sigma_1+\sigma_2+\sigma_3$.

* We shall show later that the above inequality is possible and that the region R' always exists.

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We shall consider σ_1, σ_2, σ_3, in order, and shall prove that they are all finite.

The finite sum σ_1 is evidently finite and less than K_1, suppose .....(5).

Consider σ_2.

When $m+1 \leqslant s \leqslant N$, we have, from (2),

$$\left(\frac{a_s}{a_n}\right)^{s^q} < \exp\left[-s^q.\beta'\left\{\frac{1}{s^q}+\frac{1}{(s+1)^q}+\dots+\frac{1}{(n-1)^q}\right\}\right]$$

$$< \exp\left[-s^q.\beta'\int_s^n \frac{dx}{x^q}\right] \quad\text{.....(6).}$$

If $q>1$, we have

$$s^q.\beta'.\int_s^n \frac{dx}{x^q} = \frac{\beta' s^q}{q-1}\left[s^{-(q-1)}-n^{-(q-1)}\right]$$

$$= \frac{\beta' s}{q-1}\left[1-\left(\frac{s}{n}\right)^{q-1}\right]$$

$$\geqslant \frac{\beta' s}{q-1}\left[1-\left(\frac{N}{n}\right)^{q-1}\right]$$

$$\geqslant \frac{\beta' s}{q-1}(1-\theta^{q-1})$$

$$> \lambda_1 s, \text{ where } \lambda_1 > 0 \quad\text{.....}(7_1).$$

If $q=1$,

$$s^q\beta'\int_s^n \frac{dx}{x^q} = \beta' s(\log n - \log s) \geqslant \beta' s.\log(n/N) \geqslant \beta' s \log(1/\theta)$$

$$> \lambda_2 s, \text{ where } \lambda_2 > 0 \quad\text{.....}(7_2).$$

When $q<1$,

$$s^q.\beta'\int_s^n \frac{dx}{x^q} = \frac{s^q\beta'}{1-q}.(n^{1-q}-s^{1-q})$$

$$= \frac{s\beta'}{1-q}\left[\left(\frac{n}{s}\right)^{1-q}-1\right]$$

$$\geqslant \frac{s\beta'}{1-q}\left[\left(\frac{n}{N}\right)^{1-q}-1\right]$$

$$\geqslant \frac{s\beta'}{1-q}\left[\left(\frac{1}{\theta}\right)^{1-q}-1\right]$$

$$> \lambda_3 s, \text{ where } \lambda_3 = 0 \quad\text{.....}(7_3).$$

Hence in any case,

$$s^q.\beta'\int_s^n \frac{dx}{x^q} > \lambda s, \text{ where } \lambda > 0 \quad\text{.....(7).}$$

Then from (6) and (7), we have

$$\left|\left(\frac{a_s}{z}\right)^{s^q}\right| \leqslant \left(\frac{a_s}{a_n}\right)^{s^q} < e^{-\lambda s} \quad\text{.....(8),}$$

$$< e^{-\lambda}.$$

Hence, as above, $\left|\log\left[1+\left(\frac{a_s}{z}\right)^{s^q}\right]\right| < K''.\left|\left(\frac{a_s}{z}\right)^{s^q}\right| < K''.e^{-s\lambda}.$

Therefore $$\sigma_2 < K''.\sum_{s=m+1}^{N} e^{-s\lambda} < \frac{K''}{1-e^{-\lambda}} \quad\text{.....(9).}$$

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Next consider σ_3.

When $N+1 \leqslant \theta n \leqslant s \leqslant n-1$,

$$\left(\frac{a_s}{r}\right)^{s^q} \leqslant \left(\frac{a_s}{r}\right)^{(\theta n)^q} < \left[\left\{\frac{a_s}{a_{s+1}} \cdot \frac{a_{s+1}}{a_{s+2}} \cdots \frac{a_{n-1}}{a_n}\right\}^{n^q}\right]^{\theta^q}$$

$$< \left[\left(\frac{a_s}{a_{s+1}}\right)^{s^q} \cdots \left(\frac{a_{n-1}}{a_n}\right)^{(n-1)^q}\right]^{\theta^q}$$

$$< [e^{-(n-s)\beta'}]^{\theta^q}$$

$$< e^{-\mu(n-s)}, \text{ where } \mu > 0$$

$$< e^{-\mu} \quad \text{.....(10).}$$

Hence
$$\left|\log\left[1+\left(\frac{a_s}{z}\right)^{s^q}\right]\right| < K''' \cdot \left|\left(\frac{a_s}{z}\right)^s\right| < K''' \cdot e^{-\mu(n-s)},$$

and
$$\sigma_3 = \sum_{s=1}^{n-N-1} \left|\log\left[1+\left(\frac{a_{n-s}}{z}\right)^{(n-s)^q}\right]\right|$$

$$< K''' \cdot \sum_{s=1}^{n-N-1} e^{-\mu s}$$

$$< \frac{K'''}{1-e^{-\mu}} \quad \text{.....(11).}$$

From (5), (9), and (10), $|S| \leqslant \sigma_1 + \sigma_2 + \sigma_3 < K' + \dfrac{K''}{1-e^{-\lambda}} + \dfrac{K'''}{1-e^{-\mu}}$ .....(12).

We have finally to consider T_1 and T_2.

Suppose q is an integer, so that R' is the part of the z region which is exterior to all the circles ${}_tA_s$.

The radius of ${}_tA_n$ is $\dfrac{ka_n}{n^q}$, z is exterior to all circles ${}_tA_n$, and we have further $|z| \geqslant a_n$.

Let P be the point z, and let the centres of the circles ${}_1A_n$, ${}_2A_n$... be $P_1P_2 \ldots P_{n^q}$. Let P' be the point in which OP meets the circle $|z|\,a_n$.

Then
$$|z^{n^q} + a_n^{n^q}| = PP_1 . PP_2 \ldots PP_{n^q}.$$

We assume, what we shall presently show, that the circles ${}_tA_n$ do not intersect. Then P' cannot lie inside more than one circle. If it lie inside any circle let it lie inside ${}_1A_n$, and if it do not lie in any circle, let P_1 be the centre which is nearest to P'. Let $\phi = P'\hat{O}P_1$.

We have
$$PP_2 . PP_3 \ldots PP_{n^q} \geqslant P'P_2 \ldots P'P_{n^q} \geqslant \left[\prod_{s=1}^{n^q} P'P_s\right] \Big/ P'P_1.$$

Now by Cotes' property of the circle

$$\left[\prod_{s=1}^{n^q} P'P_s\right] \Big/ a_n{}^{n^q} = 2 \sin\left(\tfrac{1}{2} n^q \phi\right),$$

and hence
$$\left[\prod_{s=1}^{n^q} P'P_s\right] \Big/ (P'P_1 a_n{}^{n^q-1}) = \frac{\sin(\frac{1}{2}n^q\phi)}{\sin \frac{1}{2}\phi}.$$

Now since P_1 is the centre nearest to P', $\phi < \dfrac{\pi}{n^q}$, and hence $\frac{1}{2}n^q\phi < \dfrac{\pi}{2}$.

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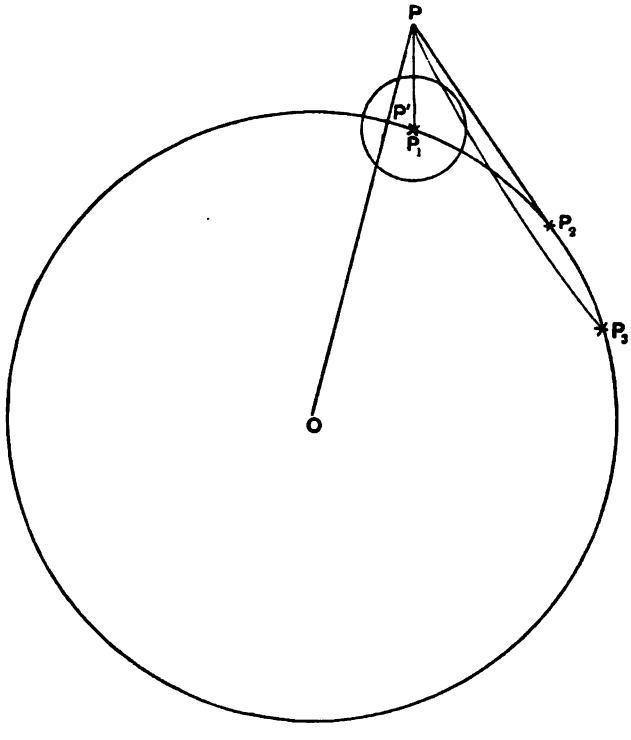
Hence
$$\frac{\sin(\frac{1}{2}n^q\phi)}{\sin\frac{1}{2}\phi} = n^q \cdot \frac{\sin(\frac{1}{2}n^q\phi)}{(\frac{1}{2}n^q\phi)} \cdot \frac{\frac{1}{2}\phi}{\sin\frac{1}{2}\phi}$$
$$> n^q \cdot \frac{1}{\pi} \cdot 1 > n^q/\pi.$$



Hence
$$\left[\prod_{s=1}^{n^q} PP_s\right] \Big/ a_n^{n^q} \geqslant \frac{PP_1}{a_n} \cdot \left[\prod_{s=1}^{n^q} P'P_s\right] \Big/ (a_n^{n^q-1} \cdot P'P_1)$$
$$> \frac{k}{n^q} \cdot \frac{n^q}{\pi} > \frac{k}{\pi},$$

or
$$\left|\frac{z^{n^q} + a_n^{n^q}}{a_n^{n^q}}\right| > \frac{k}{\pi}.$$

Hence, since $|z| \geqslant |a_n|$, $\left|1 + \left(\frac{a_n}{z}\right)^{n^q}\right| > \frac{k}{\pi}$ .....(13).

The result (13) follows more easily in the case when q is not an integer, when the region R' consists of the annuli $a_s\left(1 + \frac{k}{s^q}\right) < |z| < a_{s+1}\left(1 - \frac{k}{s^q}\right)$, and we shall omit the analysis for this case.

Again, we have clearly $\left|1 + \left(\frac{a_n}{z}\right)^{n^q}\right| < 2$ .....(14).

From (13) and (14) it follows that

$$|T_2| = \left|\log\left[1 + \left(\frac{a_n}{z}\right)^{n^q}\right]\right| < \text{a finite } K_4 \text{ .....(15).}$$

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In a similar manner we obtain

$$T_1 < K_5 \quad \text{.....(16).}$$

Now $\log F(z) = \bar{P} + \iota\gamma_z + R + S + T_1 + T_2$ where we may suppose

$$|\gamma_z| < \pi \quad \text{.....(17).}$$

From (4), (12), (15), (16), and (17), we have finally,

$$\log F(z) = \bar{P} + Q,$$

where $|Q| <$ a finite K.

The results (4) and (12) were obtained without supposing z restricted to the region R'. Then we see from (4), (12), and (17), that when z is restricted we have

$$\log F(z) = \bar{P} + \log\left[1 + \left(\frac{z}{a_{n+1}}\right)^{(n+1)^q}\right] + \log\left[1 + \left(\frac{a_n}{z}\right)^{n^q}\right] + Q',$$

where $|Q'| <$ a finite K'.

§ 17. We must show that the region R' always exists. If q is an integer, R' is the region exterior to certain circles ${}_tA_s$.

Now the distance between the centres of two consecutive circles ${}_tA_s$, ${}_{t+1}A_s$, is

$$2a_s \sin\left(\frac{\pi}{s^q}\right),$$

and the radius of each circle is $\frac{ka_s}{s^q}$. Since $k < 2\pi$ these circles do not intersect when s is sufficiently large.

Again, the sum $\sigma(s)$ of the radii of one circle ${}_tA_s$ and one circle ${}_{t'}A_{s+1}$, is

$$\frac{a_s k}{s^q} + \frac{a_{s+1} k}{(s+1)^q} < \frac{2k}{s^q} \cdot a_{s+1}.$$

Hence

$$\begin{aligned}
\frac{\sigma(s)}{a_{s+1} - a_s} &< 2k \cdot \frac{a_{s+1}}{s^q(a_{s+1} - a_s)} < \frac{2k}{s^q} \cdot \left(1 - \frac{a_s}{a_{s+1}}\right)^{-1} \\
&< \frac{2k}{s^q}\left(1 - e^{-\frac{\beta'}{s^q}}\right)^{-1}, \text{ when } s > m, \\
&< \frac{2k}{s^q} \cdot (1+\epsilon) \Big/ \left(\frac{\beta'}{s^q}\right), \text{ when } s \text{ is sufficiently large,} \\
&< \frac{2(1+\epsilon)k}{\beta'}.
\end{aligned}$$

Since $k < \frac{1}{2}\beta$, we can choose ϵ and β' so that $\frac{2(1+\epsilon)k}{\beta'} < 1$.

Then when s is sufficiently large $\frac{\sigma(s)}{a_{s+1} - a_s} < 1$, and hence the circles ${}_tA_s$ and ${}_{t'}A_{s+1}$ cannot intersect.

Hence the region R' exists when q is an integer. When q is not an integer it is again easily seen that R' exists.

In the case when $\operatorname{Lt}_{s=\infty} a_s = 1$, so that the range of z is the interior of the unit circle, the area of the region R' bears to that of the unit circle a ratio $(1 - K)$ where K

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depends on k, and where K may be made as small as we please by choosing k sufficiently small.

When $\underset{s=\infty}{\mathrm{Lt}}\, \alpha_s = \infty$, so that z ranges over the whole plane, if we choose a large circle of radius ρ, the ratio of the area of the part of R' contained in this circle, to the area of the whole circle, is $[1 - K(k, \rho)]$, where, by choosing k sufficiently small, we can ensure that $\underset{\rho=\infty}{\mathrm{Lt}}\, K(k, \rho) < \epsilon$, for any assigned ϵ.

§ 18. We shall now give some examples of the theorem of § 16. After the detailed work in §§ 9 and 11 we shall omit the analysis by which we calculate $\bar{P}$ as a function of r*, remarking only, that as we wish to obtain an approximation to $\log F(z)$ in which the error is finite, we may neglect finite terms which arise from $\bar{P}$.

I. As an example of the case $q < 1$, take the function

$$F(z) = \prod_{s=1}^{\infty}\left[1 + \left(\frac{z}{e^{\omega s^{1-q}}}\right)^{s^q}\right], \quad q < 1, \ \omega = \omega_1 + \iota\omega_2, \ \omega_1 > 0.$$

$F(z)$ is a multiform function and must be made uniform by a cross-cut.

The region R' is the sum of all the annuli,

$$e^{\omega_1 s^{1-q}}\left(1 + \frac{k}{s^q}\right) < |z| < e^{\omega_1 (s+1)^{1-q}} . \left(1 - \frac{k}{s^q}\right).$$

It is easily seen that the condition (1) of § 16 is satisfied, so that we may apply the theorem of that article.

When z is confined to R' we have

$$\log F(z) = \frac{1}{2} \cdot \frac{1-q}{1+q} \cdot \omega_1^{-\frac{1+q}{1-q}} . (\log r)^{\frac{2}{1-q}} + Q,$$

where
$$|Q| < K.$$

II. As an example of the case $q = 1$, take the function

$$F(z) = \prod_{s=1}^{\infty}\left[1 + \left(\frac{z}{s^\rho}\right)^s\right], \quad \rho = \rho_1 + \iota\rho_2, \ \rho_1 > 0,$$

which is an integral function of finite order $2/\rho_1$.

R' is the region exterior to all the circles

$$\left| z - s^\rho . e^{\frac{2t+1}{s}\pi\iota} \right| = k . s^{\rho_1 - 1}, \quad t = 0, 1 \ldots (s-1), \ s = 1, 2, 3 \ldots.$$

The necessary condition is easily seen to be satisfied.

When z is confined to R' we have

$$\log F(z) = \tfrac{1}{4}\rho_1 . r^{2/\rho_1} - \tfrac{1}{12}\log r + Q,$$

where
$$|Q| < K.$$

* The calculation in the examples given below depends on the asymptotic expansions of $\sum^n s^k$ and $\sum^n s^k \log s$, which are given in § 9. We determine in each case a function t of r, such that $n = t - \alpha$, $\alpha < 1$.

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III. The case $q>1$ is chiefly interesting when we have $\operatorname{Lt}_{s=\infty} a_s = 1$, for, as we have said in § 8, when $\operatorname{Lt}_{s=\infty} a_s = \infty$ and $q>1$, we can usually apply the general formula of § 3.

Consider the function

$$F(z) = \prod_{s=1}^{\infty} [1 + \{z \,.\, \exp(\omega s^{-(q-1)})\}^{s^q}],$$

where $q>1$, and ω is real and positive.

This is a uniform function whose Taylor series has a finite radius of convergence, when q is an integer $\geqslant 2$. When $q>1$ and is not integral, $F(z)$ is multiform.

The condition of § 16 is satisfied.

In this case the most convenient function of r, which tends to ∞ as r tends to 1, in terms of which we can express the large terms of $\bar{P}$, is $\dfrac{1}{\log(1/r)}$. We find that $n = t - \alpha$, where

$$t = \left[\frac{\omega}{\log(1/r)}\right]^{\frac{1}{q-1}}.$$

Calculating $\bar{P}$ in terms of $\dfrac{1}{\log(1/r)}$, and neglecting finite terms we obtain

$$\log F(z) = \frac{1}{2} \cdot \frac{q-1}{q+1} \cdot \omega^{\frac{q+1}{q-1}} \cdot \left[\frac{1}{\log(1/r)}\right]^{\frac{2}{q-1}} + Q,$$

where

$$|Q| < K.$$

IV. Consider $F(z) = \prod_{s=1}^{\infty} \left[1 + \left\{\exp\left(\frac{\omega}{s}\right) . z\right\}^{s^q}\right]$, $q>2$, ω positive.

The condition of § 16 is satisfied.

We obtain, when z is confined to R',

$$\log F(z) = \frac{\omega^{q+1}}{q(q+1)} \left[\frac{1}{\log(1/r)}\right]^q + \tfrac{1}{2}\alpha(1-\alpha) . \omega^{q-1} . \left[\frac{1}{\log(1/r)}\right]^{q-2} + \text{terms of lower order,}$$

where* $\alpha =$ the fractional part of $\dfrac{\omega}{\log(1/r)}$.

§ 19. In the class of functions for which the theorem of § 16 holds, and for which the general formula of § 3 does not hold, there are two types of functions for which we can obtain the finite term, as well as the large terms, of $\log F(z)$.

The first case occurs when $\operatorname{Lt}_{s=\infty} a_s = \infty$, $q \leqslant 1$, and when $\left(\dfrac{a_s}{a_{s+1}}\right)^{s^q}$ tends to a definite non-zero limit, whose modulus is less than unity, when s tends to infinity. The finite terms of $\log F(z)$ can then usually be expressed by the use of the function

$$\phi(x, \omega) = \prod_{s=1}^{\infty}\left(1 + \frac{x}{e^{s\omega}}\right) . \prod_{s=0}^{\infty}\left(1 + \frac{1}{xe^{s\omega}}\right).$$

* In the last article of this paper we shall consider the reason why terms like α appear in Case IV. and not in Cases I. to III.

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The general case is somewhat complex, and we shall content ourselves with finding the finite term of

$$\log \prod_{s=1}^{\infty}\left[1+\left(\frac{z}{s^{\rho}}\right)^{s}\right],$$

whose large terms are given in the last article, in the case when ρ and z are real and positive.

We suppose z confined to the region R'.

In the first place we must retain the finite terms arising from $\bar{P}$.

We find $$\bar{P}=\frac{\rho}{4}\left[z^{2/\rho}-\frac{1}{3\rho}\log z-\tfrac{1}{3}+4\zeta'(-1)+2\alpha-\alpha^2\right]+\epsilon(z),$$

where α is the fractional part of $z^{1/\rho}$, and where $\operatorname*{Lt}_{|z|=\infty}|\epsilon(z)|=0$.

We have $n=z^{1/\rho}-\alpha$, $z=(n+\alpha)^{\rho}$.

We must consider the terms T_1+R and T_2+S.

We have $$T_1+R=\sum_{s=1}^{\infty}\left[1+\left(\frac{z^{1/\rho}}{n+s}\right)^{\rho(n+s)}\right]=\sum_{s=1}^{\infty}\left[1+\left(\frac{n+\alpha}{n+s}\right)^{\rho(n+s)}\right] \quad\text{.....(1).}$$

We proceed to show that

$$\left|(T_1+R)-\sum_{s=1}^{\infty}\log\left(1+e^{\rho(\alpha-s)}\right)\right|$$

tends to zero when $|z|$ tends to infinity.

Let $$u_s=\log\left[1+\left(\frac{n+\alpha}{n+s}\right)^{\rho(n+s)}\right]-\log\left[1+e^{\rho(\alpha-s)}\right].$$

Choose a small positive η.

When $s<\eta\,.\,n^{\frac{1}{2}}$,

$$\left(\frac{n+\alpha}{n+s}\right)^{\rho(n+s)}=\exp\left[\rho(n+s)\log\left(1-\frac{s-\alpha}{n+s}\right)\right]$$
$$=\exp\left[\rho(\alpha-s)+\kappa\right],$$

where $$|\kappa|=\left|\rho\left[\frac{(s-\alpha)^2}{2(n+s)}+\frac{(s-\alpha)^3}{3(n+1)^2}+\dots\right]\right|$$
$$<\rho(\eta^2+\eta^3+\dots)$$
$$<2\rho\eta^2,\ \text{if}\ \eta<\tfrac{1}{2}.$$

Hence $$|u_s|=\left|\log\left(1+e^{\rho(\alpha-s)+\kappa}\right)-\log\left(1+e^{\rho(\alpha-s)}\right)\right|,$$

and we deduce that $$|u_s|<(e^{\kappa}-1)\left[-\log\left(1-e^{\rho(\alpha-s)}\right)\right]$$
$$<B\,.\,\kappa\,.\,e^{\rho(\alpha-s)},\ \text{where } B \text{ is finite.}$$

Hence $$\sum_{s=1}^{[\eta n^{\frac{1}{2}}]}|u_s|<\frac{e^{\rho\alpha}}{e^{\rho\alpha}-1}\,.\,B\,.\,\kappa<A\eta^2,\ \text{where } A \text{ is finite.}$$

Choose η so that $A\eta^2<\epsilon/3$, where ϵ is arbitrarily small.

Then $$\sum_{s=1}^{[\eta n^{\frac{1}{2}}]}|u_s|<\epsilon/3 \quad\text{.....(2).}$$

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Again, when $s > \eta . n^{\frac{1}{2}}$,

$$\left(\frac{n+\alpha}{n+s}\right)^{\rho(n+s)} = \left[1+\frac{s-\alpha}{n+\alpha}\right]^{-\rho(n+s)}$$

$$< \left[1+\frac{\eta n^{\frac{1}{2}}-\alpha}{n+\alpha}\right]^{-\rho n} < \left[1+\frac{\frac{1}{2}\eta n^{\frac{1}{2}}}{2n}\right]^{-\rho n}, \text{ when } n > \left(\frac{\alpha}{\eta}\right)^2,$$

$$< \exp(-\rho n . \tfrac{1}{8} n^{-\frac{1}{2}})^* \text{ (when } \tfrac{1}{4}\eta n^{-\frac{1}{2}} < \tfrac{1}{2})$$

$$< \exp(-\tfrac{1}{8}\eta . n^{\frac{1}{2}}).$$

Then when $n + 2\alpha > s > \eta n^{\frac{1}{2}}$,

$$|u_s| \leqslant \left|\log\left[1+\left(\frac{n+\alpha}{n+s}\right)^{\rho(n+s)}\right] + |\log(1+e^{\rho(\alpha-s)})\right|$$

$$< 2\exp(-\tfrac{1}{4}\eta n^{\frac{1}{2}}) + 2\exp[\rho(\alpha-\eta n^{\frac{1}{2}})],^*$$

and
$$\sum_{s=[\eta n^{\frac{1}{2}}]+1}^{[n+2\alpha]} |u_s| < (n+2\alpha)\{2\exp(-\tfrac{1}{4}\eta n^{\frac{1}{2}}) + 2\exp[\rho(\alpha-\eta n^{\frac{1}{2}})]\}$$

$$< \frac{\epsilon}{3}, \text{ when } n \text{ is sufficiently large} \qquad (3).$$

Lastly, when $s > n + 2\alpha$,

$$\left(\frac{n+\alpha}{n+s}\right)^{\rho(n+s)} < (\tfrac{1}{2})^{\rho(n+s)},$$

and
$$\sum_{s=[n+2\alpha]+1}^{\infty} |u_s| \leqslant \sum_{s=[n+2\alpha]+1}^{\infty} \left\{\left|\log\left[1+\left(\frac{n+\alpha}{n+s}\right)^{\rho(n+s)}\right]\right| + |\log(1+e^{\rho(\alpha-s)})|\right\}$$

$$< 2\sum_{s=n}^{\infty}\left[\frac{1}{2\rho^{n+s}} + e^{\rho\alpha-\rho s}\right]$$

$$< A' . 2^{-\rho n} + B' . e^{-\rho n}$$

$$< \frac{\epsilon}{3}, \text{ when } n \text{ is sufficiently large} \qquad (4).$$

It follows from the relations (1) to (4) that

$$\left|(R+T_1) - \sum_{s=1}^{\infty} \log[1+e^{\rho(\alpha-s)}]\right| < \epsilon,$$

when n, or r, is sufficiently large.

In a somewhat similar manner we can show that

$$\left|(S+T_2) - \sum_{s=0}^{\infty} \log[1+e^{-\rho(\alpha+s)}]\right| < \epsilon,$$

when n is sufficiently large.

Hence
$$|(R+S+T_1+T_2) - \log\phi(e^{-\rho\alpha}, \rho)|$$

tends to zero as r tends to infinity.

We therefore have

$$\log F(z) = \frac{\rho}{4}\left[z^{2/\rho} - \frac{1}{3\rho}\log z + 4\zeta'(-1) + 2\alpha - \alpha^2\right] + \log\phi(e^{-\rho\alpha}, \rho)$$
$$+ \text{terms tending to zero with } 1/z$$

* When $|x| < \frac{1}{2}$, we have $|\log(1+x)| < 2|x|$.

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When z and ρ are complex we can still find an expression for the finite part of $\log F(z)$, but it contains additional complicated imaginary terms, and the $e^{-\rho a}$ which occurs in $\phi(e^{-\rho a}, \rho)$ is replaced by a more complicated expression.

§ 20. The second case occurs when

$$\operatorname*{Lt}_{s=\infty} s^q (\log a_{s+1} - \log a_s) = \infty \quad \text{.....(1).}$$

When $q > 1$, and $\operatorname*{Lt}_{s=\infty} a_s = \infty$, we have seen in § 8 that the complete asymptotic expansion of § 3 usually holds.

When (1) is satisfied, $q \leqslant 1$, and $\operatorname*{Lt}_{s=\infty} a_s = \infty$, we can prove the following formula, when z is restricted to lie in R':

$$\log F(z) = P + \epsilon(z), \text{ where } \operatorname*{Lt}_{r=\infty} |\epsilon(z)| = 0.$$

When (1) is satisfied, and $\operatorname*{Lt}_{s=\infty} a_s = 1$, we can prove the following formula, when z is restricted to lie in R':

$$\log F(z) = P + \log \prod_{s=1}^{\infty} [1 + a_s^{s^q} . e^{-i s^q \theta}] + \epsilon(z),$$

where

$$\operatorname*{Lt}_{r=1} |\epsilon(z)| = 0.$$

We shall omit the proofs of these formulae, merely remarking that in the first case we prove that $|R + T_1|$ and $|S + T_2|$ tend to zero with $1/r$, and in the second case that $|R + T_1|$ and $\left|(S + T_2) - \sum_{s=1}^{\infty} \log [1 + a_s^{s^q} . e^{-i s^q \theta}]\right|$ tend to zero as r tends to 1.

§ 21. When $\operatorname*{Lt}_{s=\infty} a_s = 1$, and a_s increases slowly to its limit 1, it may happen that

$$\Pi \left[1 + \left(\frac{z}{a_s}\right)^{s^q}\right]$$

does not converge for any value of q. We are thus led to consider functions defined by a product form of the type

$$F(z) = \prod_{s=1}^{\infty} \left[1 + \left(\frac{z}{a_s}\right)^{f(s)}\right],$$

where $\operatorname*{Lt}_{s=\infty} f(s) . s^{-q} = \infty$ for all values of q, and $\operatorname*{Lt}_{s=\infty} a_s = 1$.

For functions of this type the following theorem applies.

If

$$\operatorname*{Lt}_{s=\infty} \frac{f(s)}{s} (\log a_{s+1} - \log a_s) \geqslant \beta > 0 \quad \text{.....(1),}$$

then, when z is restricted to lie in the region R, defined (as in § 3) as the sum of all the annuli

$$a_{s+1}^k . a_s^{1-k} < |z| < a_{s+1}^{1-k} . a_s^k, \ (k < \tfrac{1}{2}),$$

we have

$$\log F(z) = P + T + \epsilon(z),$$

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where P has its usual meaning,

$$T = \sum_{s=1}^{\infty} \log\left[1 + a_s^{f(s)} . e^{-f(s) . i\theta}\right],$$

and

$$\operatorname*{Lt}_{r=1} |\epsilon(z)| = 0.$$

Choose $\beta' < \beta$. Then when s is greater than a finite σ, we have, from the condition (1),

$$\left(\frac{\alpha_s}{\alpha_{s+1}}\right)^{f(s)} < e^{-\beta' s}.$$

Also

$$\frac{r}{\alpha_{n+s}} \leqslant \frac{\alpha_{n+1}^{1-k}\,\alpha_n^{\,k}}{\alpha_{n+s}} < \left(\frac{\alpha_n}{\alpha_{n+s}}\right)^k < \left(\frac{\alpha_{n+s-1}}{\alpha_{n+s}}\right)^k.$$

Hence

$$\left(\frac{r}{\alpha_{n+s}}\right)^{f(n+s)} < \exp\left[-k\beta'(n+s)\right],$$

whence it follows that

$$|R + T_1| \leqslant \sum_{s=1}^{\infty} \left| \log\left[1 + \left(\frac{z}{\alpha_{n+s}}\right)^{f(n+s)}\right]\right|$$

$$< 2\sum_{s=1}^{\infty} \exp\left[-k\beta'(n+s)\right], \text{ when } n \text{ is sufficiently large,}$$

$$< K . e^{-k\beta' n} \qquad \text{.....(2).}$$

Next consider the expression $S + T_2$.

Let ν be a large integer independent of n, which we shall presently choose, and which we suppose for the present to be greater than σ.

Then, when $n \geqslant s > \nu$, we have

$$\left(\frac{\alpha_s}{r}\right)^{f(s)} \leqslant \left(\frac{\alpha_s}{\alpha_{n+1}^{\,k}\,\alpha_n^{\,1-k}}\right)^{f(s)} < \left(\frac{\alpha_s^{\,k}}{\alpha_{s+1}^{\,k}}\right)^{f(s)} < e^{-k\beta' s}, \text{ from (1)} \qquad \text{.....(3),}$$

and

$$a_s^{f(s)} < \left(\frac{\alpha_s}{r}\right)^{f(s)} < e^{-k\beta' s} \qquad \text{.....(4).}$$

From (3) it follows that

$$\sum_{s=\nu+1}^{n} \left| \log\left[1 + \left(\frac{\alpha_s}{z}\right)^{f(s)}\right]\right| < K' . e^{-k\beta'\nu} \qquad \text{.....(5),}$$

and from (4) it follows that

$$\sum_{s=\nu+1}^{\infty} \left| \log\left[1 + a_s^{f(s)} . e^{-i\theta f(s)}\right]\right| < K'' . e^{-k\beta'\nu} \qquad \text{.....(6).}$$

Again we clearly have

$$\left| \sum_{s=1}^{\nu} \log\left[1 + \left(\frac{\alpha_s}{z}\right)^{f(s)}\right] - \sum_{s=1}^{\nu} \log\left[1 + a_s^{f(s)} . e^{-i\theta f(s)}\right]\right| < \chi(\nu)(1 - |z|) \qquad \text{.....(7),}$$

where $\chi(\nu)$ is a constant depending on ν.

Now if any positive ϵ be assigned, we choose ν so large that

$$K' . e^{-k\beta'\nu} < \epsilon/4 \text{ and } K'' . e^{-k\beta'\nu} < \epsilon/4.$$

We then choose n so large that $K . e^{-k\beta' n} < \epsilon/4$, and $\chi(\nu)(1 - |z|) < \epsilon/4$.

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It then follows from (2), (5), (6), and (7), that

$$|(R+T_1)+(S+T_2)-\sum_{s=1}^{\infty}\log[1+a_s^{f(s)}.e^{-\iota\theta f(s)}]|$$

$$<|R+T_1|+\left|\sum_{s=\nu+1}^{n}\log\left[1+\left(\frac{a_s}{z}\right)^{f(s)}\right]\right|+\left|\sum_{s=\nu+1}^{\infty}\log[1+a_s^{f(s)}e^{-\iota\theta f(s)}]\right|$$

$$+\left|\left\{\sum_{s=1}^{\nu}\log\left[1+\left(\frac{a_s}{z}\right)^{f(s)}\right]-\sum_{s=1}^{\nu}\log[1+a_s^{f(s)}e^{-\iota\theta f(s)}]\right\}\right|$$

$$<K.e^{-k\beta' n}+K'.e^{-k\beta'\nu}+K''.e^{-k\beta'\nu}+\chi(\nu)(1-|z|)$$

$$<\epsilon.$$

Hence the left-hand side tends to zero as r tends to 1, and we have the formula of the theorem.

We notice that $|R+T_1|$ tends to zero as r tends to unity, as is seen at once from (2).

As an example consider $F(z)=\prod_{s=1}^{\infty}[1+\{z.\exp(e^{-as})\}^{se^{as}}]$, which is an uniform function if e^a is an integer.

When z is restricted to the region R we have

$$\log F(z)=\frac{1}{2a^2}.\left[\log\left(\frac{1}{\log\frac{1}{r}}\right)\right]^2+\frac{1}{2a}\left(1-2\alpha-\frac{2e^{-a\alpha}}{1-e^{-a}}\right)\log\left(\frac{1}{\log\frac{1}{r}}\right)$$

$$+\left\{\begin{array}{c}\frac{1}{2}\alpha(1-\alpha)+\frac{\alpha e^{-a\alpha}}{1-e^{-a}}+\frac{e^{-a(1+\alpha)}}{(1-e^{-a})^2}+\iota\theta\left[\frac{(t-\alpha)}{1-e^{-a}}.e^{a(t-\alpha)}-\frac{e^{a(t-1-\alpha)}}{(1-e^{-a})^2}+\frac{e^{-a}}{(1-e^{-a})^2}\right]\\ +T\end{array}\right\}$$

$$+\epsilon(z),$$

where $t=\frac{1}{a}\log\left(\frac{1}{\log\frac{1}{r}}\right)$, $\theta=\arg(z)$, α is the fractional part of t, $T=\sum_{s=1}^{\infty}\log[1+e^{-s}.e^{-\iota\theta s e^{as}}]$, and where $\operatorname{Lt}_{r=1}|\epsilon(z)|=0$.

The terms in crooked brackets count as finite terms.

§ 22. In the case of all the types of functions we have so far considered, we have always had the inequality $|R+T_1+S+T_2|<K$, so that the expression $\bar{P}$ represents $\log F(z)$ with an error of an order at most finite. It may happen in the case of certain functions defined by product forms of the type we are considering, that $|Q|=|R+T_1+S+T_2|$ tends to infinity as r approaches its limit ∞ or 1, but is of less order than $\bar{P}$. If we then can calculate $\bar{P}$ as a function of r, with an error which is of the order of $|Q|$, obtaining $P_1(r)$ say, then we have $\log F(z)=P_1(r)+$ terms of the order of $|Q|$, and $P_1(r)$ gives an approximation for $\log F(z)$. We shall not develop a theory of this kind in detail. Any particular case can usually be worked out quite easily, and we shall content ourselves with a single example.

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Consider the integral function

$$F(z) = \prod_{s=1}^{\infty} \left[1 + \left(\frac{z}{s^{\rho}}\right)^{[s^{1/2}]}\right],$$

where $\rho = \rho_1 + \iota\rho_2$, $\rho_1 > 0$, and $[x]$ is the greatest integer contained in x.

For the present we suppose z unrestricted.

The integer n is determined by $n+1 \geqslant r^{1/\rho_1} \geqslant n$, so that $n = r^{1/\rho_1} - \alpha$, where $0 \leqslant \alpha \leqslant 1$. Let ν be the integer such that $(\nu+1)^2 \geqslant n+1 > n \geqslant \nu^2$. Then $\nu+1 > t > \nu$, where $t = r^{1/2\rho_1}$, and $\nu = t - \beta$, $0 \leqslant \beta \leqslant 1$.

Consider the expression R.

We have
$$|R| \leqslant \sum_{s=2}^{\infty} u_s,$$

where
$$|u_s| = \left|\log\left[1 - \left|\left(\frac{z}{(n+s)^{\rho}}\right)^{[(n+s)^{1/2}]}\right|\right]\right| = \left|\log\left[1 - \left(\frac{n+\alpha}{n+s}\right)^{\rho_1[(n+s)^{1/2}]}\right]\right|.$$

Now when $2 \leqslant s \leqslant n$, we have

$$\left(\frac{n+\alpha}{n+s}\right)^{\rho_1[(n+s)^{1/2}]} < \left(\frac{n+1}{n+s}\right)^{\rho_1(n^{1/2}-1)} < \left(\frac{n+1}{n+s}\right)^{\frac{1}{2}\rho_1 n^{1/2}} < \exp\left[\tfrac{1}{2}\rho_1 n^{1/2} . \left\{\log\left(1 - \frac{s-1}{n+s}\right)\right\}\right]$$
$$< \exp\left[-\tfrac{1}{2}\rho_1 n^{1/2} . \frac{s-1}{n+s}\right]$$
$$< \exp\left[-K . n^{-1/2} . s\right] \text{ (since } s-1 > \tfrac{1}{2}s,\ n+s < 2n\text{), where } K \text{ is finite.}$$

It follows easily that

$$u_s < |\log(K_1 n^{-1/2} . s)| . \exp[-Kn^{-1/2} . s], \text{ where } 2 \leqslant s \leqslant [n^{1/2}] \quad \text{.....(1)},$$

and that
$$u_s < K_2 \exp[-Kn^{-1/2} . s], \text{ where } n \geqslant s > [n^{1/2}] \quad \text{.....(2)}.$$

From (1) it follows that

$$\sum_{s=2}^{[n^{1/2}]} u_s < n^{1/2} . \log K_1 + |\log[\Gamma\{[n^{1/2}]\}/\{[n^{1/2}]\}^{n^{1/2}}]|$$
$$< K_3 . n^{1/2} \text{ (as we see by the help of Stirling's theorem)} \quad \text{.....(3)}.$$

From (2) we have

$$\sum_{s=[n^{1/2}]+1}^{n} u_s < \frac{K_4}{1 - \exp(-Kn^{1/2})} < K_5 . n^{1/2} \quad \text{.....(4)}.$$

Finally when $s > n$,

$$\left(\frac{n+\alpha}{n+s}\right)^{[(n+s)^{1/2}]} < \left(\frac{1}{2 - \frac{2}{n}}\right)^{s^{1/2}/2^{1/2}}$$
$$< e^{-\lambda s^{1/2}}, \text{ where } \lambda \text{ is a positive non-zero constant,}$$
$$< \tfrac{1}{2}, \text{ where } n \text{ is large.}$$

Hence
$$u_s < 2 . e^{-\lambda s^{1/2}},$$

and
$$\sum_{s=n+1}^{\infty} u_s < 2 . \sum_{s=n+1}^{\infty} e^{-\lambda s^{1/2}}$$
$$< 2\int_0^{\infty} e^{-\lambda x^{1/2}}\, dx$$
$$< \text{a finite constant} \quad \text{.....(5)}.$$

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From (3), (4), and (5) it follows that

$$|R| < A' n^{1/2} < A\nu \quad \text{.....(6)},$$

where A is a constant.

In a similar way we can prove that

$$|S| < B\nu \quad \text{.....(7)}.$$

We now proceed to determine the region $\mathfrak{R}$, in which z is to be restricted to lie, in such a manner that $|T_1|$ and $|T_2|$ are less than $C.\nu$ for some finite C. In this manner we obtain the most extensive region possible, within which our approximation is valid.

The function $F(z)$ has $[s^{1/2}]$ zeros on the circle $|z| = s^{\rho_1}$. They are

$${}_t b_s = s^{\rho} . e^{\frac{2t+1}{[s^{1/2}]} . \pi i}, \quad t = 1, \ 2 \ldots [s^{1/2}].$$

Choose a positive number k, which may be as small as we please, and let ${}_t A_s$ be the circle with centre ${}_t b_s$ and radius $k . s^{\rho_1 - 1/2}$. It is easily seen that when s is greater than some finite value, all the circles ${}_t A_s$ are external to each other, and that when k is chosen sufficiently small, all the circles, for all values of s, are mutually external. We define the region $\mathfrak{R}$ to be the part of the z plane exterior to all the circles ${}_t A_s$. Then when z is restricted to lie in $\mathfrak{R}$, it can be shown, by considerations similar to those employed in § 16, that $|T_1 + T_2| < C.\nu$. Then if $\iota\gamma_z$ be the imaginary part of P, and $|\gamma_z|$ is supposed less than π, we have $\log F(z) = \bar{P} + Q$, where $|Q| < H.\nu$, where H is some constant. It remains to calculate $\bar{P}$ as a function of r, with an error of order not greater than that of ν.

We have
$$\bar{P} = \sum_{s=1}^{n} [s^{1/2}] . (\log r - \rho_1 \log s).$$

It is easily shown that the $\sum\limits_{s=1}^{n}$ may be replaced by $\sum\limits_{s=1}^{\nu^2}$, when we neglect terms of order ν.

We find
$$\sum_{s=1}^{\nu^2} [s^{1/2}] = \tfrac{2}{3}\nu^3 + \tfrac{3}{2}\nu^2 + (\text{a term of order } \nu) \quad \text{.....(8)},$$

and
$$\sum_{s=1}^{\nu^2} [s^{1/2}] . \log s = \sum_{m=1}^{\nu} m [\log m^2 + \log (m^2+1) + \ldots + \log \{(m+1)^2 - 1\}]$$
$$= \sum_{m=1}^{\nu} m \log [\Gamma \{(m+1)^2\} / \Gamma (m^2)]$$

which reduces, by the help of Stirling's theorem, to

$$\tfrac{4}{3}\nu^3 \log \nu - \tfrac{4}{9}\nu^3 + 3\nu^2 \log \nu + \tfrac{1}{2}\nu^2 + (\text{a term of order } \nu) \quad \text{.....(9)}.$$

Substituting from (8) and (9) in the expression for $\bar{P}$, then substituting $\nu = t - \beta$, expanding the terms in ν in descending powers of t, and arranging, we finally obtain

$$\bar{P} = \tfrac{4}{9}\rho_1 r^{3/2\rho_1} - \tfrac{1}{2}\rho r^{1/\rho_1} + M . r^{1/2\rho_1}$$

where $|M|$ is finite.

Hence when z is confined to the region $\mathfrak{R}$ defined above, we have

$$\log F(z) = \tfrac{4}{9}\rho_1 . r^{3/2\rho_1} - \tfrac{1}{2}\rho_1 . r^{1/\rho_1} + N . r^{1/2\rho_1},$$

where $|N|$ is finite.

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§ 23. We have so far only considered functions for which the index $f(s)$ tends to infinity with s. Our methods may also be applied to approximate to $\log F(z)$ in the cases when $f(s)$ is a constant, which may, without loss of generality, be taken to be unity, and when $f(s)$ tends to zero with $1/s$.

I have shown in another paper* that when $f(s)=1$, and when $a_s = s^{\phi(s)}$, where $\operatorname{Lt}_{s=\infty} \phi(s) = \infty$, and $\phi(s)$ is a non-decreasing function of s, the expression $\bar{P}$ is of order not less than $n\phi(n)$, and $|R+T_1+S+T_2|$ is of order not greater than the greater of K and $\dfrac{Kn}{\phi(n)}$. Thus $\bar{P}$ is an approximation to $\log F(z)$ with an error of order K or $\dfrac{Kn}{\phi(n)}$. This theorem applies to all integral functions of zero order, the modulus of whose sth zero increases steadily with s.

We shall then not consider the case $f(s)=1$, but shall consider the case when $f(s)$ tends to zero with $1/s$.

We rewrite the form $\prod_{s=1}^{\infty}\left[1+\left(\dfrac{z}{a_s}\right)^{f(s)}\right]$ as $\prod_{s=1}^{\infty}\left[1+\dfrac{z^{f(s)}}{b_s}\right]$,

where $b_s = a_s^{f(s)}$. The function $F(z)$ is clearly multiform, and must be made uniform by a cross-cut from the origin.

Theorem.

Let
$$F(z) = \prod_{s=1}^{\infty}\left[1+\left(\frac{z}{a_s}\right)^{f(s)}\right] = \prod_{s=1}^{\infty}\left[1+\frac{z^{f(s)}}{b_s}\right],$$

where $|b_s| = \beta_s$, and where $\beta_s^{\frac{1}{f(s)}} = \alpha_s$ is an increasing function of s.

Suppose the following conditions are satisfied:

(α) $\operatorname{Lt}_{s=\infty} \dfrac{1}{f(s)} = \infty$.

(β) $\dfrac{1}{f(s)}$ is a non-decreasing function of s after a finite value σ of s.

(γ) $\operatorname{Lt}_{s=\infty} \dfrac{\beta_s}{\beta_{s+1}} \leqslant \lambda < 1$.

Then when z is confined to a certain region R'', we have the formula
$$\log F(z) = \bar{P} + Q,$$
where
$$\bar{P} = [\sum_{s=1}^{n} f(s)] \,.\, \log r - \sum_{s=1}^{n} \log \beta_s,$$
n is the integer such that
$$\alpha_{n+1} \geqslant |z| \geqslant \alpha_n,$$
and where $|Q| <$ a finite K.

We clearly have, from (γ), $\operatorname{Lt}_{s=\infty} \beta_s = \infty$, and hence $\operatorname{Lt}_{s=\infty} \alpha_s = \infty$, since $f(s)$ tends to zero with $1/s$. Hence the whole range of z is the whole plane.

* *loc. cit.* § 6.

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Choose $\lambda' < \lambda$. Then when s is greater than a finite m, which we may suppose greater than σ, we have $\dfrac{\beta_s}{\beta_{s+1}} < \lambda'$.

Then when $s \geqslant 2$,

$$\left|\frac{z^{f(n+s)}}{b_{n+s}}\right| = \frac{r^{f(n+s)}}{\beta_{n+s}}$$

$$< \frac{r^{f(n+1)}}{\beta_{n+s}}, \text{ from condition } (\beta),$$

$$< \frac{r^{f(n+1)}}{\beta_{n+1}} \cdot \frac{\beta_{n+1}}{\beta_{n+2}} \cdots \frac{\beta_{n+s-1}}{\beta_{n+s}}$$

$$< \lambda^{s-1} \quad \text{.....(1)},$$

$$< \lambda \quad \text{.....(2)}.$$

Hence
$$\left|\log\left[1 + \frac{z^{f(n+s)}}{b_{n+s}}\right]\right| < L \cdot \left|\frac{z^{f(n+s)}}{b_{n+s}}\right|, \text{ from (1)},$$

$$< L \cdot \lambda^{s-1}, \text{ from (2)}.$$

Consequently
$$|R| \leqslant \sum_{s=2}^{\infty} \left|\log\left[1 + \frac{z^{f(n+s)}}{b_{n+s}}\right]\right|$$

$$\leqslant L \sum_{s=2}^{\infty} \lambda^{s-1}$$

$$< \text{a finite } K_1 \quad \text{.....(3)}.$$

Again, when $m < s \leqslant n-1$,

$$\left|\frac{b_s}{z^{f(s)}}\right| = \frac{\beta_s}{r^{f(s)}}$$

$$< \frac{\beta_s}{r^{f(n)}}, \text{ from } (\beta),$$

$$< \frac{\beta_s}{\beta_{s+1}} \cdots \frac{\beta_{n-1}}{\beta_n} \cdot \frac{\beta_n}{r^{f(n)}}$$

$$< \lambda^{n-s};$$

whence, by reasoning similar to the above,

$$\sum_{s=m+1}^{n-1} \left|\log\left[1 + \frac{b_s}{r^{f(s)}}\right]\right| < \text{a finite } K_2.$$

Also when $n > m$,

$$\sum_{s=1}^{m} \left|\log\left[1 + \frac{b_s}{r^{f(s)}}\right]\right| \text{ is finite and } < K_3.$$

Hence
$$|S| \leqslant \sum_{s=1}^{m} \left|\log\left[1 + \frac{b_s}{r^{f(s)}}\right]\right| + \sum_{s=m+1}^{n-1} \left|\log\left[1 + \frac{b_s}{r^{f(s)}}\right]\right|$$

$$< \text{a finite } K_4 \quad \text{.....(4)}.$$

We now define the region R'' as follows. We choose a positive μ, which may be as near to 1 as we please, but which must in any case be greater than $\lambda^{\frac{1}{2}}$.

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Then R'' is defined as the sum of all the annuli

$$\mu^{\frac{1}{f(s+1)}} . \alpha_{s+1} \geqslant |z| \geqslant \mu^{-\frac{1}{f(s)}} . \alpha_s \quad \text{.....(5).}$$

We have

$$\operatorname*{Lt}_{s=\infty} \left[\frac{\mu^{\frac{1}{f(s+1)}} \alpha_{s+1}}{\mu^{-\frac{1}{f(s)}} \alpha_s} \right]^{f(s+1)} = \operatorname*{Lt}_{s=\infty} \left[\mu^{1+\frac{f(s+1)}{f(s)}} . \frac{\alpha_{s+1}^{f(s+1)}}{\alpha_s^{f(s)}} . \alpha_s^{f(s)-f(s+1)} \right]$$

$$\geqslant \operatorname*{Lt}_{s=\infty} \left[\mu^2 . \frac{\beta_{s+1}}{\beta_s} \right], \text{ since } f(s+1) \geqslant f(s),$$

$$\geqslant (\mu^2 \lambda)$$

$$> 1.$$

It follows that

$$\operatorname*{Lt}_{s=\infty} \frac{\mu^{\frac{1}{f(s+1)}} \alpha_{s+1}}{\mu^{-1/f(s)} . \alpha_s} = \infty$$

and therefore that the inequalities (5) are possible, so that R'' exists.

Now when z is confined to R'',

$$\left| \frac{z^{f(n+1)}}{b_{n+1}} \right| \text{ and } \left| \frac{b_n}{z^{f(n)}} \right|$$

are less than a positive constant which is less than 1.

If follows at once that $|T_1|$ and $|T_2|$ are finite.....(6).

Now we have

$$\log F(z) = \bar{P} + \iota\gamma_z + R + T_1 + S + T_2, \text{ where } |\gamma_z| \leqslant \pi \quad \text{.....(7).}$$

From (3), (4), (6), and (7), we have

$$\log F(z) = \bar{P} + Q, \text{ where } |Q| < K.$$

It is possible to show by the methods of § 6 of my paper cited above, that if $\beta_s = s^{\phi(s)}$, where $\phi(s)$ is a non-decreasing function of s, and $\operatorname*{Lt}_{s=\infty} \phi(s) = \infty$, and where $f(s)$ satisfies the conditions (α), (β) above, that the expression $\bar{P}$ is of higher order in n or r than $|R + T_1 + S + T_2|$, when z is confined to a suitable region.

Hence in this case $\bar{P}$ is an approximation for $\log F(z)$ down to terms of a certain order.

As an example of the theorem above, consider the function

$$F(z) = \prod_{s=1}^{\infty} \left[1 + \frac{z^{1/s^q}}{e^{s\omega}} \right], \text{ where } q > 0,\ \omega = \omega_1 + \iota\omega_2,\ \omega_1 > 0.$$

Calculating $\bar{P}$ in terms of r, we find :

When $q < 1$, $\log F(z) = \frac{1}{2} . \frac{1+q}{1-q} . \omega_1^{-\frac{1-q}{1+q}} . (\log r)^{\frac{2}{1+q}} + \zeta(q) . \log r + (\text{finite terms}).$

When $q = 1$, $\log F(z) = \frac{1}{2} \log \log r . \log r + [\gamma - \frac{1}{2} \log (e\omega_1)] \log r + (\text{finite terms}),$
where γ is Euler's constant.

When $q > 1$, $\log F(z) = \zeta(q) . \log r - \frac{1}{2} . \frac{q+1}{q-1} . \omega_1^{\frac{q-1}{q+1}} . (\log r)^{\frac{2}{q+1}} + (\text{finite terms}),$

Thus when $q > 1$, $|F(z)|$ behaves asymptotically like $r^{\zeta(q)}$.

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§ 24. The methods which we have used may be applied to obtain approximations to other functions than those we have considered. For instance, in the theorem of § 16, we may substitute a function like $s^q(\log s)^{q_1}(\log_2 s)^{q_2}\ldots(\log_m s)^{q_m}$ for s^q, wherever the latter expression occurs in the statement of the theorem and in the condition (1).

In general our methods may be used to approximate to the function

$$F(z)=\prod_{s=1}^{\infty}\left[1+\frac{\phi(z,\ s)}{a_s}\right]^{\chi(s)},$$

where $\phi(z,\ s)$ is such that $$\operatorname*{Lt}_{s=\infty}\left|\frac{\phi(z,\ s)}{a_s}\right|=0,$$

for all values of z, where ϕ is such that there is always a single n, such that

$$\left|\frac{\phi(z,\ n)}{a_n}\right|\geqslant 1,\ \text{and}\ \left|\frac{\phi(z,\ n+1)}{a_{n+1}}\right|\leqslant 1,$$

and where the product form converges sufficiently rapidly.

The latter condition will generally be ensured, ϕ and χ being supposed given, when $|a_s|$ increases sufficiently rapidly with s.

The approximation which we obtain is

$$P=\sum_{s=1}^{n}[\chi(s)(\log\phi(z,\ s)-\log a_s)],$$

and the general rule is as follows:

When the product form converges as rapidly as or more rapidly than $\prod_{s=1}^{\infty}\left(1+\frac{z}{e^{s\omega}}\right)$, $\Re P$ is an approximation for $\log F(z)$ with a finite error, and when the product converges as rapidly as, or more rapidly than some integral function of zero order, $\Re P$ is of higher order in n or r than $[\log F(z)-\Re P]$, and $\Re P$ is an approximation for $\log F(z)$ down to a certain order.

In certain cases we may possibly obtain some more extended formula analogous to that of § 3.

We shall not develop this theory further, but shall now show how we may approximate to certain products which are of a different type to those which we have so far considered.

§ 25. We shall consider functions of the following type:

$$F(z)=\prod_{s=1}^{\infty}[1+u_s z^{f(s)}],$$

where $$\operatorname*{Lt}_{s=1}\frac{f(s+1)}{f(s)}=\infty,$$

where $|u_s|\leqslant 1$ for all values of s, and where $\sum_{s=1}^{\infty}u_s$ is divergent.

In the case where $f(s+1)=m_s f(s)$, where m_s is an integer which becomes infinite with s, $F(z)$ is an uniform function, and the radius of convergence of its Taylor series is evidently unity. It can be shown, moreover, by a proof similar to the well-known

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proof in the analogous case of the Taylor series* $\Sigma a_s z^{m_1 m_2 \dots m_s}$, that the circle $|z|=1$ is a ring of essential singularities of $F(z)$, and that the function consequently exists only inside the circle. Since $|u_s| \leqslant 1$ the function has no zeros inside the circle, and is of a completely different type to that of the functions we have been considering.

Let us consider the general case when

$$\underset{s\ \infty}{\mathrm{Lt}} \frac{f(s+1)}{f(s)} = \infty,$$

and z is real and positive, less than unity, but near unity.

There are two cases in which we may obtain an approximation.

I. Suppose that for each value of s, one or each of the following conditions is satisfied:

(a) $|u_s| \leqslant \beta < 1$;

(b) $|\arg u_s| \leqslant \pi - \gamma < \pi$.

With z (which is real and positive) we associate an integer n defined by the relation

$$z^{f(n+1)} \leqslant e^{-1} \leqslant z^{f(n)} \quad \dots\dots\dots\dots(1),$$

n is clearly a function of z which is large when z is nearly unity.

Then we have

$$F(z) = A(z) \,.\, \prod_{s=1}^{n} (1+u_s),$$

where for some value of K we have

$$\frac{1}{K} < |A(z)| < K.$$

Since Σu_s is divergent, $\prod_{s=1}^{n} (1+u_s)$ becomes either infinite or zero with n. It may be possible to approximate to this product in terms of n and hence, possibly, in terms of z.

We proceed to the proof of the theorem.

If follows from (a) or (b), that, for all values of s,

$$|1+u_s| \text{ is greater than a non-zero constant } c \quad \dots\dots\dots\dots(2).$$

Let $\dfrac{f(s+1)}{f(s)} = m_s$, which is here not necessarily an integer. Assign a large positive constant $h > 1$. Then when s is greater than a finite μ, $m_s > h$.

Hence when $n > \mu$, $s \geqslant 2$,

$$\begin{aligned} z^{f(n+s)} &= \exp\left([\log\{z^{f(n+1)}\}] \,.\, m_{n+1} \,.\, m_{n+2} \dots m_{n+s}\right) \\ &\leqslant \exp\left(-m_{n+1} \dots m_{n+s}\right), \text{ by the definition of } n, \\ &< \exp\left(-h^{s-1}\right) \\ &< \exp\left[-(s-1)\log h\right] \quad \dots\dots\dots\dots(3), \\ &< \exp\left(-\log h\right) \quad \dots\dots\dots\dots(4). \end{aligned}$$

* Cf. for example Vivanti, *Theorie der Eindeutigen analytischen Funktionen*, pp. 411—412.

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From (4), $$|\log(1+u_{n+s}z^{f(n+s)})| \leqslant |\log(1-z^{f(n+s)})| < H \,.\, |z|^{f(n+s)}$$
$$< H \exp[-(s-1)\log h], \text{ from (3).}$$

Hence $$\sum_{s=2}^{\infty} |\log[1+u_{n+s}z^{f(n+s)}]| < H \,.\, \sum_{s=2}^{\infty} \exp[-(s-1)\log h] < \text{a finite } K_1 \;\ldots\ldots(5).$$

Again, when $n > s > \mu$,
$$1 > z^{f(s)} > \exp[\log\{z^{f(n)}\} \,.\, /(m_{n-1}\, m_{n-2} \ldots m_s)]$$
$$> \exp\left(-\frac{1}{m_s \ldots m_{n-1}}\right)$$
$$> \exp(-h^{-(n-s-1)}).$$

Hence $$[1-z^{f(s)}] < 1-\exp(-h^{-(n-s-1)})$$
$$< 2h^{-(n-s-1)}, \text{ if } h^{-1} < \tfrac{1}{2}\,^{*}.$$

Hence, when $n < s < \mu$,
$$|\log(1+u_s) - \log[1+u_s z^{f(s)}]| = \left|\log\left[1-\frac{u_s(1-z^{f(s)})}{1+u_s}\right]\right|$$
$$\leqslant \left|\log\left[1-\frac{1-z^{f(s)}}{c}\right]\right|, \text{ from (1),}$$
$$< \left|\log\left[1-\frac{2h^{-(n-s-1)}}{c}\right]\right|$$
$$< \frac{4}{c} \,.\, h^{-(n-s-1)}, \text{ provided } h > \frac{4}{c}.$$

Hence
$$\left|\sum_{s=\mu+1}^{n-1} \log(1+u_s) - \sum_{s=\mu+1}^{n-1} \log[1+u_s z^{f(s)}]\right| < \frac{4}{c} \sum_{s=\mu+1}^{n-1} h^{-(n-s-1)} < \text{a finite } K_2 \;\ldots\ldots (6).$$

Again
$$\left|\sum_{s=1}^{\mu} \log(1+u_s) - \sum_{s=1}^{\mu} \log[1+u_s z^{f(s)}]\right| < \mu \,.\, \frac{4}{c} \,.\, h^{-(n-\mu-1)} < \text{a finite } K_3 \ldots\ldots\ldots\ldots(7).$$

Finally, it follows at once from (a) or (b), and the fact that $z^{f(n)}$ and $z^{f(n+1)}$ are positive and less than unity, that
$$\log[1+u_n z^{f(n)}] + \log[1+u_n z^{f(n+1)}] < \text{a finite } K_4 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

From (5), (6), (7), and (8), we deduce that
$$\left|\sum_{s=1}^{\infty} \log[1+u_s z^{f(s)}] - \sum_{s=1}^{n} \log(1+u_s)\right| < \text{a finite } K_5,$$

whence we at once obtain the formula of the theorem.

When $\operatorname*{Lt}_{s=\infty} |u_s| = 0$, it is easily shown that the following more precise formula holds:—
$$F(z) = [1+\epsilon(z)] \prod_{s=1}^{n} [1+u_s], \text{ where } \operatorname*{Lt}_{z=1} |\epsilon(z)| = 0.$$

* We have $1-e^{-x} = x\left[1+\frac{x}{2}+\ldots\right] < 2x$, if $x < \frac{1}{2}$.

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II. Consider the case $u_s = -1$, so that $F(z) = \prod_{s=1}^{\infty} [1 - z^{f(s)}]$.

We shall prove that when z is real and positive and nearly unity, and n is determined as above, we have

$$\log [1/F(z)] = n \log f(n) - \sum_{s=1}^{n} \log f(s) + B(z) . n,$$

where

$$|B(z)| < K,$$

or

$$F(z) = e^{-nB(z)} . \frac{f(1) f(2) \ldots f(n)}{[f(n)]^n}.$$

We can prove, just as in the case I, that

$$\sum_{s=1}^{\infty} \log [1 - z^{f(n+s)}] < \text{a finite } K_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let $z^{f(n)} = e^{-\theta}$, so that $0 < \theta \leqslant 1$.

When $s < n$,

$$1 - z^{f(s)} = 1 - \exp\left[\frac{-\theta f(s)}{f(n)}\right]$$
$$= C_s . \theta \frac{f(s)}{f(n)},$$

where $|C_s|$ lies between finite constant limits, of which the lower is not zero.

Hence

$$\prod_{s=1}^{n} [1 - z^{f(s)}] = \prod_{s=1}^{n} [C_s \theta] . \frac{f(1) f(2) \ldots f(n)}{[f(n)]^n}$$
$$= (e^{-B_1 n}) \frac{f(1) f(2) \ldots f(n)}{[f(n)]^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $|B_1|$ is finite.

From (1) and (2) the formula of the theorem follows.

We shall give one or two examples.

As an example of I. take $F(z) = \prod_{s=1}^{\infty} [1 + z^{s!}]$, and suppose that z tends to the circumference of the unit circle along any radius vector whose argument bears a commensurable ratio to π.

Let this argument be $\phi = \frac{p\pi}{q}$. Then when $s > 2q$, $\phi . s!$ is a multiple of 2π and $z^{s!}$ is real and positive. The first $2q$ factors of $F(z)$ only give a finite factor in the approximation, and we obtain the same approximation, viz. $\prod_{s=1}^{n} [1 + 1] = 2^n$, as when z is real and positive.

The integer n is determined by

$$|z|^{n!} = e^{-\theta}, \quad 0 < \theta \leqslant 1, \text{ and } |z|^{(n+1)!} = e^{-\theta'}, \quad \theta' > 1;$$

or

$$n! = \theta / \log\left(\frac{1}{r}\right), \quad (n+1)! = \theta' / \log (1/r),$$

whence we obtain

$$n = [1 + \epsilon'(r)] \frac{\log\left(\frac{1}{1-r}\right)}{\log \log\left(\frac{1}{1-r}\right)}, \text{ where } \operatorname*{Lt}_{r=1} \epsilon'(r) = 0.$$

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Substituting for n in terms of r we obtain

$$F(z) = \left[\frac{1}{1-r}\right]^{\frac{\{1+\epsilon(z)\}}{\log 2 \,.\, \log\log\left[\frac{1}{1-r}\right]}}, \text{ where } \underset{r=1}{\mathrm{Lt}} \,|\, \epsilon(z)| = 0.$$

For the function $F(z) = \prod_{s=1}^{\infty}(1-z^{s!})$, $\arg z = \frac{p\pi}{q}$, which comes under case II, we find

$$F(z) = \exp\left\{-[1+\epsilon(z)] \,.\, \frac{\left[\log\left(\frac{1}{1-r}\right)\right]^2}{2\log\log\left[\frac{1}{1-r}\right]}\right\}.$$

For the functions $\prod^{\infty}\left[1 \pm \frac{1}{s} z^{s!}\right]$, which belong to case I, we find, when $\arg z = \frac{p\pi}{q}$,

$$\prod_{s=1}^{\infty}\left(1 + \frac{1}{s} z^{s!}\right) = A(z) \,.\, \frac{\log\left[\frac{1}{1-r}\right]}{\log\log\left[\frac{1}{1-r}\right]},$$

$$\prod_{s=z}^{\infty}\left(1 - \frac{1}{s} z^{s!}\right) = B(z) \,.\, \frac{\log\log\left[\frac{1}{1-r}\right]}{\log\left[\frac{1}{1-r}\right]},$$

where $\frac{1}{K} < |A(z)| < K$, and $B(z)$ satisfies a similar inequality.

The application of these methods to the case of Taylor series of the type $\Sigma u_s z^{f(s)}$, is obvious.

We find for example, when $\arg z = \frac{p\pi}{q}$,

$$\sum_{s=1}^{\infty} z^{s!} = [1+\epsilon(z)] \,.\, \frac{\log\left[\frac{1}{1-r}\right]}{\log\log\left[\frac{1}{1-r}\right]},$$

$$\sum_{s=1}^{\infty} \frac{1}{s} z^{s!} = \log\log\left[\frac{1}{1-r}\right] - \log\log\log\left[\frac{1}{1-r}\right] + C(z),$$

where $\underset{r=1}{\mathrm{Lt}} \,|\, \epsilon(z)| = 0$, and $|C(z)| < K$.

§ 26. We shall conclude with some general remarks on a certain phenomenon which presents itself in the subject of this paper, and of which the various particular functions which we have considered afford illustration. We consider for the present only those types of functions considered up to § 24, i.e. the functions which have zeros. It will have been noticed that in the final approximation in terms of r, sometimes, (A), none but the ordinary analytic functions of analysis, powers, exponentials, and logarithms, occur in the expression, while in other cases, (B), we require a function α, defined as the

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fractional part of some analytic function of r, or in some analogous manner, for the complete expression of our approximation*, or else we found that we could in no way express our approximation completely in terms of the ordinary functions. This fact appears to be governed by a certain rule, to which I have not found any exception. This rule refers to the rapidity of the convergence of the product form, and may be roughly stated as follows.

When the product-form† converges more rapidly than the form $\Pi\left[1+\frac{z}{e^{\delta\omega}}\right]$, we are in case (B).

When the product-form converges less rapidly than $\Pi\left[1+\frac{z}{e^{\delta\omega}}\right]$, the circumstances are as follows.

It may be possible to invert the relation $a_n \leqslant r \leqslant a_{n+1}$, into an equation, $n = \chi(r) + \alpha$, where $\chi(r)$ is an expression in finite terms containing only the ordinary analytic functions, and where the modulus of the undetermined number α is less than some finite constant.

When this is the case, the problem of effecting the substitution is practically soluble, and we obtain an approximation for $\log F(z)$ in finite terms, and containing only the ordinary analytic functions. The number α disappears from the final approximation.

On the other hand, we may not be able to invert the relation $a_n \leqslant r \leqslant a_{n+1}$ in the above manner. In this case we cannot find an approximation for $\log F(z)$ with only the theoretical error, but the approximation which we do obtain (which usually takes the form of an asymptotic expansion in descending powers of some function of r, in which we can determine any numerically assigned number of the coefficients) never contains α terms.

Concerning the question as to when we can, and when we cannot, invert the relation $a_n \leqslant r \leqslant a_{n+1}$ into the desired form, we may give the following examples.

If $\log a_n$ is equal to $n^\rho +$ an expression of order not greater than $n^{\rho'}$, where $\rho' < \rho$, we can effect the inversion with the necessary degree of approximation.

If $\log a_n = n^\rho (\log n)^{\rho'}$, we can only obtain an equation $n = \chi(r) + \chi_1(r)$, where $\chi_1(r)$ is of order $(\log r)^{1/\rho}/(\log\log r)^p$, p being any numerically assigned integer.

The statements concerning the rapidity of the convergence may be made more precise, as follows:

In the product form $\Pi\left[1+\left(\frac{z}{a_s}\right)^{f(s)}\right]$, if $u_s = \left|\left(\frac{z}{a_s}\right)^{f(s)}\right|$, there is an integer n such that $u_n \geqslant 1$, $u_{n+1} \leqslant 1$. We say that the product converges more rapidly than $F_0(z) = \Pi\left(1+\frac{z}{e^{\delta\omega}}\right)$, when we have $\operatorname*{Lt}_{n=\infty} \frac{u_{n+2}}{u_{n+1}} = 0$ (the z which enters into the expressions u_{n+1}, u_{n+2}, of course depends on n), and we say that the product converges less rapidly than $F_0(z)$, when

$$\operatorname*{Lt}_{n=\infty} \frac{u_{n+2}}{u_{n+1}} = 1.$$

* Only one instance of this case occurs in this paper, viz. in § 11. It is easy, however, to multiply instances.

† It is assumed that a_s and $f(s)$ are analytic functions of s, and are not defined, e.g., by a Taylor series in s. The above properties, however, continue to hold when $f(s)$ is such a function as $[s^q]$.

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When we have $\underset{n=\infty}{\mathrm{Lt}} \frac{u_{n+2}}{u_{n+1}} = \beta$, where $0 < \beta < 1$, we say that the product converges like $F_0(z)$.

There is a simple criterion for these cases.

When $F(z)$ converges less rapidly than $F_0(z)$, $\underset{r=\infty}{\mathrm{Lt}} |R| = \infty$.

When $F(z)$ converges like $F_0(z)$, $|R|$ is finite.

And when $F(z)$ converges more rapidly than $F(z)$, $\mathrm{Lt}\,|R| = 0$.

We might expect that when the convergence of the product form is like that of $F_0(z)$, intermediate characteristics will present themselves, and this is in fact the case. The terms of P which the general theory permits us to retain are all the terms which tend to infinity. But we have seen that it is sometimes possible to find the finite term of $\log F(z)$. Now when this is the case, this finite term involves terms in α while the large terms are ordinary analytic functions.

We should add to our statement of the behaviour of the functions in case (A), the remark that α terms (or unknown functions), always appear in the *large* terms of the approximation.

Moreover when the rate of convergence of $F(z)$, while remaining more rapid than that of $F_0(z)$, approaches the latter rate, the terms in α occurring in the large terms of $\log F(z)$, are of less and less order in r when compared with the dominant term. Thus the nature of the approximation is subject to a certain amount of continuity.

With regard to the functions of § 25, which have no zeros, we may remark that they converge more rapidly than $F_0(z)$, and also that they require unknown functions or α terms* for the complete expression of our theoretical approximation in terms of r.

I am not clear, however, as to how far the general rule applies to this type of functions.

The appended table gives all the particular functions with zeros, which have been considered in this paper, and in my previous paper† on integral functions of zero order, with the nature of the approximation, the order of $|R|$, and the article where the function is considered. The number I. refers to the previous paper, and II. to the present paper.

* Such terms do not occur in the particular cases considered in § 28, but the approximations which we have given have an error of greater order than that of tn theoretical error.

† *loc. cit.*

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u_s	Order of $\lvert R \rvert$	Nature of the final approximation for $\log F(z)$	Article
$\dfrac{z}{\exp\left[(a+s\omega)^k\right]}$, $k<1$	$\lvert R \rvert$ tends to infinity	Only known functions occur	I. § 15
$\dfrac{z}{\exp\left[(\log s)^{1+k}\right]}$			I. § 16
$\left[\dfrac{z}{s^\rho}\right]^{[s^{1/2}]}$			II. § 22
* $\dfrac{z}{e^{s\omega}}$	$\lvert R \rvert$ is finite both ways	The large terms contain only known functions	I. § 12
* $\dfrac{z}{(a+vs)^k e^{s\omega}}$			I. § 14
$\left[\dfrac{z}{\exp(\omega s^{1-q})}\right]^{s^q}$, $q<1$			II. § 18
* $\left[\dfrac{z}{s}\right]^s$		In the cases marked with a star the finite term contains terms in a	II. §§ 18, 19
$\left[z\exp\{\omega s^{-(q-1)}\}\right]^{s^q}$, $q>1$			II. § 18
z^{1/s^q} $e^{s\omega}$, $q>0$			II. § 23
$\dfrac{z}{\exp(e^s)}$	$\lvert R \rvert$ tends to zero	Terms in a occur in the large terms	I. § 16
$\dfrac{z}{\exp\left[(a+s\omega)^k\right]}$, $k>1$			I. § 15
$\left[\dfrac{z}{\exp(e^{s\omega})}\right]^s$			II. § 6
$\left[\dfrac{z}{s^\rho}\right]^{s^q}$, $q>1$			II. § 9
$\left[z\exp\dfrac{\omega}{s}\right]^{s^q}$, $q>2$			II. § 18
$\left[z\exp(e^{-as})\right]^{e^{as}}$			II. § 21
$\left[\dfrac{z}{(\log s)^\rho}\right]^{s^q}$, $q>1$		$\bar{P}$ requires new functions for its complete expression	II. § 11

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THE REALITY OF THE ROOTS OF CERTAIN TRANSCENDENTAL EQUATIONS OCCURRING IN THE THEORY OF INTEGRAL EQUATIONS

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XV. *The Reality of the Roots of certain Transcendental Equations occurring in the theory of Integral Equations.* By H. BATEMAN, M.A., Fellow of Trinity College.

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§ 1. In a former paper* I showed that if $\phi_\lambda(s)$ is the solution of the integral equation

$$f(s) = \phi_\lambda(s) - \lambda \int_a^b \kappa(s, t)\, \phi_\lambda(t)\, dt \qquad (1),$$

the coefficients in the expansion of an arbitrary function $\chi(s)$ in a series of the form

$$\chi(s) = \Sigma\, a_\lambda \phi_\lambda(s) \qquad (2),$$

where the summation extends over all values of λ for which $\phi_\lambda(s)$ satisfies a given linear condition

$$\phi_\lambda(*) = c,$$

are given by the formula

$$a_\lambda = \frac{\int_a^b K(*, t)\, \chi(t)\, dt}{\frac{\partial}{\partial \lambda} \phi_\lambda(*)} \qquad (3),$$

where $K(s, t)$ is the solving function of the integral equation and $F(*)$ is a convenient symbol for the result of imposing a linear condition of a distributive nature upon the function $F(s)$.

The results obtained, however, were in some respects incomplete because the roots of the equation

$$\phi_\lambda(*) = c$$

are not necessarily real and there is nothing to show that they are infinite in number. The latter condition is necessary if the expansion (2) is to be used to represent a very general type of function. The object of this paper is to carry the analysis a step further. The particular case that is considered still leads to a very general class of expansions which is seen to include many of the expansions which occur in the boundary problems of mathematical physics.

* *Camb. Phil. Trans.*, Vol. XX. No. XII.

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The particular condition that is chosen is

$$\int_a^b f(s)\,\phi_\lambda(s)\,ds = c \quad\text{.....(4)},$$

and we limit ourselves to the case in which $\kappa(s, t)$ is a real symmetric function of s and t which is continuous for $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$.

On this supposition the equation (1) is analogous to the system of linear equations

$$V_r = \sum_{s=1}^{n} p_{rs} E_s, \qquad (r = 1, 2, \dots n),$$

which connect the potentials and charges of a system of electrified conductors. The function

$$w(\lambda) = \int_a^b f(s)\,\phi_\lambda(s)\,ds \quad\text{.....(5)},$$

corresponds to twice the energy ($2W = \Sigma E_r V_r$).

We shall show in the first place that if c is either negative or zero, the roots of the equation

$$w(\lambda) = c,$$

are all real.

If possible let $\lambda = \lambda_1 + i\lambda_2$ be a complex root and let $\phi(s) = \phi_1(s) + i\phi_2(s)$ be the corresponding complex expression for $\phi_\lambda(s)$.

Substituting in equation (1) and equating real and imaginary parts we obtain

$$\left.\begin{aligned} f(s) &= \phi_1(s) - \lambda_1 \int_a^b \kappa(s, t)\,\phi_1(t)\,dt + \lambda_2 \int_a^b \kappa(s, t)\,\phi_2(t)\,dt \\ 0 &= \phi_2(s) - \lambda_1 \int_a^b \kappa(s, t)\,\phi_2(t)\,dt - \lambda_2 \int_a^b \kappa(s, t)\,\phi_1(t)\,dt \end{aligned}\right\} \quad\text{.....(6)}.$$

Also equation (4) gives

$$\left.\begin{aligned} \int_a^b f(s)\,\phi_1(s)\,ds &= c \\ \int_a^b f(s)\,\phi_2(s)\,ds &= 0 \end{aligned}\right\} \quad\text{.....(7)}.$$

Multiplying the first of equations (6) by $\phi_2(s)$, the second by $\phi_1(s)$ and subtracting we obtain on integration*

$$\lambda_2 \int_a^b \int_a^b \kappa(s, t)\,[\phi_1(s)\,\phi_1(t) + \phi_2(s)\,\phi_2(t)]\,ds\,dt = 0.$$

Similarly, if we multiply the first equation by $\phi_1(s)$ the second by $\phi_2(s)$, integrate and add, we obtain

$$c = \int_a^b [\phi_1^2(s) + \phi_2^2(s)]\,ds - \lambda_1 \int_a^b \int_a^b \kappa(s, t)\,[\phi_1(s)\,\phi_1(t) + \phi_2(s)\,\phi_2(t)]\,ds\,dt.$$

Now $\int_a^b [\phi_1^2(s) + \phi_2^2(s)]\,ds - c$ is positive since c is not positive and $b > a$, hence the double integral on the right-hand side cannot be zero. The previous equation then shows that λ_2 must be zero, that is, λ must be real.

* This method is used by Picard for a different purpose. *Palermo Rend.* t. XXII. (1906).

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If $\kappa(s, t)$ is a *definite function**, that is a function such that the double integral

$$\int_a^b \int_a^b \kappa(s, t)\, x(s)\, x(t)\, ds dt,$$

is positive for every continuous function $x(s)$; the equation

$$c = \int_a^b f(s)\, \phi_\lambda(s)\, ds = \int_a^b \phi_\lambda^2(s)\, ds - \lambda \int_a^b \int_a^b \kappa(s, t)\, \phi_\lambda(s)\, \phi_\lambda(t)\, ds dt,$$

shows that λ is necessarily positive.

Our next step is to find how the roots of the equation $w(\lambda) = c$ are distributed. We recall that if $K(s, t)$ is the solving function of the equation

$$f(s) = \phi_\lambda(s) - \lambda \int_a^b \kappa(s, t)\, \phi_\lambda(t)\, dt \quad \text{.....(1)},$$

we have

$$\phi_\lambda(s) = f(s) + \lambda \int_a^b K(s, t) f(t)\, dt \text{.....(8)},$$

$$\frac{\partial}{\partial \lambda} \phi_\lambda(t) = \int_a^b K(s, t)\, \phi_\lambda(s)\, ds \quad \text{.....(9)}.$$

Now

$$\frac{d}{d\lambda} w(\lambda) = \int_a^b f(t) \frac{\partial}{\partial \lambda} \phi_\lambda(t)\, dt$$

$$= \int_a^b \int_a^b K(s, t)\, \phi_\lambda(s) f(t)\, ds dt,$$

or

$$\frac{d}{d\lambda} \{w(\lambda)\} = \int_a^b \int_a^b \kappa(s, t)\, \phi_\lambda(s)\, \phi_\lambda(t)\, ds dt \text{.....(10)},$$

since

$$\int_a^b K(s, t) f(t)\, dt = \int_a^b \kappa(s, t)\, \phi_\lambda(t)\, dt,$$

from (1) and (8).

If $\kappa(s, t)$ is a definite function the double integral is positive and so $w(\lambda)$ increases with λ. Now $\phi_\lambda(t)$ is finite except possibly at the singular values of λ for the integral equation, these being the poles of the function $K(s, t)$. The function $w(\lambda)$ may become infinite for one of these values and change sign in passing through it. Since it increases continually with λ it must pass through every real value before it can become infinite again. Hence a root of the equation $w(\lambda) = c$ lies between every consecutive pair of poles. This is true whether c is positive or negative, but when c is positive we cannot assert that *all* the roots of the equation are real.

When $\kappa(s, t)$ is not necessarily a definite function a somewhat similar procedure may be adopted. In this case we calculate the rate of change of the function $\lambda w(\lambda)$. Since

$$w(\lambda) = \int_a^b f(s)\, \phi_\lambda(s)\, ds = \int_a^b \phi_\lambda^2(s)\, ds - \lambda \int_a^b \int_a^b \kappa(s, t)\, \phi_\lambda(s)\, \phi_\lambda(t)\, ds dt$$

$$= \int_a^b \phi_\lambda^2(s)\, ds - \lambda \frac{d}{d\lambda} w(\lambda),$$

we have

$$\frac{d}{d\lambda} \{\lambda w(\lambda)\} = \int_a^b \phi_\lambda^2(s)\, ds \text{.....(11)}.$$

* Hilbert, *Gött. Nachr.* (1904), Heft. 1.

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The integral on the right-hand side has a positive value since $b > a$ and so the function $\lambda w(\lambda)$ increases continually with λ. The points where $\lambda w(\lambda)$ becomes infinite are singular values of λ for the integral equation but it does not necessarily become infinite for every such value. As λ passes through a pole $\lambda w(\lambda)$ becomes infinite and changes sign and will then take every value between $-\infty$ and $+\infty$ as λ passes from this pole to the next. We thus have the theorem.

The zeros and poles of the function $\lambda w(\lambda) - c$ occur alternately.*

As λ passes from one positive pole to another or from one negative pole to another, $w(\lambda)$ will pass from $-\infty$ to $+\infty$ and so will assume every real value at least once in the interval. Hence we have the theorem.

Between every pair of positive poles of the function $w(\lambda)$ there lies at least one root of the equation $w(\lambda) = c$, also between every pair of negative poles there is at least one root of $w(\lambda) = c$. In the interval between the smallest negative pole and the smallest positive pole the number of roots is either zero or an even number.

Since every pole of $w(\lambda)$ is a singular value of λ for the integral equation, we may conclude that if the equation $\lambda w(\lambda) = c$ has an infinite number of roots, the number of singular values of λ is also infinite.

The expansion of an arbitrary function in a series of the form

$$\sum_{\lambda} a_\lambda \phi_\lambda(s),$$

where the values of λ included in the summation are given by $\lambda w(\lambda) = c$ may be investigated as follows.

Since
$$f(s) = \phi_\lambda(s) - \lambda \int_a^b \kappa(s, t)\, \phi_\lambda(t)\, dt,$$

and
$$c = \lambda \int_a^b f(t)\, \phi_\lambda(t)\, dt$$

we have
$$0 = \phi_\lambda(s) - \lambda \int_a^b \left[\kappa(s, t) + \frac{f(s) f(t)}{c}\right] \phi_\lambda(t)\, dt.$$

This is a homogeneous integral equation of the form

$$\phi_\lambda(s) - \lambda \int_a^b h(s, t)\, \phi_\lambda(t)\, dt = 0,$$

in which $h(s, t)$ is a symmetric function, consequently the values of λ are all real and Hilbert's general expansion theorem† may be applied.

* *If $\kappa(s, t)$ is a definite function the singular values of λ are all positive and the roots of $\lambda w(\lambda) = c$ are all positive when c is positive.*

To prove this we observe that $\lambda w(\lambda)$ is zero when λ is zero and decreases continually as λ recedes to $-\infty$. Now $\lambda w(\lambda)$ cannot become infinite for a negative value of λ and so will not change sign. Consequently, it cannot take the positive value c for any negative finite value of λ.

When c is negative there can only be one negative root of the equation, for $\lambda w(\lambda)$ takes each negative value once in the interval $(-\infty, 0)$.

† *Göttinger Nachrichten* (1904), Heft. I. See also a paper by Erhard Schmidt, *Diss. Göttingen* (1905) and *Math. Ann.* LXII. (1907), in which a limitation introduced by Hilbert is shown to be unnecessary.

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This theorem tells us that a function $F(s)$ which can be expressed in the form

$$F(s) = \int_a^b h(s, t)\,\psi(t)\,dt,$$

where $\psi(t)$ is a continuous function, is capable of being expanded in an absolutely and uniformly convergent series of the form

$$F(s) = \Sigma\, a_\lambda \phi_\lambda(s),$$

the coefficients being determined by Fourier's rule.

The theorems we have just proved may be illustrated by means of Hilbert's integral equation*

$$\left.\begin{aligned} f(s) &= \phi(s) - \lambda \int_0^1 \kappa(s, t)\,\phi(t)\,dt \\ \text{in which} \quad \kappa(s, t) &= \left.\begin{matrix} (1-s)\,t, & t \leqslant s \\ (1-t)\,s, & t \geqslant s \end{matrix}\right\} \end{aligned}\right\} \quad \text{.....(12).}$$

The solving function is in this case

$$\left.\begin{aligned} K(s, t) &= \frac{\sin\sqrt{\lambda}\,(1-s)\,.\,\sin\sqrt{\lambda}\,.\,t}{\sqrt{\lambda}\,\sin\sqrt{\lambda}}, & t \leqslant s \\ &= \frac{\sin\sqrt{\lambda}\,(1-t)\,.\,\sin\sqrt{\lambda}\,.\,s}{\sqrt{\lambda}\,.\,\sin\sqrt{\lambda}}, & t \geqslant s \end{aligned}\right\} \quad \text{.....(13),}$$

and the singular values of λ are the roots of the equation

$$\delta(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}},$$

i.e. the positive quantities π^2, $2^2\pi^2$, $3^2\pi^2$, ...

Since the function $1-s$ decreases as s increases the function $\kappa(s, t)$ is *definite*, for it is of the form

$$\kappa(s, t) = \left.\begin{matrix} \alpha(s)\,\beta(t), & t \leqslant s \\ \alpha(t)\,\beta(s), & t \geqslant s \end{matrix}\right\},$$

and we have shown elsewhere† that such a function is definite if $\alpha(s)$ continually decreases and $\beta(s)$ continually increases as s increases, both functions being constantly positive.

If in the first place we put $f(s) = s$, the formula

$$\phi(s) = f(s) + \lambda \int_0^1 K(s, t) f(t)\,dt,$$

gives

$$\phi(s) = \frac{\sin s\sqrt{\lambda}}{\sin\sqrt{\lambda}}.$$

Thus

$$\begin{aligned} w(\lambda) &= \int_0^1 f(s)\,\phi(s)\,ds = \operatorname{cosec}\sqrt{\lambda} \int_0^1 s \sin s\sqrt{\lambda}\,.\,ds \\ &= \frac{1}{\lambda} - \frac{\cot\sqrt{\lambda}}{\sqrt{\lambda}}. \end{aligned}$$

* *Göttinger Nachrichten*, Heft. III. p. 277 (1904).

† *Messenger of Mathematics*, (1907).

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Putting $\lambda = m^2$, we have

$$w(\lambda) - c = \frac{1}{m^2} - c - \frac{\cot m}{m} \quad \text{.....(14)},$$

$$\lambda w(\lambda) - c = 1 - cm^2 - m \cot m \quad \text{.....(15)}.$$

The theorems that have been proved tell us in the first place that if c is negative the values of m^2 which satisfy

$$\cot m = \frac{1}{m} - cm \quad \text{.....(16)},$$

are all positive and lie in the intervals $(0, \pi^2)$, $(\pi^2, 2\pi^2)$....

The corresponding values of m are thus all real and lie in the intervals

$$\ldots(-\pi, 0),\ (0, \pi),\ (\pi, 2\pi)\ldots.$$

The class of expansions is of the type

$$F(s) = \Sigma\, c_m \sin ms \quad \text{.....(17)},$$

where the summation is for values of m^2 which satisfy the above equation. Our theorem enables us to determine the coefficients in this expansion.

Secondly we learn from the other theorem that the values of m^2 which satisfy the equation

$$1 - c - m \cot m = 0,$$

are all real and that if c is positive they are all positive. Further, if m satisfy this equation the functions of type $\sin ms$ form an orthogonal system of functions and we may deduce from Hilbert's expansion theorem that if $F(s)$ can be expressed in the form

$$F(s) = \int_0^1 \left[\kappa(s, t) + \frac{st}{c}\right] \psi(t)\, dt,$$

where $\psi(t)$ is continuous in the interval $(0 \leqslant t \leqslant 1)$ then $F(s)$ can be expanded in the form

$$F(s) = \Sigma\, c_m \sin ms,$$

the expansion being absolutely and uniformly convergent for the interval $(0 \leqslant s \leqslant 1)$.

Putting $1 - c = \mu$ the equation in m becomes

$$m \cot m = \mu, \qquad \mu < 1,$$

and the definite integral takes the form

$$F(s) = \int_0^1 h(s, t)\, \psi(t)\, dt,$$

where

$$\left.\begin{aligned} h(s, t) &= t\left(1 + \frac{s\mu}{1-\mu}\right), \qquad t \leqslant s \\ &= s\left(1 + \frac{t\mu}{1-\mu}\right), \qquad t \geqslant s \end{aligned}\right\}.$$

The coefficients in the expansion are obtained by multiplying it by $\sin ms$ and integrating between 0 and 1.

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Since
$$\int_0^1 \sin^2 ms \,.\, ds = \frac{m - \sin m \cos m}{2m},$$
we have
$$c_m = \frac{2m}{m - \sin m \cos m} \int_0^1 F(s) \sin ms \,.\, ds.$$

Next, if we put $f(s) = 1$ in the equation
$$\phi(s) = f(s) + \lambda \int_0^1 K(s, t) f(t)\, dt,$$
we get
$$\phi(s) = \frac{\cos \sqrt{\lambda}\,(\frac{1}{2} - s)}{\cos \frac{1}{2}\sqrt{\lambda}},$$
and
$$\int_0^1 f(s)\, \phi(s)\, ds = 2 \tan \frac{\sqrt{\lambda}}{2}.$$

The equation $\lambda w(\lambda) = c$ is in this case
$$2\sqrt{\lambda} \tan \frac{\sqrt{\lambda}}{2} = c.$$

Putting $\begin{matrix} \lambda = 4m^2 \\ c = 4\mu \end{matrix}$ and applying Hilbert's expansion theorem we find that any function $F(s)$ which can be expressed in the form
$$F(s) = \int_0^1 h(s, t)\, \psi(t)\, dt, \qquad (0 \leqslant s \leqslant 1),$$
where
$$\left.\begin{aligned} h(s, t) = (1 - s)\, t + \frac{1}{4\mu}, \qquad & t \leqslant s \\ (1 - t)\, s + \frac{1}{4\mu}, \qquad & t \geqslant s \end{aligned}\right\},$$
and $\psi(t)$ is continuous in the interval $0 \leqslant t \leqslant 1$, can be expanded in an absolutely and uniformly convergent series of the form
$$\Sigma\, c_m \cos m\,(1 - 2s) = F(s), \qquad (0 \leqslant s \leqslant 1),$$
in which the summation extends over the values of m that satisfy the transcendental equation
$$m \tan m = \mu.$$

The coefficients as determined by Fourier's rule are
$$c_m = \frac{2m}{m + \sin m \cos m} \int_0^1 F(s) \cos m\,(1 - 2s)\, ds.$$

§ 2. The fact that the function
$$\lambda w(\lambda) = \lambda \int_a^b f(s)\, \phi(s)\, ds,$$
increases continually with λ may be used to obtain some properties of the solving function $K(s, t)$. This function is the solution of the integral equation
$$f(s) = \phi(s) - \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt,$$

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in the particular case when $f(s)=\kappa(s, r)$, accordingly we have

$$\kappa(s, r)=K(s, r)-\lambda\int_a^b \kappa(s, t)\, K(t, r),$$

and in this case

$$\lambda w(\lambda)=\lambda\int_a^b \kappa(s, r)\, K(s, r)\, ds=K(r, r)-\kappa(r, r).$$

Our theorem thus tells us that the function $K(r, r)$ increases continually with λ. It will in general be infinite at the singular values of λ and so will take every real value once and once only in passing from one infinity to the next*.

Next, let
$$\Omega(\lambda)=\int_a^b\int_a^b K(s, t)\, x(s)\, x(t)\, ds\, dt.$$

The formula
$$\frac{\partial}{\partial\lambda} K(s, t)=\int_a^b K(s, x)\, K(x, t)\, dx,$$

which is deduced from

$$\frac{\partial}{\partial\lambda}\phi_\lambda(s)=\int_a^b K(s, x)\,\phi_\lambda(x)\, dx,$$

by putting $f(s)=\kappa(s, t)$, $\phi_\lambda(s)=K(s, t)$, enables us to calculate the rate of increase of $\Omega(\lambda)$. We have in fact

$$\frac{\partial}{\partial\lambda}\Omega(\lambda)=\int_a^b\int_a^b \frac{\partial}{\partial\lambda} K(s, t)\, x(s)\, x(t)\, ds\, dt$$

$$=\int_a^b\int_a^b\int_a^b K(s, r)\, K(r, t)\, x(s)\, x(t)\, dr\, ds\, dt.$$

Now $K(s, t)$ like $\kappa(s, t)$ is a symmetric function of s and t and so the triple integral on the right-hand side may be written

$$\int_a^b [\chi(r)]^2\, dr,$$

where
$$\chi(r)=\int_a^b K(r, t)\, x(t)\, dt.$$

* If $\delta(\lambda)$ is the determinant of the integral equation the relation

$$\frac{\delta'(\lambda)}{\delta(\lambda)}=-\int_a^b K(r, r)\, dr$$

shows that the function $\frac{d}{d\lambda}\{\log\delta(\lambda)\}$ decreases as λ increases.

Again, if with Fredholm's notation

$$\kappa\begin{pmatrix} s, x_1 \dots x_n \\ t, y_1 \dots y_n \end{pmatrix}=\begin{vmatrix} \kappa(s, t), & \kappa(s, y_1), & \dots, & \kappa(s, y_n) \\ \kappa(x_1, t), & \kappa(x_1, y_1), & \dots & \\ \dots & \dots & \dots & \dots \\ \kappa(x_n, t), & \dots & & \kappa(x_n, y_n) \end{vmatrix},$$

we form an integral equation

$$f(s)=\phi(s)-\lambda\int_a^b h(s, t)\,\phi(t)\, dt,$$

in which
$$h(s, t)=\frac{\kappa\begin{pmatrix} s, x_1 \dots x_n \\ t, y_1 \dots y_n \end{pmatrix}}{\kappa\begin{pmatrix} x_1 \dots x_n \\ y_1 \dots y_n \end{pmatrix}},$$

it is easy to verify that the solving function is

$$H(s, t)=\frac{K\begin{pmatrix} s, x_1, \dots x_n \\ t, y_1, \dots y_n \end{pmatrix}}{K\begin{pmatrix} x_1 \dots x_n \\ y_1 \dots y_n \end{pmatrix}}.$$

Now when $x_1=y_1, \dots x_n=y_n$, the function

$$h(s, t)=\frac{\kappa\begin{pmatrix} s, x_1 \dots x_n \\ t, x_1 \dots x_n \end{pmatrix}}{\kappa\begin{pmatrix} x_1 \dots x_n \\ x_1 \dots x_n \end{pmatrix}},$$

is a symmetric function of s and t. The theorem that $H(s, s)$ increases with λ thus tells us that the function

$$\frac{K\begin{pmatrix} s, x_1 \dots x_n \\ s, x_1 \dots x_n \end{pmatrix}}{K\begin{pmatrix} x_1 \dots x_n \\ x_1 \dots x_n \end{pmatrix}},$$

increases with λ. The roots of $K\begin{pmatrix} s, x_1 \dots x_n \\ s, x_1 \dots x_n \end{pmatrix}=0$ are therefore separated by those of $K\begin{pmatrix} x_1 \dots x_n \\ x_1 \dots x_n \end{pmatrix}=0$.

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Also, $\chi(r)$ cannot be identically zero for all values of r in the interval (a, b) unless a function $x(t)$ exists for which

$$0 = \int_a^b \kappa(r, t)\, x(t)\, dt,$$

for this equation is obtained by putting $\lambda = 0$.

The quantity $\int_a^b [\chi(r)]^2\, dr$ is thus positive and so $\frac{d}{d\lambda}\Omega(\lambda)$ is positive, this implies that $\Omega(\lambda)$ increases continually with λ.

If $\Omega(\lambda)$ is zero for $\lambda = -\infty$ this theorem can be used to determine a range of values for which the double integral is positive and consequently for which $K(s, t)$ is a definite function. $\Omega(\lambda)$ will clearly be positive until the first singular value of λ is reached.

The values of λ for which $\Omega(\lambda)$ takes an assigned value are separated by the singular values of λ. This fact is of some significance in connection with the problem of representing an arbitrary function $F(\lambda)$ in the form

$$F(\lambda) = \int_a^b \int_a^b Q(\lambda; s, t)\, x(s)\, x(t)\, ds\, dt,$$

where $x(s)$ is a continuous function*. Thus in the particular case when $Q(\lambda; s, t)$ has the form $K(s, t)$ the equation can only possess a solution if $F(\lambda)$ increases continually with λ.

§ 3. We next inquire whether any information can be obtained with regard to the values of λ for which the homogeneous integral equation†

$$0 = \int_a^b [f(s, t) - \lambda g(s, t)]\, \phi(t)\, dt,$$

can be satisfied.

Let us suppose that $f(s, t)$ and $g(s, t)$ are real, continuous, symmetric functions of s and t for $(a \leqslant s \leqslant b)$, $(a \leqslant t \leqslant b)$ and that $f(s, t)$ is a definite function.

If λ and μ are two different singular values for which the equation can be satisfied and $\phi_\lambda(s)$, $\phi_\mu(s)$ the corresponding functions, we have

$$0 = \int_a^b \int_a^b \phi_\mu(s)\, [f(s, t) - \lambda g(s, t)]\, \phi_\lambda(t)\, dt,$$

$$0 = \int_a^b \int_a^b \phi_\lambda(s)\, [f(s, t) - \mu g(s, t)]\, \phi_\mu(t)\, dt.$$

Consequently, $$(\mu - \lambda) \int_a^b \int_a^b f(s, t)\, \phi_\lambda(s)\, \phi_\mu(t)\, ds\, dt = 0.$$

* If we attempt to solve a non-linear differential equation such as

$$F(\lambda) = \alpha(\lambda) \left(\frac{dy}{d\lambda}\right)^2 + \beta(\lambda)\, y \frac{dy}{d\lambda} + \gamma(\lambda)\, y^2,$$

by a definite integral

$$y(\lambda) = \int_a^b f(\lambda, t)\, x(t)\, dt,$$

we are evidently led to an integral equation of the above type.

† An equation of this type occurs whenever we attempt to solve a linear functional equation

$$L_s u - \lambda a(s)\, u = 0,$$

by means of a definite integral

$$u(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt.$$

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Let us suppose now that $\lambda = \lambda_1 + i\lambda_2$ and that $\mu = \lambda_1 - i\lambda_2$, the conjugate complex value, then if

$$\phi_\lambda(s) \equiv \phi_1(s) + i\phi_2(s),$$
$$\phi_\mu(s) = \phi_1(s) - i\phi_2(s),$$

we have
$$0 = \lambda_2 \int_a^b \int_a^b f(s, t)\,[\phi_1(s) + i\phi_2(s)]\,[\phi_1(t) - i\phi_2(t)]\,ds\,dt.$$

Equating real and imaginary parts we have

$$0 = \lambda_2 \int_a^b \int_a^b f(s, t)\,[\phi_1(s)\,\phi_1(t) + \phi_2(s)\,\phi_2(t)]\,ds\,dt.$$

Now the double integral is positive since $f(s, t)$ is a definite function, consequently $\lambda_2 = 0$, in other words, the singular values of λ are all real.

If $g(s, t)$ is also a definite function the singular values of λ are seen to be all positive.

§ 4. It is known that when $\kappa(s, t)$ is a real symmetric function, the values of λ for which the homogeneous integral equation

$$\phi(s) - \lambda \int_a^b \kappa(s, t)\,\phi(t)\,dt = 0 \quad \text{.....(1)},$$

can be satisfied are all real. When $\kappa(s, t)$ is not a symmetric function the singular values of λ may be complex quantities and some assumption as to the character of $\kappa(s, t)$ must be made if further information is required.

Let us suppose in the first place that

$$\kappa(s, t) = -\kappa(t, s) \quad \text{.....(2)},$$

we have then

$$\phi(t) + \lambda \int_a^b \kappa(r, t)\,\phi(r)\,dr = 0,$$

hence
$$\phi(s) + \lambda^2 \int_a^b \int_a^b \kappa(s, t)\,\kappa(r, t)\,\phi(r)\,dr\,dt = 0.$$

Now $\int_a^b \kappa(s, t)\,\kappa(r, t)\,dt$ is a symmetric function of r and t, consequently λ^2 and $\phi(s)$ are real.

Further, the above equation gives

$$\int_a^b \phi^2(s)\,ds + \lambda^2 \int_a^b \int_a^b \int_a^b \kappa(s, t)\,\kappa(r, t)\,\phi(s)\,\phi(r)\,dr\,ds\,dt = 0,$$

or
$$\int_a^b \phi^2(s)\,ds + \lambda^2 \int_a^b \psi^2(t)\,dt = 0,$$

where
$$\psi(t) = \int_a^b \kappa(s, t)\,\phi(s)\,ds.$$

This equation shows that λ^2 is negative, consequently the singular values of λ are purely imaginary quantities*.

* This is simply an extension of Weierstrass's theorem for a bilinear form.

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Next, consider the case in which

$$\kappa(s, t) = \int_a^b f(s, x)\, g(x, t)\, dx \quad \text{.....(3)},$$

$f(s, x)$ and $g(x, t)$ being *definite* functions.

It is clear that if

$$\phi(s) = \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt \quad \text{.....(4)},$$

the function

$$\psi(x) = \int_a^b g(x, t)\, \phi(t)\, dt \quad \text{.....(5)},$$

is a solution of the adjoint homogeneous equation

$$\psi(r) = \lambda \int_a^b \psi(s)\, \kappa(s, r)\, ds \quad \text{.....(6)},$$

for we have

$$\psi(r) = \int_a^b g(r, t)\, \phi(t)\, dt = \lambda \int_a^b \int_a^b \int_a^b g(r, s) f(s, x)\, g(x, t)\, \phi(t)\, ds\, dx\, dt$$

$$= \lambda \int_a^b \int_a^b g(r, s) f(s, x)\, \psi(x)\, ds\, dx$$

$$= \lambda \int_a^b \psi(x)\, \kappa(x, r)\, dx.$$

Now let

$$\left.\begin{aligned} \lambda &\equiv \lambda_1 + i\lambda_2 \\ \phi(s) &= \phi_1(s) + i\phi_2(s) \\ \psi(s) &= \psi_1(s) + i\psi_2(s) \end{aligned}\right\} \quad \text{.....(7)},$$

then

$$\psi_1(x) = \int_a^b g(x, t)\, \phi_1(t)\, dt \quad \text{.....(8)},$$

$$\psi_2(x) = \int_a^b g(x, t)\, \phi_2(t)\, dt \quad \text{.....(9)}.$$

Also since

$$\phi(s) = \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt$$

$$= \lambda \int_a^b \int_a^b f(s, x)\, g(x, t)\, \phi(t)\, dx\, dt,$$

we have

$$\phi(s) = \lambda \int_a^b f(s, x)\, \psi(x)\, dx \quad \text{.....(10)}.$$

Equating real and imaginary parts we get

$$\left.\begin{aligned} \phi_1(s) &= \lambda_1 \int_a^b f(s, x)\, \psi_1(x)\, dx - \lambda_2 \int_a^b f(s, x)\, \psi_2(x)\, dx \\ \phi_2(s) &= \lambda_1 \int_a^b f(s, x)\, \psi_2(x)\, dx + \lambda_2 \int_a^b f(s, x)\, \psi_1(x)\, dx \end{aligned}\right\} \quad \text{.....(11)}.$$

Now it follows from (8) and (9) that

$$\int_a^b [\phi_2(x)\, \psi_1(x) - \phi_1(x)\, \psi_2(x)]\, dx = 0,$$

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hence if we multiply the first of equations (11) by $\psi_2(s)$, the second by $\psi_1(s)$, integrate and subtract, we obtain

$$0 = \lambda_2 \int_a^b \int_a^b f(s, x) [\psi_1(s) \psi_1(x) + \psi_2(s) \psi_2(x)] \, ds \, dx.$$

The double integral cannot be zero since $f(s, x)$ is a *definite* function, consequently λ_2 is zero, that is λ is real.

Again, the equations

$$\psi(x) = \int_a^b g(x, t) \phi(t) \, dt,$$

$$\phi(s) = \lambda \int_a^b f(s, x) \psi(x) \, dx,$$

give

$$\int_a^b \int_a^b g(x, t) \phi(x) \phi(t) \, dx \, dt = \int_a^b \phi(x) \psi(x) \, dx$$

$$= \lambda \int_a^b \int_a^b f(s, x) \psi(s) \psi(x) \, ds \, dx.$$

The double integrals are both positive, and so λ is positive. Thus we have the theorem

If

$$\kappa(s, t) = \int_a^b f(s, x) g(x, t) \, dx,$$

where $f(s, x)$ and $g(x, t)$ are definite functions the singular values of λ for the homogeneous integral equation

$$\phi(s) - \lambda \int_a^b \kappa(s, t) \phi(t) \, dt = 0,$$

are all real and positive.

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ON THE SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS HAVING DOUBLY-PERIODIC COEFFICIENTS

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XVI. *On the Solutions of Ordinary Linear Differential Equations having Doubly-Periodic Coefficients.*

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INTRODUCTION.

The systematic study of linear differential equations, having their coefficients doubly-periodic and solutions all uniform, owes its inception to Picard, who, in a note to *Comptes Rendus** shewed that, in general, the integrals of such equations are linear combinations of doubly-periodic functions of the second species. A subsequent note from Mittag-Leffler† pointed out the theorem, now known by Picard's name‡, to the effect that in all cases there is at least *one* solution which is a doubly-periodic function of the second species. Neither of the writers however gave an account§ of the effect which multiple roots of the fundamental systems of period equations have on the analytic form of the integrals. This remained for Floquet|| who, by means of an induction, is able to establish that in the most general case the integrals are linear functions of polynomials in u and v (I, § 35) having coefficients which are doubly-periodic of the second species. But the investigations of this writer have the defect that no precise information as to the degrees of these polynomials in u and v is attained: we are never sure how many of the coefficients are evanescent. Since Floquet many writers have dealt with the subject, notably Jordan¶, Picard**, Schlesinger†† and Forsyth‡‡; but nowhere is the point just mentioned made definite.

In section I. of the present memoir we attempt to make good the omission just noticed. The method used may best be described as the two dimensional analogue of that which would have to be employed for the case of a single period, supposing the

* t. xc. (1880), pp. 128—131.

† *Comptes Rendus*, t. xc. pp. 299, 300.

‡ This is only just since Picard in a previous note to the same publication (pp. 293—295) shews it to be true for equations of the second order.

§ We refer to equations whose order is general. Picard in the place just quoted considers equations of the second order.

|| *Comptes Rendus*, t. xcviii. (1884), pp. 82—85; *Ann de l'Éc. Norm. Sup.*, iii.me séries, t. i. (1884), pp. 181—238.

¶ *Cours d'Analyse*, t. iii.

** *Traité d'Analyse*, t. iii. (1896), pp. 406—417.

†† *Theorie der Linearen Differentialgleichungen*, ii. 2 (1898), pp. 403—424.

‡‡ *Theory of Differential equations*, Vol. iv. (1902), pp. 441—477. In addition to the writers quoted mention may be made of a paper by Stenberg, *Acta Mathematica*, b. 15, pp. 259—278.

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theory of elementary divisors unknown. There are however two directions in which we make the problem more general than that considered by previous writers. In the first place we do not limit ourselves to the case in which the coefficients of our equation are mere elliptic functions. We adopt the wider condition that they should have for their singularities a reducible set of points, and then, thanks to the theorem quoted in I, § 1, we are able to obtain our results with no more difficulty than when their number is finite. The second generalisation follows as a result of our investigations in §§ 1—5, where we find that the relation

$$aba^{-1}b^{-1} = 1,$$

on which depends the theory of the equations studied by Picard, is true of a much more general class. The only drawback is that we are limited to the consideration of particular branches of solutions, localised in a doubly-periodic region (§ 7) and are therefore unable to define precisely the nature of the remaining ones.

The defect, just mentioned, being kept in mind, section II. is devoted to the consideration of a very particular class of solutions—exclusive solutions as we call them (§ 8). A detailed discussion of Halphen equations (§ 10) is given, and as a result of it we are able to see that Halphen's conclusions in respect to the particular case are still true of our more general one: we are able however to state our results in a somewhat neater form, by the aid of the artifice of § 12. Finally we consider the most general possible exclusive solution these equations can have; finding that it must be a solution of one of Halphen type (§ 22).

We may observe in conclusion that the results of section II. justify in some measure the more general point of view adopted in section I.: in effect the solutions dealt with are integrals of equations of the type there discussed.

I.

§ 1. Suppose that we have a linear differential equation of order m, say

$$\frac{d^m w}{dz^m} = p_1(z)\frac{d^{m-1}w}{dz^{m-1}} + \ldots + p_r(z)\frac{d^{m-r}w}{dz^{m-r}} + \ldots + p_m(z),$$

where $p_1 p_2, \ldots p_m$ are doubly-periodic functions (common periods ω, ω') which are (1) uniform and (2) such that their singularities are a reducible set of points: in the sequel we will describe functions of this type merely as uniform doubly-periodic* functions.

Instead of restricting ourselves to the case in which the solutions are all uniform, it will at least be instructive if, in the first instance, we consider the most general kind of solutions possible. Take therefore any base point z_0, which is not a singularity of any of the coefficients $p_1, p_2, \ldots p_m$. At this point there will exist m linearly independent power series in $(z - z_0)$ which are solutions of the equation; and the radius of convergence of each will be $\nless |c - z_0|$ where "c" is that singularity of the p's which is nearest to z_0. Let us call these

$$P_1(z - z_0),\ P_2(z - z_0), \ldots P_m(z - z_0),$$

* These functions include *elliptic* functions as a particular case: the functions $\wp(\wp(u))$, $\wp(\wp(\wp(\wp u))$ etc., where $\wp$ is the usual Weierstrassian elliptic function, are doubly-periodic but not elliptic.

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where
$$P_r(z-z_0) = \sum_{n=0}^{\infty} \lambda_{rn}(z-z_0)^n.$$

Any one of them $P_r(z-z_0)$ together with its continuations will constitute a certain monogenic* function (f_r) whose singularities are among those of the coefficients of the equation. As the latter consist of m reducible sets they are themselves a reducible set (S); and so the points of the plane, other than these, form a completely open region or domain† (Π): obviously (Π) is doubly-periodic‡. In general the set of singularities of each function f_r coincides with (S); but in particular cases it may happen that the points of a certain component of (S) are apparent singularities, so far as regards (f_r); the set will then consist of those points of (S) which do not belong to this component.

It is a fundamental property of monogenic functions that if $\bar{\omega}_r(z-\lambda)$ be an element of (f_r) whose base is λ, that is to say an element derived by continuation from $P_r(z-z_0)$, then it also satisfies§ the equation. This property of (f_r) is expressed shortly in the statement that (f_r) is a solution of the equation.

§ 2. Let us now fix on a definite path (γ) from z_0 to $z_0+\omega$, which passes through no singularity of the coefficients; and let us continue the series $P_1(z-z_0), \ldots P_m(z-z_0)$‖ along it so as to obtain new power series in $z-(z_0+\omega)$ say

$$P_r(z-(z_0+\omega)\,|\,\gamma\,|\,z_0) \equiv \sum_0^{\infty} \mu_{rn}^{\gamma}(z-(z_0+\omega))^n.$$
$$(r=1,2,\ldots m)\,|\,c-z_0\,|>|\,z-(z_0+\omega)\,|$$

Each of these, as we have said, satisfies the differential equation of § 1. Thus, if we write $z=\omega+t$, we see that

$$\sum_0^{\infty} \mu_{rn}^{\gamma}(t-z_0)^n \qquad (r=1,2,\ldots m)$$

are solutions of the equation

$$\frac{d^m w}{dt^m} = p_1(t+\omega)\frac{d^{m-1}w}{dt^{m-1}} + \ldots + p_m(t+\omega),$$

i.e. of
$$\frac{d^m w}{dt^m} = p_1(t)\frac{d^{m-1}w}{dt^{m-1}} + \ldots + p_m(t),$$

since the coefficients are periodic. It follows therefore, replacing t by z, that each of the power series $\sum_0^{\infty} \mu_{rn}^{\gamma}(z-z_0)^n$ (say $Q_r(z-z_0)$) which have z_0 for their base and a radius of convergence $\nless |\,c-z_0\,|$ are solutions of our equation.

* "Monogenic function" is here used in precisely the same sense as "analytic function" in Harkness and Morley's *Introduction to the Theory of Analytic Functions* (e.g. p. 154, § 90 of the 1898 edition). Cf. Baker, *Proc. Lond. Math. Soc.* 2nd series, Vol. IV. p. 116.

† These words are used in the strict sense of Young, *Sets of Points*, Ch. IX. The reader who is acquainted with the theory of regions will not find much difficulty in establishing, by means of an induction, that if R is any region and β a reducible set of points internal to it, then the set of points $(R-\beta)$ which belong to R but not to β, is a region. This theorem is a particular case of a more general one.

‡ A region or domain is said to be doubly-periodic (periods ω, ω') when it is such that if z_0 be a point of it $z_0 \pm \omega$, $z_0 \pm \omega'$ are points of it.

§ At all points which are not singularities of the coefficients, i.e. not apparent singularities above-mentioned. In such a case the expression—in the strict sense at any rate—is meaningless. We always suppose this taken into account when speaking of a function as satisfying a differential equation.

‖ This continuation is possible because (Π) is a domain.

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These series are all linearly independent. For if not we would have a relation of the form

$$\sum_1^m l_r \sum_0^\infty \mu_{rn}{}^\gamma (z-z_0)^n = 0, \qquad |c-z_0| > |z-z_0|$$

where the l's are certain constants which do not all vanish. Consequently we would have

$$\sum_1^m l_r \sum_0^\infty \mu_{rn}{}^\gamma (z-(z_0+\omega))^n = 0, \qquad |c-z_0| > |z-(z_0+\omega)|$$

or

$$\sum_1^m l_r P_r (z-(z_0+\omega) \,|\, \gamma \,|\, z_0) = 0.$$

Continuing the left-hand member of this equation back along the path (γ) to z_0* we deduce

$$\sum_1^m l_r P_r (z-z_0) = 0,$$

which is contrary to our hypothesis that the series $P(z-z_0)$ are linearly independent. It follows from the general theory that we have

$$Q_r(z-z_0) = a_{r1}P_1(z-z_0) + a_{r2}P_2(z-z_0) + \ldots + a_{rm}P_m(z-z_0), \qquad (r=1, 2, \ldots m)$$

where the a's are constants and the determinant of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1m} \\ a_{21} & a_{22} & \ldots\ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{r1} & a_{r2} & \ldots\ldots & a_{rm} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mm} \end{pmatrix},$$

called a† matrix of the system for the period ω is not zero: symbolically this result may be expressed as

$$Q(z-z_0) = aP(z-z_0)$$

where (a) is the matrix just written.

§ 3. It is obvious that a similar result applies to the other period (ω'), viz. if we take a definite path (γ') from z_0 to $z_0 + \omega'$ and continue along this, the series $P_r(z-z_0)$ becomes

$$P_r(z-(z_0+\omega') \,|\, \gamma' \,|\, z_0): \qquad (r=1, 2, \ldots m) \,|\, c-z_0| > |z-(z_0+\omega')$$

the latter gives rise to a power series

$$\sum_0^\infty \mu_{rn}{}^{\gamma'} (z-z_0)^n,$$

* We can always suppose that our continuation along (γ) from z_0 to $(z_0+\omega)$ has been by a standard chain. Harkness and Morley, *op. cit.* § 90, p. 156.

† Of course it is obvious that this matrix may, and in general will, vary with the path γ.

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say $Q_r'(z-z_0)$, which is a solution of the equation, and such that

$$Q_r'(z-z_0)=b_{r1}P_1(z-z_0)+b_{r2}P_2(z-z_0)+\ldots+b_{rm}P_m(z-z_0);$$
$$(r=1,2,\ldots m)$$

or briefly
$$Q'(z-z_0)=bP(z-z_0),$$

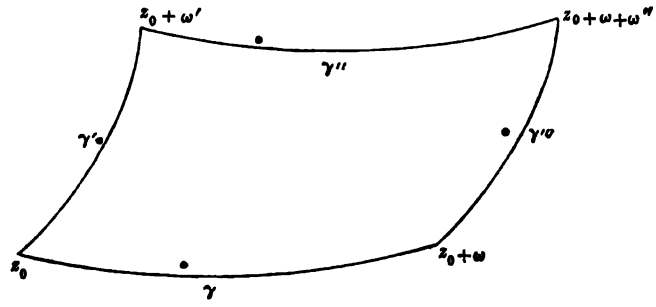
where b is a matrix of the system for the period (ω').

§ 4. Consider now what happens when we continue the series $P_r(z-(z_0+\omega)\,|\,\gamma\,|\,z_0)$ from $(z_0+\omega)$ to $(z_0+\omega)+\omega'$ along the path γ''' congruent* with γ'. Since

$$P_r(z-(z_0+\omega)\,,\,\gamma\,|\,z_0),$$

is obtained from $Q_r(z-z_0)$, i.e. from

$$a_{r1}P_1(z-z_0)+a_{r2}P_2(z-z_0)+\ldots+a_{rm}P_m(z-z_0)$$



by writing $z-\omega$ for z, it is clear that

$$P_r(z-(z_0+\omega+\omega')\,|\,\gamma'''\,|\,z_0+\omega\,|\,\gamma\,|\,z_0)$$

is obtained from
$$a_{r1}P_1(z-(z_0+\omega')\,|\,\gamma'\,|\,z_0)$$
$$+a_{r2}P_2(z-(z_0+\omega')\,|\,\gamma'\,|\,z_0)+$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+a_{rm}P_m(z-(z_0+\omega')\,|\,\gamma'\,|\,z_0),$$

by the same substitution: i.e. is obtained from

$$a_{r1}Q_1'(z-z_0)+a_{r2}Q_2'(z-z_0)+\ldots+a_{rm}Q_m'(z-z_0),$$

or from
$$\sum_{s=1}^{m} a_{rs}\sum_{t=1}^{n} b_{st}P_t(z-z_0),\qquad (r=1,2,\ldots m)$$

by the substitution of $(z-\omega-\omega')$ for z. This result may be symbolically written

$$P(z-(z_0+\omega+\omega')\,|\,\gamma'''\,|\,z_0+\omega\,|\,\gamma\,|\,z_0)=abP(z-(z_0+\omega+\omega')).$$

In a similar manner if (γ'') is the path congruent with (γ) which joins $(z_0+\omega')$ and $(z_0+\omega+\omega')$ we prove that

$$P(z-(z_0+\omega+\omega')\,|\,\gamma''\,|\,z_0+\omega'\,|\,\gamma'\,|\,z_0)=baP(z-(z_0+\omega+\omega')).$$

Moreover, since $P(z-z_0)$, continued to $(z_0-\omega)$ or $(z_0-\omega')$ along a path congruent with

* i.e. γ''' consists of points congruent with those of γ'. For the definition of congruent points *vide* Harkness and Morley, *op. cit.* p. 243.

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γ or γ' becomes $a^{-1}P(z-(z_0-\omega))$ or $b^{-1}P(z-(z_0-\omega'))$ as the case may be, we shew similarly that

$$P(z-(z_0+\omega')\,|\,\gamma''\,|\,z_0+\omega+\omega'\,|\,\gamma'''\,|\,z_0+\omega\,|\,\gamma\,|\,z_0)=aba^{-1}P(z-(z_0+\omega')),$$

$$P(z-z_0\,|\,\gamma'\,|\,z_0+\omega'\,|\,\gamma''\,|\,z_0+\omega+\omega'\,|\,\gamma'''\,|\,z_0+\omega\,|\,\gamma\,|\,z_0)=aba^{-1}b^{-1}P(z-z_0).$$

§ 5. From the equation just written it appears that if the integrals of our equation are all uniform we must have

$$aba^{-1}b^{-1}=1;$$

but the converse is not necessarily true. We proceed to find a necessary and sufficient condition that this relation between the period matrices a and b may hold.

Imagine the pseudo-parallelogram, whose sides are $\gamma, \gamma'', \gamma''', \gamma'$, shrunk in any manner so as not to pass over any singularity of the coefficients $p_1, p_2, \ldots p_m$; in this way the interior will become a certain region or set of points Θ. Since the set (S) is closed Θ may be chosen so that the points external to it and congruent sets is a region: for instance, if the singularities in a parallelogram are finite in number, say $c_1, c_2 \ldots c_t$, Θ may be taken to be the lines joining c_1c_2, c_2c_3, $\ldots$ $c_{t-1}c_t$ provided they do not cross. In any case however this region is doubly-periodic: let us call it Φ. The necessary and sufficient condition above referred to is that *the branches of $f_1, f_2, \ldots f_m$, deduced by continuation from the series $P_1(z-z_0)$, $\ldots$ $P_m(z-z_0)$ and considered as localised in Φ, should be uniform.*

The condition is obviously sufficient; for by hypothesis the series $P(z-z_0)$ when continued round $\gamma, \gamma''', \gamma'', \gamma'$ returns unchanged, i.e. by the last equation of the last paragraph

$$aba^{-1}b^{-1}=1.$$

As to the necessity of the condition, consider any closed path Γ passing through z_0 and lying within Φ. Without passing over any of the points of S, this may be deformed into a path consisting of, firstly a path joining z_0 to $z_0+p_1\omega$, which consists of $|p_1|$ paths congruent with γ; then a path joining $z_0+p_1\omega$ to $(z_0+p_1\omega)+q_1\omega'$ consisting of $|q_1|$ paths congruent with γ'; then a path joining $z_0+p_1\omega+q_1\omega'$ and $(z_0+p_1\omega+q_1\omega')+p_2\omega$ consisting of $|p_2|$ paths congruent with γ; and so on. Thus the series $P(z-z_0)$ when continued round Γ become

$$a^{p_1}b^{q_1}a^{p_2}b^{q_2}\ldots a^{p_s}b^{q_s}P(z-z_0),$$

where $\sum_1^s p_r=\sum_1^s q_r=0$, since the path is closed.

Supposing now that $$aba^{-1}b^{-1}=1,$$

we have of course $ab=ba$. As matrices obey the associative law it is easily shewn from this that a product of λ a's and μ b's written in *any* order is equal to $a^\lambda b^\mu$. Thus the series $P(z-z_0)$ when continued round Γ become

$$a^{\sum_1^s p_r}b^{\sum_1^s q_r}P(z-z_0),$$

i.e. because $$P(z-z_0).$$

Thus the condition stated above is at once necessary and sufficient.

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Equations which possess this property will in future be called *generalised Picard equations*: they include ordinary Picard equations, i.e. those whose integrals are all uniform, as a particular case.

§ 6. It is an easy matter to express the condition

$$aba^{-1}b^{-1} = 1,$$

in terms of the behaviour of the series $P(z-z_0)$ when continued round the various singularities of the coefficients when the number of the latter contained in a given period parallelogram is finite. For we have seen that when they are continued round a closed path enclosing Θ, the series $P(z-z_0)$ become $aba^{-1}b^{-1}P$. But this path may be shrunk so that it becomes a series of loops enclosing successively the various singularities $c_1, c_2 \dots c_t$ contained in Θ. Consequently, if the series $P(z-z_0)$ become $m_r P(z-z_0)$ after continuation round the loop enclosing c_r, they will become

$$m_1 m_2 \dots m_t P(z-z_0),$$

after describing the series of loops just mentioned. Thus we have

$$aba^{-1}b^{-1} = m_1 m_2 \dots m_t.$$

The condition that the equation should be one of generalised Picard type is therefore expressed by

$$m_1 m_2 \dots m_t = 1.$$

In particular, if there is only one singularity c_1 we must have

$$m_1 = 1,$$

i.e. in the case of a single singularity the only generalised Picard equations are those for which the integrals are all uniform functions, i.e. those of the ordinary Picard type.

§ 7. Suppose now that our equation is of the generalised Picard type and that Φ is the region within which the branches of $f_1, f_2, \dots f_m$ are uniform: Φ *may be called a doubly-periodic region* for the equation or simply a doubly-periodic region. Denote by $g_1(z)$, $g_2(z), \dots g_m(z)$, those branches which are deduced from $P_1(z-z_0), \dots P_m(z-z_0)$ by continuation within Φ*. These localised functions will be single-valued and will have a definite value for each value of the argument z corresponding to points within this region, though of course $f_1, f_2, \dots f_m$ may have any number. *We say that the g's are a fundamental system of solutions of the equation, localised in* Φ, or briefly a fundamental system of solutions, it being understood throughout that such systems are so localised. From what has been said in §§ 2 and 3,

$$g(z+\omega) = a\, g(z),$$

$$g(z+\omega') = b\, g(z),$$

where now we can speak of a and b as *the* matrices for the periods ω, ω' respectively, since, as we have remarked, the g's are single-valued.

* We suppose z_0 a point within Φ.

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Now consider any linear combination of the g's, say

$$F(z) = \lambda_1 g_1(z) + \lambda_2 g_2(z) + \ldots + \lambda_m g_m(z),$$

or as we can write symbolically

$$F(z) = \lambda\, g(z).$$

The function $F(z)$ is of course only defined in Φ; but since the g's are expressible as power series about any point of the region, F is so expressible, and so can, like the g's, be regarded as a branch of a monogenic function localised in Φ.

Obviously
$$F(z+\omega) = \lambda a\, g(z),$$
$$F(z+\omega') = \lambda b\, g(z).$$

Let us enquire whether we can choose the numbers λ in such a way that $F(z)$ is a doubly-periodic function of the second species. If $F(z)$ is of this character we shall have

$$F(z+\omega) = \theta\, F(z), \quad F(z+\omega') = \theta'\, F(z),$$

where θ and θ' are certain constants. Recalling that the g's are linearly independent, it is seen that we must have

$$\left.\begin{aligned} \sum_{r=1}^{n} \lambda_r a_{rs} &= \theta\ \lambda_s \\ \sum_{r=1}^{n} \lambda_r b_{rs} &= \theta'\ \lambda_s \end{aligned}\right\} (s = 1, 2, \ldots m) \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

It follows then that if the λ's are not all zero—and this is the only case of any use—θ and θ' are necessarily such that the system of determinants

$$\left\|\begin{matrix} a_{11}-\theta & a_{12} & a_{13} & \ldots\ldots & a_{1m} & b_{11}-\theta' & b_{12} & \ldots\ldots & b_{1m} \\ a_{21} & a_{22}-\theta & a_{23} & \ldots\ldots & a_{2m} & b_{21} & b_{22}-\theta' & \ldots\ldots & b_{2m} \\ a_{31} & a_{32} & a_{33}-\theta & \ldots\ldots & a_{3m} & b_{31} & b_{32} & \ldots\ldots & b_{3m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{r1} & a_{r2} & a_{r3} \ldots & a_{rr}-\theta & a_{rm} & b_{r1} & b_{r2} \ldots & b_{rr}-\theta' & \ldots b_{rm} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mm}-\theta & b_{m1} & b_{m2} & \ldots\ldots & b_{mm}-\theta' \end{matrix}\right\|,$$

or as we write for brevity

$$\| a_{1m}, \theta : b_{1m}, \theta' \|,$$

all vanish. Conversely if we can find values θ and θ' so that this actually does happen, we can choose at least one system of values λ which are not all zero, and which satisfy the system of equations (i).

We are led therefore to consider systems of determinants of this type. This is our object in the next few paragraphs, where, in view of future applications, we consider these

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systems apart from the theory of differential equations, in order that we may appeal to it, not only immediately, but also at a later stage, when the circumstances are somewhat different. It would be interesting to consider the theory of these systems for its own sake; but as we have in view its applications to generalised Picard equations, we here confine ourselves to the bare minimum of properties necessary for our purpose.

§ 8. Let x and y* be any two matrices of m rows and columns, so that x_{rs} is the sth element of the rth row of x, and y_{rs} the corresponding element of y. The system of determinants

$$\| x_{1m}, \quad \theta : y_{1m}, \quad \theta' \|$$

will be called a *conjoint system*, provided that neither $|x|$ nor $|y|$† are zero and that

$$xy = yx^{\ddagger}_{+}.$$

For example, the system

$$\| a_{1m}, \quad \theta : b_{1m}, \quad \theta' \|$$

considered in the preceding paragraph is conjoint. In fact we have seen that $|a| \neq 0$, $|b| \neq 0$ and as

$$aba^{-1}b^{-1} = 1,$$

it follows that

$$ab = ba.$$

If for any values κ of θ and κ' of θ' the determinants

$$\| x_{1m}, \quad \kappa : y_{1m}, \quad \kappa' \|$$

are all zero, we will say that κ, κ' are an *associated pair of roots* of the system of equations

$$\| x_{1m}, \quad \theta : y_{1m}, \quad \theta' \| = 0.$$

It may happen in any particular case that not only all these determinants but also all their minors up to the $(\iota - 1)$th inclusive vanish, whilst the ιth do not all do so; we then speak of "ι" as the *index* of the pair of roots (κ, κ') of the above system.

§ 9. We proceed to prove that associated pairs of roots of conjoint systems exist and to give a method of finding them and their indices.

Using the notation

$$| x_{1m}, \quad \theta |$$

for the determinant

$$\begin{vmatrix} x_{11} - \theta & x_{12} & x_{13} \dots x_{1m} \\ x_{21} & x_{22} - \theta & x_{23} \dots x_{2m} \\ x_{31} & x_{32} & \dots\dots x_{3m} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ x_{m1} & x_{m2} \dots\dots\dots & x_{mm} - \theta \end{vmatrix}$$

* The matrices a and b will be used to refer to period matrices only. To prevent confusion we use x and y, ξ and η so long as the theory of conjoint systems is considered without reference to the differential equation.

† $|x|$ is a shorthand notation for the determinant of the x's.

‡ It is on account of this intimate connection between the x's and y's that the adjective conjoint has been used.

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it is clear that if (κ, κ') is an associated pair of roots of

$$\| x_{1m}, \quad \theta : y_{1m}\theta' \| = 0,$$

then κ must be a root of

$$| x_{1m}, \quad \theta | = 0^*.$$

Suppose, therefore, that we take κ to be one of the roots of this equation, and let the minors of order τ in

$$| x_{1m}, \quad \kappa |$$

be the first which do not all vanish; moreover, let one of these non-vanishing τth minors be contained in the n_1th, n_2th, ... $n_{m-\tau}$th columns and in the $p_{\tau+1}$th, $p_{\tau+2}$th, ... p_mth rows of the determinant; $n_1, n_2, \ldots n_m$, $p_1, p_2, \ldots p_m$, each consisting of the first m integers arranged in a certain order.

Every set of values of the λ's which satisfy

$$\lambda_{n_r}\kappa = \lambda_1 x_{1n_r} + \lambda_2 x_{2n_r} + \ldots + \lambda_m x_{mn_r} \quad (r = 1, 2, \ldots m - \tau) \ldots\ldots\ldots \text{(ii)}$$

will necessarily satisfy the system

$$\lambda_r\kappa = \lambda_1 x_{1r} + \lambda_2 x_{2r} + \ldots + \lambda_m x_{mr} \quad (r = 1, 2, \ldots m) \ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

Hence this latter system may be solved by taking any numbers we please for

$$\lambda_{p_1}, \lambda_{p_2}, \ldots \lambda_{p_\tau},$$

and then determining the remainder of the λ's as definite linear functions of them, say

$$\lambda_{p_{\tau+s}} = k_{s1}\lambda_{p_1} + k_{s2}\lambda_{p_2} + \ldots + k_{s\tau}\lambda_{p_\tau}. \qquad (s = 1, 2, \ldots m - \tau)$$

Now let us write

$$\mu_r = \lambda_1 y_{1r} + \lambda_2 y_{2r} + \ldots + \lambda_m y_{mr}.$$

Then if we take the system of equations (iii) and multiply them by $y_{1r}, y_{2r} \ldots y_{mr}$ respectively and add, we obtain

$$\begin{aligned}\mu_r\kappa = \lambda_1 &(x_{11} y_{1r} + x_{12} y_{2r} + \ldots + x_{1m} y_{mr})\\ + \lambda_2 &(x_{21} y_{1r} + x_{22} y_{2r} + \ldots + x_{2m} y_{mr})\\ + \lambda_3 &(\qquad\qquad\qquad\qquad)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ + \lambda_m &(x_{m1} y_{1r} + x_{m2} y_{2r} + \ldots + x_{mm} y_{mr}).\end{aligned}$$

Since

$$xy = yx,$$

it follows that

$$\sum_{t=1}^{m} x_{rt} y_{ts} = \sum_{t=1}^{m} y_{rt} x_{ts}.$$

Consequently

$$\begin{aligned}\mu_r\kappa = \lambda_1 &(y_{11} x_{1r} + y_{12} x_{2r} + \ldots + y_{1m} x_{mr})\\ + \lambda_2 &(y_{21} x_{1r} + y_{22} x_{2r} + \ldots + y_{2m} x_{mr})\\ + \lambda_3 &(\qquad\qquad\qquad\qquad)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ + \lambda_m &(y_{m1} x_{1r} + y_{m2} x_{2r} + \ldots + | y_{mm} + x_{mr} |),\end{aligned}$$

or

$$\mu_r\kappa = \mu_1 x_{1r} + \mu_2 x_{2r} + \ldots + \mu_m x_{mr} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

* It may be unnecessary to point out that as $| x_{1m}, 0 | = | x | \neq 0$ then $\kappa \neq 0$. Similarly $\kappa' \neq 0$.

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Now comparing the systems (iii) and (iv) we are not justified in assuming—as we might at first sight be tempted to do—that λ_r/μ_r has the same value for $r = 1, 2, \ldots m$. All that we can conclude is that

$$\mu_{p_{\tau+s}} = k_{s1}\,\mu_{p_1} + k_{s2}\,\mu_{p_2} + \ldots + k_{s\tau}\,\mu_{p_\tau}.$$

If however we have

$$\frac{\mu_{p_r}}{\lambda_{p_r}} = \theta' \qquad (r = 1, 2, \ldots \tau) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

it clearly follows that

$$\frac{\mu_{p_{\tau+s}}}{\lambda_{p_{\tau+s}}} = \theta' \quad (s = 1, 2, \ldots m - \tau) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}.$$

§ 10. Let us see whether we can choose the numbers λ so as to ensure this. Take the τ equations

$$\lambda_1\, y_{1p_r} + \lambda_2\, y_{2p_r} + \ldots + \lambda_{p_r}(y_{p_rp_r} - \theta') + \ldots + \lambda_m\, y_{mp_r} = 0 \quad (r = 1, 2, \ldots \tau)$$

together with the system (ii); we have thus m equations; label them (vii).

In order that we may be able to choose numbers $\lambda_1, \lambda_2 \ldots \lambda_m$ so as to satisfy (vii), we see that θ' must satisfy

$$\begin{vmatrix} y_{1p_1} & y_{1p_2} \ldots\ldots & y_{1p_\tau} & x_{1n_1} & x_{1n_2} \ldots\ldots & x_{1n_{m-\tau}} \\ y_{2p_1} & y_{2p_2} \ldots\ldots & y_{2p_\tau} & x_{2n_1} & x_{2n_2} \ldots\ldots & x_{2n_{m-\tau}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & x_{n_1n_1} - \kappa & \ldots & \ldots \\ y_{p_1p_1} - \theta' & y_{p_1p_2} \ldots\ldots & \ldots & x_{p_1n_1} & \ldots & x_{p_1n_{m-\tau}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ y_{p_2p_1} & y_{p_2p_2} - \theta' & \ldots & \ldots & \ldots & x_{p_2n_{m-\tau}} \\ \ldots & \ldots & \ldots & \ldots & x_{n_2n_2} - \kappa & \ldots\ x_{n_2n_{m-\tau}} \\ y_{mp_1} & y_{mp_2} \ldots\ldots & y_{mp_\tau} & x_{mn_1} & \ldots & x_{mn_{m-\tau}} \end{vmatrix} = 0 \ldots\ldots\text{(viii)}.$$

As the coefficient of $(\theta')^\tau$ in this equation does not vanish it is clear that it is of degree τ in θ'. Let κ' be one of its roots, and suppose that κ' is written for θ' in the determinant which is the left-hand member of (viii), the σth minors are the first which do not all vanish.

The system of equations (vii) may be satisfied by taking arbitrarily σ of the λ's, say $\lambda_{q_1}, \lambda_{q_2}, \ldots \lambda_{q_\sigma}$, and then determining the remainder as definite linear functions of these, say

$$\lambda_{q_{\sigma+r}} = l_{r_1}\,\lambda_{q_1} + l_{r_2}\,\lambda_{q_2} + \ldots + l_{r_\sigma}\,\lambda_{q_\sigma} \qquad (r = 1, 2, \ldots m - \sigma);$$

where $q_1, q_2, \ldots q_m$ are the integers $1, 2 \ldots m$ arranged in a certain order. Now whatever values have been chosen for $\lambda_{q_1}, \lambda_{q_2}, \ldots \lambda_{q_\sigma}$, the system of equations (iii) is satisfied since that labelled (ii) is; moreover, from the first set of the equations (vii) we have

$$\frac{\mu_{p_r}}{\lambda_{p_r}} = \kappa' \qquad (r = 1, 2, \ldots \tau).$$

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From the remark made at the end of the preceding paragraph it follows that

$$\frac{\mu_r}{\lambda_r} = \kappa' \qquad (r = 1, 2, \dots m);$$

and therefore that any system of values of the λ's, chosen as above, satisfies the system of $2m$ equations

$$\left.\begin{aligned} \lambda_r \kappa &= \lambda_1 x_{1r} + \lambda_2 x_{2r} + \lambda_3 x_{3r} + \dots + \lambda_m x_{mr} \\ \lambda_r \kappa' &= \lambda_1 y_{1r} + \lambda_2 y_{2r} + \dots\dots\dots\dots + \lambda_m y_{mr} \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots\text{(ix)}.$$

We see from this that the system

$$\| x_{1m}, \kappa : y_{1m}, \kappa' \|$$

all vanish, i.e. *with every root κ of*

$$| x_{1m}, \theta | = 0$$

we can associate at least one number κ', so that (κ, κ') are an associated pair of roots of the system of equations

$$\| x_{1m}, \theta : y_{1m}, \theta' \| = 0.$$

The number κ' is of course a root of

$$| y_{1m}, \theta' | = 0.$$

It is not difficult to prove that the index ι of the pair is equal to σ. In the first place, since σ of the λ's can be chosen arbitrarily and then the rest determined as above, so as to satisfy the system of $2m$ equations (ix), we see that $\iota \geqslant \sigma$. Again, since there is at least σth minor of a particular determinant of the system

$$\| x_{1m}, \kappa : y_{1m}, \kappa' \|$$

which does not vanish, viz. a σth minor of the determinant on the left-hand side of (viii) after θ' has been replaced by κ', we cannot have $\iota > \sigma$. The only remaining alternative is therefore $\iota = \sigma$.

§ 11. One more point has to be attended to: we must convince ourselves that if we take all the roots of

$$| x_{1m}, \theta | = 0$$

in succession and proceed as above, we determine all possible pairs of roots.

Let (κ, κ') be an associated pair of roots, no matter how determined, so that we have

$$\| x_{1m}, \kappa : y_{1m}, \kappa' \| = 0.$$

In the first place it is clear that κ is a root of

$$| x_{1m}, \theta | = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(x)}.$$

Suppose that the τth minors of $| x_{1m}, \kappa |$ are the first which do not all vanish; and, adopting the notation of § 9, let one of these non-vanishing minors belong to the n_1th, n_2th, ... $m_{m-\tau}$th columns and the $p_{\tau+1}$th, $p_{\tau+2}$th, ... p_mth rows. Since the determinant which the left-hand member of (viii) becomes after writing κ' for θ', is one of the system

$$\| x_{1m}, \kappa : y_{1m}, \kappa' \|$$

it vanishes, i.e. κ' is a root of (viii). The method of §§ 9 and 10 may therefore be regarded as a practical method of determining all associated pairs of roots.

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In passing we may notice that associated pairs may be arrived at in a similar way by starting with any root κ' of

$$| y_{1m}, \theta' | = 0 \quad \text{(xi)},$$

and then determining κ on lines analogous to the above; in fact the system is not altered if we interchange x's and y's, θ and θ'. The remarks we have just made prove conclusively that the pairs arrived at in this way are the same as before.

In particular, on referring back to the results of the preceding paragraph, it becomes apparent that if either of the equations (x) or (xi) has all its roots unequal, we shall have m distinct associated pairs of roots of the system

$$\| x_{1m}, \theta : y_{1m}, \theta' \| = 0.$$

The converse of this however is not true: for instance, if $m = 3$ we have the three distinct *pairs* (κ_1, κ_1'), (κ_2, κ_1'), (κ_1, κ_2'), which are associated roots of

$$\left\| \begin{matrix} \kappa_1 - \theta & 0 & 0 & \kappa_1' - \theta' & 0 & 0 \\ 0 & \kappa_2 - \theta & 0 & 0 & \kappa_1' - \theta' & 0 \\ 0 & 0 & \kappa_1 - \theta & 0 & 0 & \kappa_2' - \theta' \end{matrix} \right\| = 0,$$

but κ_1 is a double root of

$$\begin{vmatrix} \kappa_1 - \theta & 0 & 0 \\ 0 & \kappa_2 - \theta & 0 \\ 0 & 0 & \kappa_1 - \theta \end{vmatrix} = 0,$$

and κ_1' is a double root of

$$\begin{vmatrix} \kappa_1' - \theta' & 0 & 0 \\ 0 & \kappa_1' - \theta' & 0 \\ 0 & 0 & \kappa_2' - \theta' \end{vmatrix} = 0.$$

In any case however we are certain of the existence of at least one associated pair of roots.

§ 12. Having considered these associated pairs we pass on to consider a class of conjoint systems whose members are all closely connected with one another.

Let ρ be a matrix of m rows and columns

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \cdots & \rho_{1m} \\ \rho_{21} & \rho_{22} & \rho_{23} \cdots & \rho_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{r1} & \rho_{r2} & \rho_{r3} \cdots & \rho_{rm} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{m1} & \rho_{m2} & \rho_{m3} \cdots & \rho_{mm} \end{pmatrix},$$

which is such that $| \rho | = R$ does not vanish. Suppose, moreover, that ξ and η are two matrices such that

$$\xi\rho = \rho x,$$

$$\eta\rho = \rho y.$$

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The system $$\| \xi_{1m}, \theta : \eta_{1m}, \theta' \|$$
is said to be a *transform* of the system
$$\| x_{1m}, \theta : y_{1m}, \theta' \|.$$
Since
$$x\rho^{-1} = \rho^{-1}\xi,$$
$$y\rho^{-1} = \rho^{-1}\eta,$$
it is obvious that the latter is a transform of the former.

We proceed to prove two theorems of considerable importance.

§ 13. *Theorem I.* The transform of a conjoint system is a conjoint system.

Adopting the notation already used, let
$$\| x_{1m}, \theta : y_{1m}, \theta' \|$$
be the original system, and
$$\| \xi_{1m}, \theta : \eta_{1m}, \theta' \|$$
its transform. In the first place we observe that
$$|\xi| = |x|, \quad |y| = |\eta|,$$
so that
$$|\xi| \neq 0, \quad |\eta| \neq 0.$$
Again we have
$$\eta\rho = \rho y,$$
and so
$$\eta\rho x = \rho y x,$$
i.e. since
$$\rho x = \xi\rho \text{ and } xy = yx,$$
$$\eta\xi\rho = \rho x y.$$
Consequently
$$\eta\xi\rho = \xi\rho y$$
$$= \xi\eta\rho.$$
As $|\rho| \neq 0$ we must have
$$\eta\xi = \xi\eta,$$
which proves the proposition.

§ 14. *Theorem II.* The associated pairs of roots of any conjoint system are the same as those of all its transforms and the index of any pair is the same for each.

Take a determinant of the original system which involves p columns of the x's and m-p columns of the y's, say

$$\begin{vmatrix} x_{1n_1} & x_{1n_2} & \cdots\cdots & x_{1n_p} & y_{1n_{p+1}} & & y_{1n_{p+2}} & \cdots & y_{1n_m} \\ x_{2n_1} & x_{2n_2} & \cdots\cdots & x_{2n_p} & y_{2n_{p+1}} & & y_{2n_{p+2}} & \cdots & y_{2n_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n_1n_1} - \theta & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & y_{n_{p+1}n_{p+1}} - \theta' & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & x_{n_2n_2} - \theta & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{mn_1} & x_{mn_2} & & x_{mn_p} & y_{mn_{p+1}} & \cdots & \cdots & \cdots & y_{mn_m} \end{vmatrix};$$

and call it D.

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We have

$$RD = \begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \cdots & \rho_{1m} \\ \rho_{21} & \rho_{22} & \rho_{23} & \cdots & \rho_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{r1} & \rho_{r2} & \cdots & \cdots & \rho_{rm} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{m1} & \rho_{m2} & \cdots & \cdots & \rho_{mm} \end{vmatrix} \begin{vmatrix} x_{1n_1} & x_{2n_1} & x_{3n_1} & \cdots & x_{n_1n_1} - \theta & \cdots & x_{mn_1} \\ x_{1n_2} & x_{2n_2} & \cdots & \cdots & \cdots & x_{n_2n_2} - \theta & \cdots & x_{mn_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{1n_{p+1}} & y_{2n_{p+1}} & \cdots & \cdots & y_{n_{p+1}n_{p+1}} - \theta & \cdots & \cdots & y_{mn_{p+1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{1n_m} & y_{2n_m} & \cdots & \cdots & \cdots & \cdots & \cdots & y_{mn_m} \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{u=1}^{m} \rho_{1u}x_{un_1} - \rho_{1n_1}\theta & \sum_{u=1}^{m} \rho_{1u}x_{un_2} - \rho_{1n_2}\theta & \cdots\cdots & \sum_{u=1}^{m} \rho_{1u}y_{un_{p+1}} - \rho_{1n_{p+1}}\theta' & \cdots\cdots \\ \sum_{u=1}^{m} \rho_{2u}x_{un_1} - \rho_{2n_1}\theta & \sum_{u=1}^{m} \rho_{2u}x_{un_2} - \rho_{2n_2}\theta & \cdots\cdots & \sum_{u=1}^{m} \rho_{2u}y_{un_{p+1}} - \rho_{2n_{p+1}}\theta' & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{u=1}^{m} \rho_{ru}x_{un_1} - \rho_{rn_1}\theta & \sum_{u=1}^{m} \rho_{ru}x_{un_2} - \rho_{rn_2}\theta & \cdots\cdots & \sum_{u=1}^{m} \rho_{ru}y_{un_{p+1}} - \rho_{rn_{p+1}}\theta' & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{u=1}^{m} \rho_{mu}x_{un_1} - \rho_{mn_1}\theta & \sum_{u=1}^{m} \rho_{mu}x_{un_2} - \rho_{mn_2}\theta & \cdots\cdots & \sum_{u=1}^{m} \rho_{mu}y_{un_{p+1}} - \rho_{mn_{p+1}}\theta' & \cdots\cdots \end{vmatrix}$$

Recalling that $\rho x = \xi\rho$ $\rho y = \eta\rho$, this determinant is equal to

$$\begin{vmatrix} \sum_{u=1}^{m} \xi_{1u}\rho_{un_1} - \rho_{1n_1}\theta & \sum_{u=1}^{m} \xi_{1u}\rho_{un_2} - \rho_{1n_2}\theta & \cdots\cdots & \sum_{u=1}^{m} \eta_{1u}\rho_{un_{p+1}} - \rho_{1n_{p+1}}\theta' & \cdots\cdots \\ \sum_{u=1}^{m} \xi_{2u}\rho_{un_1} - \rho_{2n_1}\theta & \sum_{u=1}^{m} \xi_{2u}\rho_{un_2} - \rho_{2n_2}\theta & \cdots\cdots & \sum_{u=1}^{m} \eta_{2u}\rho_{un_{p+1}} - \rho_{2n_{p+1}}\theta' & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{u=1}^{m} \xi_{ru}\rho_{un_1} - \rho_{rn_1}\theta & \sum_{u=1}^{m} \xi_{ru}\rho_{un_2} - \rho_{rn_2}\theta & \cdots\cdots & \sum_{u=1}^{m} \eta_{ru}\rho_{un_{p+1}} - \rho_{rn_{p+1}}\theta' & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{u=1}^{m} \xi_{mu}\rho_{un_1} - \rho_{mn_1}\theta & \sum_{u=1}^{m} \xi_{mu}\rho_{un_2} - \rho_{mn_2}\theta & \cdots\cdots & \sum_{u=1}^{m} \eta_{mu}\rho_{un_{p+1}} - \rho_{mn_{p+1}}\theta' & \cdots\cdots \end{vmatrix}$$

The determinant just written may obviously be expanded as a sum of multiples of determinants of the transform-system

$$\| \xi_{1m}, \theta : \eta_{1m}, \theta' \|.$$

If then (κ, κ') are an associated pair of roots of the latter, it follows that, as $|\rho| \neq 0$, we have

$$D' = 0,$$

where D' is the determinant derived from D by writing κ for θ and κ' for θ'. Similarly every other determinant of the system

$$\| x_{1m}, \kappa : y_{1m}, \kappa' \|$$

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vanishes: in other words (κ, κ') are an associated pair of roots of the system of equations

$$\| x_{1m}, \theta : y_{1m}, \theta' \| = 0.$$

Now from a remark made above (§ 12) the system which is the left-hand member of the equation just written is a transform of

$$\| \xi_{1m}, \theta : \eta_{1m}, \theta' \|$$

and so by an application of the above reasoning we prove that every associated pair of roots of the former are a pair for the latter. We conclude therefore that the system of associated pairs of roots is the same for each.

§ 15. We have next to prove that the index of the pair (κ, κ') is the same in each. Take any nth minor of the product (RD). When the latter is written in the second form of § 14 it is clear that we can expand it as a linear function of the nth minors of determinants of the system

$$\| \xi_{1m}, \theta : \eta_{1m}, \theta' \|.$$

Let n be less than ι' the index of the pair (κ, κ') for the system just written. Since the nth minors of determinants of the system

$$\| \xi_{1m}, \kappa : \eta_{1m}, \kappa' \|$$

all vanish, it is clear that the nth minors of the product RD' all vanish when it is written in the first form.

Now the nth minors of a determinant of m rows and columns are μ^2 in number, where

$$\mu = \frac{m!}{m-n!\ n!}:$$

suppose them to be $A_{\iota\gamma}$, $R_{\iota\gamma}$, $P_{\iota\gamma}$ $(\iota\gamma = 12 \ldots \mu)$ when formed from D', R, RD' respectively. We have

$$P_{\iota\gamma} = R_{\iota 1} A_{\gamma 1} + R_{\iota 2} A_{\gamma 2} + \ldots + R_{\iota\mu} A_{\gamma\mu}{}^{*}.$$

As the numbers $P_{\iota\gamma}$ are by hypothesis all zero it follows that

$$\begin{aligned} 0 &= R_{11} A_{\gamma 1} + R_{12} A_{\gamma 2} + \ldots + R_{1\mu} A_{\gamma\mu}, \\ 0 &= R_{21} A_{\gamma 1} + R_{22} A_{\gamma 2} + \ldots + R_{2\mu} A_{\gamma\mu}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 &= R_{\mu 1} A_{\gamma 1} + R_{\mu 2} A_{\gamma 2} + \ldots + R_{\mu\mu} A_{\gamma\mu}. \end{aligned}$$

The determinant of the coefficients of the A's is equal to R^v†, where $v = \dfrac{m-1!}{m-n-1!\ n!}$, and consequently does not vanish. It must therefore be that

$$A_{\gamma 1}, A_{\gamma 2}, \ldots A_{\gamma\mu}$$

are all zero. In a similar way we prove that the nth minors of every other determinant of the system

$$\| x_{1m}, \kappa : y_{1m}, \kappa' \|$$

vanishes. This proves that the index (ι) of the pair (κ, κ') for the original system is $\geqslant \iota'$. Since the original system is a transform of the second it follows that

$$\iota' \geqslant \iota.$$

We have therefore $\iota = \iota'$, which completes the proof of the theorem.

* Scott and Matthews, *Theory of Determinants* (2nd edition, 1904), p. 58. † *Ibid.* pp. 65, 66.

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§ 16. The relevance of the work of the preceding paragraphs (§§ 8—15) will be best appreciated as we proceed; but before doing so it will be well to make one or two observations which will be useful to us later and which at the same time bear testimony to its close connection with the subject before us.

In § 7 we were led to consider the conjoint system

$$\| a_{1m}, \theta : b_{1m}, \theta' \|,$$

where a and b are the matrices for the periods ω and ω' respectively. We will call it the *fundamental period system* and the system of equations

$$\| a_{1m}, \theta : b_{1m}, \theta' \| = 0$$

we will call the *fundamental system of period equations* corresponding to the system of solutions

$$g_1(z), g_2(z), \dots g_m(z).$$

In § 11 we saw that this system of period equations has at least one pair of associated roots say (κ, κ'); and so, from what was said in § 7, we can choose at least one system of values for $\lambda_1, \lambda_2, \dots \lambda_m$ (which are not all zero) such that

$$\left.\begin{aligned} \sum_{r=1}^{m} \lambda_r a_{rs} &= \kappa \lambda_s \\ \sum_{r=1}^{m} \lambda_r b_{rs} &= \kappa' \lambda_s \end{aligned}\right\}. \qquad (s = 1, 2, \dots m)$$

It follows that $F(z) (= \lambda\, g(z))$ has the property

$$F(z + \omega) = \kappa\, F(z), \quad F(z + \omega') = \kappa'\, F(z).$$

In other words we have Picard's theorem or rather a slightly more general one, viz., every generalised Picard equation has at least one solution, a branch of which considered as localised in the periodic region of the equation is a doubly-periodic function of the second species.

§ 17. The following theorem throws light on the bearing which the theory of transforms has on our work.

Theorem. Of any two fundamental period systems corresponding to two different fundamental systems of solutions of the equation of § 1, considered as localised in the same periodic region Φ, each is the transform of the other.

Let one fundamental system of solutions be

$$g_1(z), g_2(z), \dots g_m(z),$$

and let any other be

$$\gamma_1(z), \gamma_2(z), \dots \gamma_m(z).$$

The period matrices of the first system are a and b: suppose that those of the second are α and β, so that

$$\gamma(z + \omega) = \alpha\, \gamma(z),$$

$$\gamma(z + \omega') = \beta\, \gamma(z).$$

The g's and γ's being each fundamental systems we must have

$$\gamma(z) = \rho\, g(z),$$

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where ρ is a matrix such that $|\rho|$ does not vanish. In less condensed form, the equation just written expresses, of course, that

$$\gamma_r(z) = \rho_{r1}\, g_1(z) + \rho_{r2}\, g_2(z) + \ldots + \rho_{rm}\, g_m(z). \qquad (r = 1, 2, \ldots m)$$

We have
$$\gamma(z+\omega) = \alpha\,\gamma(z) = \alpha\rho\, g(z),$$
and again
$$\gamma(z+\omega) = \rho\, g(z+\omega) = \rho a\, g(z).$$
Remembering that the g's are a fundamental system, it follows that
$$\alpha\rho = \rho a.$$
Mutatis mutandis we prove that
$$\beta\rho = \rho b.$$
The theorem is thus established.

Referring back to Theorem II, §§ 14 and 15, we conclude that all associated pairs of roots of a fundamental system of period equations and their indices are independent of the particular fundamental system of solutions from which they are derived—they are functions merely of the coefficients of the differential equation and the region Φ.

§ 18. We now take up the thread of our work as it was left in § 7. Suppose that (κ, κ') are a pair of associated roots of the fundamental system of period equations
$$\| a_{1m},\ \theta : b_{1m},\ \theta' \| = 0,$$
and that σ_1 is their index. We will call σ_1 the first invariant of the pair, when considered in connection with the differential equation: 'invariant' implying the property at which we arrived in the previous paragraph, viz. that σ_1 is the same for all fundamental systems localised in Φ: the reason for using the word 'first' will become apparent as our method develops.

From what has been said above (§§ 9, 10) we can solve the system of $2m$ equations
$$\left.\begin{aligned} \lambda_1 a_{1r} + \lambda_2 a_{2r} + \ldots + \lambda_m a_{mr} &= \lambda_r \kappa' \\ \lambda_1 b_{1r} + \lambda_2 b_{2r} + \ldots + \lambda_m b_{mr} &= \lambda_r \kappa \end{aligned}\right\} \ (r = 1, 2, \ldots m) \ldots\ldots \text{(xii)},$$
by taking arbitrary values for σ_1 of the λ's and then determining the remainder as definite linear functions of these. Without loss of generality we can suppose the σ_1 λ's to be
$$\lambda_1, \lambda_2, \ldots \lambda_{\sigma_1}{}^*,$$
the other λ's will then be determined by
$$\lambda_{\sigma_1+s} = k_{s1}\lambda_1 + k_{s2}\lambda_2 + \ldots k_{s\sigma_1}\lambda_{\sigma_1}. \qquad (s = 1, 2, \ldots m - \sigma_1)$$
Then taking any values we please for $\lambda_1, \lambda_2 \ldots \lambda_{\sigma_1}$,
$$\begin{aligned} F(z) &= \lambda_1 g_1(z) + \lambda_2\, g_2(z) + \ldots + \lambda_m g_m(z) \\ &= \sum_{t=1}^{\sigma_1} \lambda_t \left(g_t(z) + \sum_{s=1}^{m-\sigma_1} k_{st}\, g_{s+\sigma_1}(z) \right) \\ &= \sum_{t=1}^{\sigma_1} \lambda_t\, G_t(z) \quad \text{say;} \end{aligned}$$
where
$$G_t(z) = g_t(z) + \sum_{s=1}^{m-\sigma_1} k_{st}\, g_{s+\sigma_1}(z).$$

* It is clear that the interchange of any two members of a fundamental system of solutions merely interchanges two corresponding rows of the system of period determinants.

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Since the equations (xii) above are satisfied we have

$$F(z+\omega)=\kappa\, F(z), \quad F(z+\omega')=\kappa'\, F(z),$$

whatever $\lambda_1, \lambda_2 \dots \lambda_{\sigma_1}$ may be. We must have therefore

$$G_t(z+\omega)=\kappa\, G_t(z), \quad G_t(z+\omega')=\kappa'\, G_t(z). \qquad (t=1, 2, \dots \sigma_1)$$

From the form of the functions G it is clear that any non-evanescent linear relation among them would involve one of a similar nature between the g's. As the latter are a fundamental system no such relation can exist among them. It follows that the G's are linearly independent.

We conclude that *corresponding to any associated pair of roots* (κ, κ') *of the fundamental system of period equations we have* σ_1 *solutions of the equation which are such that one branch of each considered as localised in* Φ *is a doubly-periodic function of the second species with multipliers* (κ, κ'). *Moreover these branches are all linearly independent.*

In particular if the equation is one of ordinary Picard type these solutions will be single-valued doubly-periodic functions of the second species, whose domain* is the whole plane with the exception of some or all of the singularities of the coefficients. The total number of integrals of this kind will evidently be equal to the sum of the first invariants of all associated pairs. It will be proved below (§§ 29, 30) that these are all linearly independent. Consequently whenever the sum of the first invariants is equal to m *every* integral will be a sum of multiples of doubly-periodic functions of the second species. For instance this will be the case whenever either of the equations

$$|\, a_{1m}, \theta \,| = 0, \quad |\, b_{1m}, \theta' \,| = 0$$

has all its roots distinct (§ 11).

§ 19. Take the system of solutions

$$G_1(z),\ G_2(z),\ \dots\ G_{\sigma_1}(z),\ g_{\sigma_1+1}(z),\ g_{\sigma_1+2}(z),\ \dots\ g_m(z).$$

As any linear relation between these would involve one between the g's it is clear that this system is fundamental. It will be convenient to say that any such system is *normal* with respect to the first invariant of the associated pair (κ, κ').

Re-label this system

$$h_1(z),\ h_2(z),\ \dots\ h_{\sigma_1}(z),\ h_{\sigma_1+1}(z),\ \dots\ h_m(z)$$

respectively.

As before we shall have

$$h(z+\omega)=c\, h(z),$$
$$h(z+\omega')=d\, h(z),$$

where c and d are new period matrices. But now we have $c_{rs}=d_{rs}=0$ provided $r \neq s$ and $r \leqslant \sigma_1$, whilst $c_{11}=c_{22}=\dots\dots=c_{\sigma_1\sigma_1}=\kappa : d_{11}=d_{22}=\dots\dots=d_{\sigma_1\sigma_1}=\kappa'$ so that the new fundamental period system is

* The domain of a single-valued function is the completely open region whose points are internal to one or other of the circles of convergence of its elements.

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$$\left\| \begin{array}{cccccccccccc}
\kappa-\theta & 0 & 0 & \cdots & 0 & \kappa'-\theta' & 0 & 0 & \cdots & 0 \\
0 & \kappa-\theta & 0 & \cdots & 0 & 0 & \kappa'-\theta' & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 \cdots \kappa-\theta & \cdots & 0 & 0 & 0 & 0 \cdots \kappa'-\theta & \cdots & 0 \\
c_{\sigma_1+1,1} & c_{\sigma_1+1,2} & c_{\sigma_1+1,3} & \cdots & c_{\sigma_1+1,\sigma_1+1}-\theta \cdots c_{\sigma_1+1,m} & d_{\sigma_1+1,1} & d_{\sigma_1+1,2} & \cdots & d_{\sigma_1+1,\sigma_1+1}-\theta' \cdots & d_{\sigma_1+1,m} \\
c_{\sigma_1+2,1} & c_{\sigma_1+2,2} & \cdots & \cdots & c_{\sigma_1+2,m} & d_{\sigma_1+2,1} & d_{\sigma_1+2,2} & \cdots & \cdots & d_{\sigma_1+2,m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
c_{m1} & c_{m2} & \cdots & \cdots & c_{mm}-\theta & d_{m1} & d_{m2} & \cdots & \cdots & d_{mm}-\theta
\end{array} \right\|$$

From the theorems proved in § 17 and § 13 or by a direct appeal to the theory of § 5 it is clear that this is a conjoint system. But before we consider it in detail it will be necessary to make a further excursion into the general theory of such systems.

§ 20. Consider a conjoint system

$$\| x_{1m}, \theta : y_{1m}, \theta' \|,$$

where $x_{rs} = y_{rs} = 0$ provided $r \neq s$ and $\leqslant n\,(\leqslant m)$, whilst

$$x_{11} = x_{22} = \ldots\ldots = x_{nn} = \kappa, \quad y_{11} = y_{22} = \ldots\ldots = y_{nn} = \kappa'.$$

We will say that the first n rows of this system are *regular* with respect to the associated pair (κ, κ')*. For instance the period system considered at the end of the preceding paragraph has its first σ_1 rows regular with respect to the pair (κ, κ').

The system of determinants of $m-n$ rows and columns, obtained by erasing the first n rows and also the first n columns of x's and y's, viz. the system

$$\left\| \begin{array}{cccccccccc}
x_{n+1,\,n+1}-\theta & x_{n+1,\,n+2} & \cdots & x_{n+1,\,m} & y_{n+1,\,n+1}-\theta' & y_{n+1,\,n+2} & \cdots & y_{n+1,\,m} \\
x_{n+2,\,n+1} & x_{n+2,\,n+2}-\theta & \cdots & x_{n+2,\,m} & y_{n+2,\,n+1} & y_{n+2,\,n+2}-\theta' & \cdots & y_{n+2,\,m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_{n+r,\,n+1} & x_{n+r,\,n+2} \cdots & x_{n+r,\,n+r}-\theta \cdots & x_{n+rm} & y_{n+r,\,n+1} & y_{n+r,\,n+2} \cdots & y_{n+r,\,n+r}-\theta' \cdots & y_{n+r,\,m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_{m,\,n+1} & x_{m,\,n+2} & \cdots & x_{mm}-\theta & y_{m,\,n+1} & y_{m,\,n+2} & \cdots & y_{mm}-\theta'
\end{array} \right\|$$

will be found to be of particular interest: we will call it the *irregular component* or briefly the *component*† of the original system. For brevity we will denote the component by

$$\| x_{n+1,m}, \theta : y_{n+1,m}, \theta' \|.$$

We proceed to prove certain theorems about these components.

* It is of course quite clear that $(\kappa\kappa')$ are an associated pair of roots of the system.

† We use the word in no other sense than the present so that when we speak of a component it will be understood that it is obtained from a conjoint system as above.

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§ 21. *Theorem I.* The component of a conjoint system is a conjoint system.

Adopting the above notation we have to prove that

$$\| x_{n+1,m}, \theta \ : \ y_{n+1,m}, \theta' \|$$

is conjoint.

Since $xy = yx$ we have the relations

$$\sum_{u=1}^{m} x_{ru} y_{us} = \sum_{u=1}^{m} y_{ru} x_{us}, \qquad \begin{pmatrix} r \\ s \end{pmatrix} = 1, 2, \dots m$$

which in the case before us become effectively

$$\sum_{u=1}^{m-n} x_{n+r,n+u}\, y_{n+u,s} = \sum_{u=1}^{m-n} y_{n+r,n+u}\, x_{n+u,s}.$$

$$(r = 1, 2, \dots m-n, \quad s = 1, 2, \dots m)$$

In particular we have

$$\sum_{u=1}^{m-n} x_{n+r,n+u}\, y_{n+u,n+s} = \sum_{u=1}^{m-n} y_{n+r,n+u}\, x_{n+u,n+s}, \qquad \begin{pmatrix} r \\ s \end{pmatrix} = 1, \dots m-n$$

which proves the theorem.

§ 22. *Theorem II.* Any associated pair of roots of the component of a conjoint system whose first n rows are regular with respect to an associated pair (κ, κ') are an associated pair for the complete system.

If (κ, κ') are the associated pair of the component the theorem is obvious: we confine ourselves therefore to the case in which this is not so. It will then follow that if (kk') are the pair of roots of the component we may have $k = \kappa$ or $k' = \kappa'$, but not *both* of these equalities can hold. Suppose that we have $k \neq \kappa$, and let

$$\| x_{1m}, \theta \ : \ y_{1m}, \theta' \|$$

be the complete system,

$$\| x_{n+1,m}, \theta \ : \ y_{n+1,m}, \theta' \|$$

its component.

We see at once that as k is a root of

$$| x_{n+1,m}, \theta | = 0,$$

it is also a root of

$$| x_{1m}, \theta | = 0.$$

Take any pth minor of the determinant

$$| x_{1m}, k |.$$

A little consideration will shew that if it does not vanish identically it must be equal to

$$(\kappa - k)^t \times a\,(p + t - n)\text{th minor of } | x_{n+1,m}, k |,$$

where $t \leqslant n$. It follows that if the τth minors of $| x_{n+1,m}, k |$ are the first which do not all vanish, the corresponding number τ' for $| x_{1m}, k |$ is $\geqslant \tau$. As, however, one particular τth minor of $| x_{1m}, k |$ is not zero, viz. one of the τth minors of $| x_{n+1,m}, k |$ which does not vanish, bordered by elements of the first n rows and columns of $| x_{1m}, k |$, it is clear that $\tau' = \tau$.

Let us refer back to §§ 9, 10, and, in the notation there employed, suppose that the non-vanishing minor just mentioned is common to the n_1th, n_2th, ... $n_{m-\tau}$th* columns, and

* Of course τ of these numbers are 1, 2, 3, ... τ.

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the $p_{\tau+1}$th, $p_{\tau+2}$th, ... p_mth* rows. Clearly the determinant which is the left-hand member of (viii) § 10 is equal to

$$(k-\kappa)^n \times \text{a determinant of the system } \| x_{n+1,m}, k : y_{n+1,m}, \theta' \|.$$

As k' substituted for θ' in the latter makes it vanish, we see that (k, k') are an associated pair of roots of

$$\| x_{1m}, \theta : y_{1m}, \theta' \| = 0.$$

The theorem is therefore proved.

§ 23. *Theorem III.* If of two conjoint systems one is the transform of the other, and at the same time the first n rows of each are regular with respect to the associated pair $(\kappa\kappa')$, then the component of the transform is a transform of the component of the original system.

Suppose that $\| x_{1m}, \theta : y_{1m}, \theta' \|$

is the original† system and that $\| \xi_{1m}, \theta : \eta_{1m}, \theta' \|$

is its transform.

Then by definition
$$\xi\rho = \rho x,$$
$$\eta\rho = \rho y,$$

where ρ is a matrix of m rows and columns whose determinant $|\rho|$ does not vanish (§ 12).

The first of these relations written at full length gives

$$\sum_{u=1}^{m} \xi_{ru}\rho_{us} = \sum_{u=1}^{m} \rho_{ru}x_{us} \quad (r, s = 1, 2, \dots m) \text{ .....(xiii).}$$

Hence if $r \leqslant n$, we have among other relations

$$\rho_{rn+1}x_{n+1,n+s} + \rho_{rn+2}x_{n+2,n+s} + \dots + \rho_{rm}x_{m,n+s} = 0.$$
$$(s = 1, 2, \dots m-n)$$

Considering this system of $m-n$ equations we remark that the determinant of the coefficients of the ρ's is

$$\begin{vmatrix} x_{n+1,n+1} & x_{n+1,n+2} \dots\dots & x_{n+1,m} \\ x_{n+2,n+1} & x_{n+2,n+2} \dots\dots & x_{n+2,m} \\ x_{n+3,n+1} & x_{n+3,n+2} \dots\dots & x_{n+3,m} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ x_{m,n+1} & x_{m,n+2} \dots\dots & x_{m,m} \end{vmatrix},$$

which is $\dfrac{|x|}{\kappa^n}$ and so, since $|x|$ does not vanish by definition of a conjoint system, cannot be zero.

Thus
$$\rho_{r,n+1} = \rho_{r,n+2} = \dots = \rho_{rm} = 0. \qquad (r \leqslant n)$$

* Of course τ of these numbers are 1, 2, 3, ... τ.

† It will be remembered that each of the systems is a transform of the other, so that it is entirely arbitrary which we regard as the "original" system. The adjective is merely convenient for distinguishing one system from another.

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The relations (xiii) therefore become effectively

$$\sum_{u=1}^{m-n} \xi_{n+r,n+u}\, \rho_{n+u,s} = \sum_{u=1}^{m-n} \rho_{n+r,n+u}\, x_{n+u,s}.$$

$$(r = 1, 2, \ldots m-n)(s = 1, 2, \ldots m)$$

In particular

$$\sum_{u=1}^{m-n} \xi_{n+r,n+u}\, \rho_{n+u,n+s} = \sum_{u=1}^{m-n} \rho_{n+r,n+u}\, x_{n+u,n+s}. \qquad \left(\begin{matrix} r \\ s \end{matrix} = 1, 2, \ldots m-n\right)$$

If then we write x' for the matrix whose elements are those of the determinant written above, and ξ', ρ' for the corresponding matrices in ξ's and ρ's, we have

$$\xi'\rho' = \rho' x' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xiv)}.$$

In a similar way

$$\eta'\rho' = \varphi' y' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xv)}.$$

Again

$$|\rho| = |\rho'| \times \begin{vmatrix} \rho_{11} & \rho_{12} & \rho_{13} \cdots\cdots \rho_{1n} \\ \rho_{21} & \rho_{22} & \rho_{23} \cdots\cdots \rho_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \rho_{r1} & \rho_{r2} & \rho_{r3} \cdots\cdots \rho_{rn} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \rho_{n1} & \rho_{n2} & \rho_{n3} \cdots\cdots \rho_{nn} \end{vmatrix}.$$

As $|\rho|$ does not vanish it is clear that $|\rho'|$ cannot, and so taking this in conjunction with (xiv) and (xv), we have the theorem.

§ 24. We now take up our theory again as we left it in § 19.

The system $\qquad \| c_{1m}, \theta : d_{1m}, \theta' \|$

has its first σ_1 rows regular with regard to (κ, κ'): its irregular component

$$\| c_{\sigma_1+1,m}, \theta : d_{\sigma_1+1,m}, \theta' \|$$

considered in connection with the differential equation will be called a *first derived period system* with regard to the associated pair (κ, κ').

By theorem I, § 21, this derived system is conjoint, and so (§§ 9, 10) has associated pairs of roots. It may or may not be that (κ, κ') are one of these pairs. In the latter case our process terminates; but if they are such a pair let their index be σ_2. We call σ_2 the *second invariant of the associated pair of roots* (κ, κ') of the system of period equations. The justification of the word "invariant" is fairly clear. For it results from theorem III, § 23, and theorem II, § 14, that σ_2 is the same for all first derived systems with regard to (κ, κ').

From the theory of §§ 9, 10 it appears that we can satisfy the system of $2(m-\sigma_1)$ equations

$$\lambda_{\sigma_1+1} c_{\sigma_1+1,\sigma_1+r} + \lambda_{\sigma_1+2} c_{\sigma_1+2,\sigma_1+r} + \ldots + \lambda_{\sigma_1+r}(c_{\sigma_1+r,\sigma_1+r} - \kappa) + \ldots + \lambda_m c_{m,\sigma_1+r} = 0.$$

$$\lambda_{\sigma_1+1} d_{\sigma_1+1,\sigma_1+r} + \lambda_{\sigma_1+2} d_{\sigma_1+2,\sigma_1+r} + \ldots + \lambda_{\sigma_1+r}(d_{\sigma_1+r,\sigma_1+r} - \kappa) + \ldots + \lambda_m d_{m,\sigma_1+r} = 0,$$

$$(r = 1, 2, \ldots m-\sigma_1)$$

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by taking arbitrarily σ_2 of the λ's and then determining the remainder as definite linear functions of them. Without loss of generality we can suppose that the σ_2 λ's are

$$\lambda_{\sigma_1+1},\ \lambda_{\sigma_1+2}, \dots \lambda_{\sigma_1+\sigma_2}.$$

The other λ's are then determined by

$$\lambda_{\sigma_1+\sigma_2+r} = l_{r1}\lambda_{\sigma_1+1} + l_{r2}\lambda_{\sigma_1+2} + \dots + l_{r\sigma_2}\lambda_{\sigma_1+\sigma_2}, \qquad (r = 1, 2, \dots m - \sigma_1 - \sigma_2)$$

where the l's are definite functions of the c's and d's.

Consider the function

$$F(z) = \lambda_{\sigma_1+1} h_{\sigma_1+1}(z) + \lambda_{\sigma_1+2} h_{\sigma_1+2}(z) + \dots + \lambda_m h_m(z).$$

Selecting the λ's as above we have

$$F(z) = \sum_{t=1}^{\sigma_2} \lambda_{\sigma_1+t} H_{\sigma_1+t}(z),$$

where
$$H_{\sigma_1+t}(z) = h_{\sigma_1+t}(z) + \sum_{s=1}^{m-\sigma_1-\sigma_2} l_{st} h_{\sigma_1+\sigma_2+s}(z).$$

Then whatever $\lambda_{\sigma_1+1}, \dots \lambda_{\sigma_1+\sigma_2}$ may be

$$F(z+\omega) = \kappa' F(z) + \text{linear functions of } h_1(z), \dots h_{\sigma_1}(z),$$
$$F(z+\omega') = \kappa' F(z) + \quad \text{"} \quad \text{"} \quad \text{"}$$

It follows easily that we must have

$$H_{\sigma_1+t}(z+\omega) = \kappa\ H_{\sigma_1+t}(z) + \bar{H}_{\sigma_1+t}(z),$$
$$H_{\sigma_1+t}(z+\omega') = \kappa' H_{\sigma_1+t}(z) + \bar{H}'_{\sigma_1+t}(z), \qquad (t = 1, 2, \dots \sigma_2)$$

where $H_{\sigma_1+t}(z)$ and $\bar{H}'_{\sigma_1+t}(z)$ are linear functions of $h_1(z), h_2(z) \dots h_{\sigma_1}(z)$.

A little consideration will shew that we cannot have both these linear combinations evanescent: for if any pair were, it is clear that we would have at least one more integral of the type of $h_1(z), h_2(z), \dots h_{\sigma_1}(z)$, which is independent of them. We would therefore be able to find a fundamental system such that the corresponding period system had its first $\sigma_1 + 1$ rows regular with respect to (κ, κ'). The index of any such system is clearly $> \sigma_1$, which is impossible as we have proved that it must be σ_1 for *all* fundamental systems (§ 18).

For a similar reason it is easily proved that there exists no linear combination of the functions $\bar{H}_{\sigma_1+t}(z)$ such that it and the same linear combination of the functions $\bar{H}'_{\sigma_1+t}(z)$ both vanish.

The functions $H(z)$ are of course only defined in Φ: but as we remarked in § 7, any such localised function may be regarded as a branch of a certain monogenic function localised in Φ.

§ 25. From the form of the functions $H(z)$ at which we arrived in the preceding paragraph, it is easily seen (cf. §§ 18, 19) that they are linearly independent, and further that the system of solutions

$$h_1(z), h_2(z), \dots h_{\sigma_1}(z), H_{\sigma_1+1}(z), H_{\sigma_1+2}(z), \dots H_{\sigma_1+\sigma_2}(z), h^{(z)}_{\sigma_1+\sigma_2+1}, \dots h_m(z)$$

is a fundamental system.

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The members of this system may be re-labelled

$$k_1(z),\ k_2(z),\ \dots\ k_{\sigma_1}(z),\ k_{\sigma_1+1}(z),\ k_{\sigma_1+2}(z),\ \dots\ k_{\sigma_1+\sigma_2}(z),\ \dots\ k_m(z)$$

respectively. We will say that any system of this character is *normal* with respect to the *first two* invariants of the associated pair (κ, κ').

As in the preceding case we shall have

$$k(z+\omega) = e\,k(z),$$
$$k(z+\omega') = f\,k(z),$$

where the new period matrices e and f are such that

(1) $e_{rs} = f_{rs} = 0$ provided $r \neq s$ and $r \leqslant \sigma_1$, whilst

$$e_{11} = e_{22} = \dots = e_{\sigma_1\sigma_1} = \kappa,\ f_{11} = f_{22} = \dots = f_{\sigma_1\sigma_1} = \kappa'.$$

(2) $e_{\sigma_1+r,\sigma_1+s} = f_{\sigma_1+r,\sigma_1+s} = 0$, provided $r \neq s$ and $r \leqslant \sigma_2$, whilst

$$e_{\sigma_1+1,\sigma_1+1} = e_{\sigma_1+2,\sigma_1+2} = \dots = e_{\sigma_1+\sigma_2} = \kappa,\quad f_{\sigma_1+1,\sigma_1+1} = f_{\sigma_1+2,\sigma_1+2} = \dots = f_{\sigma_1+\sigma_2} = \kappa'.$$

If we consider the first derived period system of

$$\| e_{1m}, \theta : f_{1m}, \theta' \|,$$

viz. the system $\| e_{\sigma_1+1,m}, \theta : f_{\sigma_1+1,m}, \theta' \|$,

it is clear that its first σ_2 rows are regular with respect to the associated pair (κ, κ'). We use the notation

$$\| e_{\sigma_1+\sigma_2+1,m}, \theta : f_{\sigma_1+\sigma_2+1,m}, \theta' \|$$

for its irregular component. Considered in connection with the differential equation we will call the latter a *second* derived period system with regard to the associated pair (κ, κ').

It is a very easy matter to see that any such second derived period system is conjoint. The proof follows in fact from a double application of theorem I, § 21. For the system under consideration is the component of a first derived system: and we saw above by an application of the theorem mentioned that the latter was a conjoint system. Consequently a second application of the same theorem proves what we desire.

In a similar way by using theorem III, § 23, we prove that all second derived period systems with regard to (κ, κ') are transforms of one another. It follows therefore by theorem II, §§ 14, 15, that the associated pairs and their indices are the same for all.

§ 26. It may or may not be that (κ, κ') are an associated pair of roots of the second derived system of equations

$$\| e_{\sigma_1+\sigma_2+1,\,m},\ \theta : f_{\sigma_1+\sigma_2+1,\,m},\ \theta' \| = 0.$$

If not then our process terminates: but if they are, let σ_3 be their index. We call σ_3 the *third invariant of the associated pair* (κ, κ'). After what has been said in the previous paragraph it will be sufficient if we merely remark that σ_3 is the same for all fundamental systems which are normal with respect to the first two invariants of the associated pair (κ, κ').

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From the general theory of §§ 9, 10, it appears that we can satisfy the system of $2(m-\sigma_1-\sigma_2)$ equations

$$\lambda_{\sigma_1+\sigma_2+1}e_{\sigma_1+\sigma_2+1,\,\sigma_1+\sigma_2+r}+\lambda_{\sigma_1+\sigma_2+2}e_{\sigma_1+\sigma_2+2,\,\sigma_1+\sigma_2+r}+\dots$$
$$+\lambda_{\sigma_1+\sigma_2+r}(e_{\sigma_1+\sigma_2+r,\,\sigma_1+\sigma_2+r}-\kappa)+\dots+\lambda_m e_{m,\,\sigma_1+\sigma_2+r}=0,$$
$$(r=1,\ 2,\dots m-\sigma_1-\sigma_2)$$
$$\lambda_{\sigma_1+\sigma_2+1}f_{\sigma_1+\sigma_2+1,\,\sigma_1+\sigma_2+r}+\lambda_{\sigma_1+\sigma_2+r}f_{\sigma_1+\sigma_2+2,\,\sigma_1+\sigma_2+r}+\dots$$
$$+\lambda_{\sigma_1+\sigma_2+r}(f_{\sigma_1+\sigma_2+r,\,\sigma_1+\sigma_2+r}-\kappa')+\dots+\lambda_m f_{m,\,\sigma_1+\sigma_2+r}=0,$$

by taking arbitrarily σ_3 of the λ's—which may be assumed to be $\lambda_{\sigma_1+\sigma_2+1}$, $\lambda_{\sigma_1+\sigma_2+2}$, ... $\lambda_{\sigma_1+\sigma_2+\sigma_3}$—and determining the rest as definite linear functions of these say

$$\lambda_{\sigma_1+\sigma_2+\sigma_3+r}=|m_{r_1}|\lambda_{\sigma_1+\sigma_2+1}+|m_{r_2}|\lambda_{\sigma_1+\sigma_2+2}+\dots+m_{r\sigma_3}\lambda_{\sigma_1+\sigma_2+\sigma_3}.$$
$$(r=1,\ 2,\dots m-\sigma_1-\sigma_2-\sigma_3)$$

Take the function

$$F(z)=\lambda_{\sigma_1+\sigma_2+1}k_{\sigma_1+\sigma_2+1}(z)+\lambda_{\sigma_1+\sigma_2+2}k_{\sigma_1+\sigma_2+2}(z)+\dots+\lambda_m k_m(z),$$

where the λ's are selected as above.

Then

$$F(z)=\sum_{t=1}^{\sigma_3}\lambda_{\sigma_1+\sigma_2+t}K_{\sigma_1+\sigma_2+t}(z),$$

where

$$K_{\sigma_1+\sigma_2+t}(z)=k_{\sigma_1+\sigma_2+t}(z)+\sum_{s=1}^{m-\sigma_1-\sigma_2-\sigma_3}m_{st}k_{\sigma_1+\sigma_2+\sigma_3+s}(z).$$

Proceeding in the manner of § 24 we arrive at σ_3 solutions which are linearly independent and which have the properties represented by

$$K_{\sigma_1+\sigma_2+t}(z+\omega)=\kappa K_{\sigma_1+\sigma_2+t}(z)+\bar{K}_{\sigma_1+\sigma_2+t}(z),$$
$$K_{\sigma_1+\sigma_2+t}(z+\omega')=\kappa' K_{\sigma_1+\sigma_2+t}(z)+\bar{K}'_{\sigma_1+\sigma_2+t}(z),$$

where $\bar{K}_{\sigma_1+\sigma_2+t}(z)$ and $\bar{K}'_{\sigma_1+\sigma_2+t}(z)$ are linear functions of $k_1(z)$, $k_2(z)$, ... $k_{\sigma_1+\sigma_2}(z)$.

By reasoning analogous to that used in the corresponding case in § 24 we may shew that *one* but *not both* of these functions may be evanescent, or a linear function merely of $k_1(z)$, $k_2(z)$, ... $k_{\sigma_1}(z)$: that is to say one at least must involve members of the second set of integrals (of § 24). In a similar way we shew that corresponding linear functions of the $\bar{K}$'s and $\bar{K}''$s have the same property.

It is moreover easily shewn that the solutions

$$k_1(z),\ k_2(z),\dots k_{\sigma_1+\sigma_2}(z),\ K_{\sigma_1+\sigma_2+1}(z),\ K_{\sigma_1+\sigma_2+2}(z),\ K_{\sigma_1+\sigma_2+\sigma_3}(z),\ k_{\sigma_1+\sigma_2+\sigma_3+1}(z),\dots k_m(z)$$

constitute a fundamental system.

§ 27. The general course of procedure is now obvious: but before stating the complete result it will be useful to give a definition.

We have seen that corresponding to an associated pair of roots (κ, κ') we have a number of invariants σ_1, σ_2, σ_3, &c. Let σ_s be the last of these so that (κ, κ') are *not* an associated pair of the sth derived period system with regard to (κ, κ'). The sum $\sigma_1+\sigma_2+\dots+\sigma_s$ is defined to be the multiplicity of the pair (κ, κ'). From what has

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been said above the multiplicity of the pair of roots is independent of the fundamental system from which we start.

Slightly altering our previous notation our final result then is this:—

Corresponding to an associated pair of roots (κ, κ') of a system of period equations whose multiplicity is n we have a group of n integrals of the differential equation which are linearly independent. This group may be divided into s sets where s is the number of invariants of the pair. The first set consists of integrals $w_{11}(z), w_{12}(z), \ldots w_{1\sigma_1}(z)$, equal in number to the first invariant (σ_1), which have the properties expressed by

$$w_{1r}(z+\omega) = \kappa w_{1r}(z), \quad w_{1r}(z+\omega') = \kappa' w_{1r}(z), \qquad (r = 1, 2, \ldots \sigma_1)$$

or as we may write symbolically

$$w_1(z+\omega) = \kappa w_1(z), \quad w_1(z+\omega') = \kappa' w_1(z).$$

The second set consists of the σ_2 integrals $w_{21}(z), w_{22}(z), \ldots w_{2\sigma_2}(z)$ where σ_2 is the second invariant of the pair. These have the properties

$$w_{2r}(z+\omega) = \kappa w_{2r}(z) + v_{1r}(z) \;:\; w_{2r}(z+\omega') = \kappa' w_{2r}(z) + v'_{1r}(z), \qquad (r = 1, 2, \ldots \sigma_2)$$

$v_{1r}(z)$, $v'_{1r}(z)$ being linear functions of the $w_{1r}(z)$'s, *both* of which cannot be evanescent. Symbolically we may write

$$w_2(z+\omega) = \kappa w_2(z) + \rho_{21} w_1(z) \;:\; w_2(z+\omega') = \kappa' w_2(z) + \rho'_{21} w_1(z),$$

where ρ_{21}, ρ'_{21} are matrices of σ_2 rows and σ_1 columns which are such that if we take corresponding rows of each, the elements of both cannot all vanish.

The third set consists of integrals $w_{31}(z), w_{32}(z), \ldots w_{3\sigma_3}(z)$ whose number is equal to the third invariant of the pair. These have the properties expressed by

$$w_{3r}(z+\omega) = \kappa w_{3r}(z) + v_{2r}(z) + u_{1r}(z) \;:\; w_{3r}(z+\omega') = \kappa' w_{3r}(z) + v'_{2r}(z) + u'_{1r}(z),$$

$v_{2r}(z)$ and $v'_{2r}(z)$ being linear functions of integrals belonging to the second set of which one but not both may be evanescent: $u_{1r}(z)$, $u'_{1r}(z)$ are linear functions of integrals belonging to the first set. In the language of matrices

$$w_3(z+\omega) = \kappa w_3(z) + \rho_{32} w_2(z) + \rho_{31} w_1(z),$$

$$w_3(z+\omega') = \kappa' w_3(z) + \rho'_{32} w_2(z) + \rho'_{31} w_1(z),$$

where ρ_{rs}, ρ'_{rs}* are matrices of σ_r rows and σ_s columns: and so on. The properties of the rth set have for their expression

$$w_r(z+\omega) = \kappa w_r(z) + \rho_{r,r-1} w_{r-1}(z) + \rho_{r,r-2} w_{r-2}(z) + \ldots + \rho_{r1} w_1(z),$$

$$w_r(z+\omega') = \kappa' w_r(z) + \rho'_{r,r-1} w_{r-1}(z) + \rho'_{r,r-2} w_{r-2}(z) + \ldots + \rho'_{r1} w_1(z),$$

where $\rho_{r,r-1} w_{r-1}(z)$, $\rho'_{r,r-1} w_{r-1}(z)$ are such that no two corresponding functions are evanescent and further no linear combination of either of the sets vanishes simultaneously with the same linear combination of the other (cf. §§ 24, 26).

* The matrices ρ_{rs}, ρ'_{rs} are not independent (see § 34 below).

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§ 28. The functions at which we arrived in the preceding paragraph, it will be remembered, are only defined in Φ; but from the remark made in § 7 we see that they are branches—*principal branches* it will be convenient to call them—of n monogenic functions localised in Φ. In general the remaining branches of any one of these functions when regarded as localised in Φ will have properties different from those of the principal branch; but in the case of ordinary Picard equations the solutions are all uniform and so only principal branches arise.

The group of localised functions arrived at in § 27 will be said to be *normal for the associated pair* (κ, κ'). Since every pair of associated roots can be treated in the same way there will obviously be a set of normal groups whose total number is that of the distinct pairs of associated roots of the fundamental system of period equations.

§ 29. The choice of the word "multiplicity" in § 27 will probably have prepared the reader for the theorem to the proof of which we now proceed, viz. that the sum of the multiplicities of the various associated pairs is equal to m, the order of the differential equation.

First of all we shall require the following

Theorem. No linear relation can exist between the members of the various normal groups of integrals.

We have seen already that no such relation can exist within a group, so our attention may be confined to the case in which the members of more than one group are involved. Suppose then, if possible, that a linear relation, embracing the members of two or more groups, exists, say

$$W_1(z) + W_2(z) + \ldots + W_t(z) = 0 \quad \text{(xvi)},$$

where $W_1(z)$, $W_2(z)$, ... $W_t(z)$ are each linear functions of members of a group corresponding to a definite associated pair. It will be convenient to define the order of any such function to be p, where integrals of the pth set are the latest involved: e.g. if the integrals involved are all doubly-periodic of the second species, the order will be unity. The relation (xvi) above written will be said to be of the pth order when p is the greatest of the orders of the various functions $W_r(z)$.

In the first place let us prove that every non-evanescent linear relation (xvi) of order p involves one of order $p-1$ provided that $p > 1$.

From (xvi) we have

$$\left.\begin{aligned} W_1(z+\omega) + W_2(z+\omega) + \ldots + W_t(z+\omega) &= 0 \\ W_1(z+\omega') + W_2(z+\omega') + \ldots + W_t(z+\omega') &= 0 \end{aligned}\right\} \quad \text{(xvii)}.$$

Suppose that the order of W_n is > 1. An inspection of the results of § 27, shews that

$$W_n(z+\omega) = \kappa_n W_n(z) + V_n(z),$$
$$W_n(z+\omega') = \kappa_n' W_n(z) + V_n'(z),$$

where one at least of the functions $V_n(z)$, $V_n'(z)$, which are linear functions of integrals belonging to the same group as the terms in W_n, is of order less by unity (and no

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more) than $W_n(z)$. Suppose that $V_n(z)$ is such when $W_n(z)$ is one of the functions of order p.

From (xvii) we have

$$\kappa_1 W_1(z) + \kappa_2 W_2(z) + \ldots + \kappa_t W_t(z) + V_1(z) + \ldots + V_t(z) = 0,$$

where one of the V's at least is of order $p-1$.

In a similar way since

$$W_1(z+2\omega) + W_2(z+2\omega) + \ldots + W_t(z+2\omega) = 0,$$

we have

$$\kappa_1^2 W_1(z) + \kappa_2^2 W_2(z) + \ldots + \kappa_t^2 W_t(z) + 2\kappa_1 V_1(z) + 2\kappa_2 V_2(z) + \ldots + 2\kappa_t V_t(z) + U_2 = 0,$$

where U_2 is a sum of functions which may all vanish, but if they do not they are of order $\leqslant p-2$.

Generally we have

$$\kappa_1^r W_1(z) + \ldots + \kappa_t^r W_t(z) + r\kappa_1^{r-1} V_1(z) + \ldots + r\kappa_t^r V_t(z) + U_r = 0 \quad \ldots\ldots\text{(xviii)},$$

where U_r is of the same character as U_2.

Now it may or may not be that all the numbers κ_r are different: if they are not we can group together those terms for which the multiplier is the same. Let this be done. Since the equations (xviii) are unaltered in form we can suppose them as written only now the κ's are all different, and the W's and V's may involve members of more than one group.

§ 30. Taking the system of t equations obtained by writing $r = 1, 2, \ldots t$ in (xviii) together with the equation (xvi) we can eliminate the W's. We thus obtain a linear relation

$$\sum_{r=1}^{t} \Delta_r V_r(z) + U = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xix)},$$

where U either vanishes or else is a sum of integrals of order $\leqslant p-2$ and Δ_r is the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \ldots\ldots & 0 \\ \kappa_1 & \kappa_2 & \kappa_3 & \ldots\ldots & 1 \\ \kappa_1^2 & \kappa_2^2 & \kappa_3^2 & \ldots\ldots & 2\kappa_r \\ \kappa_1^3 & \kappa_2^3 & \kappa_3^3 & \ldots\ldots & 3\kappa_r^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \kappa_1^t & \kappa_2^t & \kappa_3^t & \ldots\ldots & t\kappa_t^{t-1} \end{vmatrix}.$$

It is an easy matter to shew that this determinant does not vanish. For let X be the same determinant as above save that its last column has for its elements $1, x, x^2, \ldots x^t$ reading downwards. Then as is well known

$$X = \prod_{q=1}^{t} (x - \kappa_q) \times P,$$

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where P is the product of all terms $(\kappa_r - \kappa_s)\,(r \neq s,\ r,\ s = 1,\ 2, \ldots t)$. Differentiating with regard to x and then writing $x = \kappa_r$, it is clear that the left side reduces to Δ_r. Thus

$$\Delta_r = \prod_{q=1}^{t}{}' \,(\kappa_r - \kappa_q) \times P,$$

where Π' indicates that the term corresponding to $q = r$ is omitted. Since the κ's are all different it follows that Δ_r is not zero.

Returning now to the linear equation (xix) we know that at least one of the V's involves a non-evanescent function of order $p-1$ from one of the groups. As no two V's involve members of the same group and none of the Δ's vanish, (xix) must be a non-evanescent linear relation of order $p-1$. It appears therefore that every linear relation of order $p\,(>1)$ involves one of order $p-1$, and so by repeated applications whatever be the order of the relation (xvi) from which we start, we eventually arrive at a non-evanescent linear relation of order unity.

§ 31. Let the relation just mentioned be

$$u_1(z) + u_2(z) + \ldots + u_l(z) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xx)},$$

where each of the u's is a linear combination of doubly-periodic functions of the second species which have the same multipliers.

The multiplier corresponding to the period ω may be the same for more than one of the groups u: if so, then we group together those of the u's for which this happens. The relation (xx) then becomes

$$v_1(z) + v_2(z) + \ldots + v_j(z) = 0, \qquad (j \leqslant l)$$

where each member of the group $v_r(z)$ has the same multiplier for the period ω.

Since we have

$$v_1(z + r\omega) + v_2(z + r\omega) + \ldots + v_j(z + r\omega) = 0,$$

it follows that

$$\kappa_1^r\, v_1(z) + \kappa_2^r\, v_2(z) + \ldots + \kappa_j^r\, v_j(\gamma) = 0,$$

for all values of r: κ_t being the common multiplier for the period ω of the terms of the group $v_t(z)$.

In particular we have the system of j equations

$$\begin{array}{ccccccc}
v_1(z) & + & v_2(z) & + \ldots + & v_j(z) & = 0, \\
\kappa_1 v_1(z) & + & \kappa_2 v_2(z) & + \ldots + & \kappa_j v_j(z) & = 0, \\
\kappa_1^2 v_1(z) & + & \kappa_2^2 v_2(z) & + \ldots + & \kappa_j^2 v_j(z) & = 0, \\
\ldots & & \ldots & & \ldots & \\
\ldots & & \ldots & & \ldots & \\
\kappa_1^{j-1} v_1(z) & + & \kappa_2^{j-1} v_2(z) & + \ldots + & \kappa_j^{j-1} v_j(z) & = 0.
\end{array}$$

The determinant of the coefficients of the v's is not zero* so we must have

$$v_r(z) = 0. \qquad (r = 1,\ 2, \ldots j)$$

* Cf. § 30 above.

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Now considering the left-hand member of any one of these relations we perceive that it is equal to

$$w_1(z) + w_2(z) + \ldots + w_k(z),$$

where $w_1(z), \ldots w_k(z)$ are each linear combinations of integrals belonging to the first set of integrals of *different* groups. Since each of them had the same multiplier for the period ω they must have different multipliers for the period ω'. Applying an argument similar to that just used, it will follow that the w's are all zero. But this is impossible, since as we have said they are linear combinations of integrals of the *same* group. Our initial assumption that there exists a relation of the form (xvi) is therefore proved false, i.e. no linear relation can exist between the various members of normal groups.

§ 32. We are now in a position to prove the theorem mentioned in § 29.

Theorem. The sum of the multiplicities of the various associated pairs of roots of a fundamental system of period equations is equal to the order (m) of the differential equation.

Obviously the sum mentioned cannot be greater than m: for if it were the theorem of §§ 29—31 would shew that we had more than m linearly independent solutions of the equation which is known not to be possible.

Suppose that it should happen that the sum is n where $n < m$. As these n are linearly independent we can find $m - n$ other integrals which with them constitute a fundamental system. Let this latter be

$$\begin{gathered}
\gamma_{11}(z), \gamma_{12}(z), \ldots \gamma_{1n_1}(z), \gamma_{21}(z), \gamma_{22}(z), \ldots \gamma_{2n_2}(z), \ldots \gamma_{t_1 n_{t_1}}(z), \\
\delta_{11}(z), \delta_{12}(z), \ \ldots\ldots \ \delta_{1p_1}(z), \delta_{21}(z), \ \ldots\ldots \ \delta_{2p_2}(z), \ldots \delta_{t_2 p_{t_2}}(z), \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\chi_{11}(z), \chi_{12}(z), \ \ldots\ldots \ \chi_{1r_1}(z), \chi_{21}(z), \ \ldots\ldots \ \chi_{2r_2}(z), \ldots \chi_{t_s r_{t_s}}(z), \\
\psi_{n+1}(z), \psi_{n+2}(z), \ \ldots\ldots\ldots \ \psi_m(z),
\end{gathered}$$

where the γ's are integrals of one group, for which there are t_1 sets, the first consisting of n_1 integrals, the second of n_2, and so on, the total number of sets t_1 being the number of invariants. Similarly for the δ's ... χ's. As for the ψ's they are the $m - n$ integrals mentioned above.

Let the period matrices for this system be g and h, so that the period system is

$$\| g_{1m}, \theta : h_{1m}, \theta' \| \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxi)}.$$

By blotting out the first n rows and the first n columns of g's and h's from this it is clear that we get a system of $m - n$ rows and $2(m - n)$ columns; for brevity we call this the *residuary system.* It is easy to see by repeated applications of theorem I, § 21, that this residuary system is conjoint. It will therefore have at least one pair of associated roots (κ, κ'). There are two cases *à priori* possible—either (κ, κ') may be an associated pair of roots of the system (xxi), or they may not. The latter cannot be the case, for by repeated applications of theorem II, § 22, any associated pair of the residuary system is an associated pair for (xxi). Suppose then that (κ, κ') are an

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associated pair both for the complete system and for the residuary. Without loss of generality we can suppose that the group of γ's belongs to this pair. Repeated applications of the theorem just mentioned shews that (κ, κ') are an associated pair for the system obtained by blotting out the first $(n_1 + n_2 + \ldots + n_{t_1})$ rows and the corresponding columns of the system (xxi), i.e. a t_1th derived system of (xxi) with respect to the associated pair (κ, κ'). By using theorem III, § 23, we prove that all such systems are transforms of one another: and further, from § 27 it appears that one of them has *not* (κ, κ') for an associated pair of roots. It follows therefore from theorem II, §§ 14, 15, that (κ, κ') cannot be an associated pair both for (xxi) and the residuary system. As we have seen already that there is no pair for the latter which is not one for the former, the only alternative is that the ψ's are non-existent, i.e. $n = m$, which proves the theorem.

Summing up our results thus far, we conclude that if we take normal groups corresponding to each of the various associated pairs of the fundamental system of period equations, they together constitute a fundamental system of integrals of the differential equation.

§ 33. The reader will now perceive the analogy that exists between the theory we have developed and that of the corresponding one dimensional problem in which the coefficients are simply periodic. It is true that our investigation is more cumbrous, but that is to be expected both from the nature of the case and from the fact that we have not had anything like the theory of equations and of elementary divisors on which to fall back. Moreover, we have not obtained anything analogous to Hamburger's subgroups. Two facts seem to conspire against this. The first is that the integrals are connected together in a much more intimate manner. For instance, if we consider the integral $w_{3r}(z)$ of § 27, the functions $v_{2r}(z)$ and $v'_{2r}(z)$ may be distinct linear functions of integrals belonging to the second set, and similarly $v_{1r}(z)$ and $v'_{1r}(z)$ may be distinct linear functions of integrals of the first set. Consequently, when we wish to express

$$w_{3r}(z + t\omega + t'\omega')$$

as a function of integrals whose argument is z we will in general require *two* independent integrals of the second set and *four* from the first, besides $w_{3r}(z)$. In the corresponding one dimensional case we would have to replace *two* and *four* by *one* and *two* respectively. The second fact is that the numbers $\sigma_1, \sigma_2, \ldots \sigma_s$ may be an increasing set. All that we can be sure of is that

$$\sigma_s \leqslant 2\sigma_{s-1} \leqslant 2^2\sigma_{s-2} \leqslant \ldots \leqslant 2^{s-1}\sigma_1.$$

In the first place we will shew that

$$\sigma_2 \leqslant 2\sigma_1.$$

Take the period system of § 19 and use the notation there employed.

From the definition of σ_1 we know that there is a certain σ_1th minor of a determinant of this system which does not vanish. Suppose that α of its columns have elements belonging to the first σ_1 columns of the c's and that β of them have elements belonging to the first σ_1 columns of the d's. The minor (M) in question may then be expanded by Laplace's rule as a product of minors belonging to these $\alpha + \beta$ columns

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and their complementary minors which belong to the remaining columns. These latter, if they do not vanish identically, are minors of determinants of the component system and cannot all vanish, since M does not do so. It follows therefore that as they consist of $m-\sigma_1-\alpha-\beta$ rows and columns we have

$$m-\sigma_1-\alpha-\beta \leqslant m-\sigma_1-\sigma_2,$$

i.e.

$$\sigma_2 \leqslant \alpha+\beta,$$

and as

$$\alpha \leqslant \sigma_1, \quad \beta \leqslant \sigma_1,$$

we have

$$\sigma_2 \leqslant 2\sigma_1.$$

In a similar way we can prove that

$$\sigma_3 \leqslant 2\sigma_2,$$

hence

$$\sigma_3 \leqslant 2\sigma_2 \leqslant 2^2\sigma_1;$$

and so on: the final result being as is easily seen

$$\sigma_s \leqslant 2\sigma_{s-1} \leqslant 2^2\sigma_{s-2} \leqslant \ldots \leqslant 2^{s-1}\sigma_1.$$

§ 34. We proceed to give some indication of the analytic form of our integrals, or more precisely of their principal branches, which we recall are considered as being localised in Φ.

As a preparation for this we will consider the relation, to which reference was made in a footnote attached to § 27, as existing between the matrices ρ and ρ'.

As the functions $w_r(z)$ are uniform in Φ we have

$$w_r((z+\omega)+\omega') = w_r((z+\omega')+\omega),$$

i.e.

$$\begin{aligned}\kappa' w_r(z+\omega) + \rho'_{r,r-1} w_{r-1}(z+\omega) + \rho'_{r,r-2} w_{r-2}(z+\omega) + \ldots + \rho'_{r1} w_1(z+\omega)\\ = \kappa w_r(z+\omega') + \rho_{r,r-1} w_{r-1}(z+\omega') + \rho_{r,r-2} w_{r-2}(z+\omega') + \ldots + \rho_{r1} w_1(z+\omega'),\end{aligned}$$

or

$$\begin{aligned}&\kappa'(\kappa w_r(z) + \rho_{r,r-1} w_{r-1}(z) + \ldots + \rho_{r1}\omega_1(z)) + \rho'_{r,r-1}(\kappa w_{r-1}(z))\\ &\quad + \rho_{r-1,r-2} w_{r-2}(z) + \ldots + \rho_{r-1,1} w_1(z)) + \rho'_{r,r-2}(\kappa w_{r-2}(z) + \ldots + \rho_{r-2,1} w_1(z))\\ &\quad + \ldots + \rho'_{r,1}\kappa w_1(z)\\ &= \kappa(\kappa' w_r(z) + \rho'_{r,r-1} w_{r-1}(z) + \ldots + \rho'_{r1} w_1(z)) + \rho_{r(r-1)}(\kappa' w_{r-1}(z)\\ &\quad + \rho'_{r-1,r-2} w_{r-2}(z) + \ldots + \rho'_{r-1,1} w_1(z)) + \rho_{r,r-2}(\kappa' w_{r-2}(z)\\ &\quad + \ldots + \rho'_{r-2,1} w_1(z)) + \ldots + \rho_{r,1}\kappa' w_1(z).\end{aligned}$$

The functions w being linearly independent the equation just written must be an identity.

Equating coefficients of the various w's we obtain

$$\rho'_{rs}\rho_{st} = \rho_{rs}\rho'_{st}, \qquad (r>s>t)$$

which are the relations referred to.

It is a very simple deduction from this to prove that if we take any product of matrices

$$\rho_{r,s_1}\rho_{s_1s_2}\rho_{s_2s_3}\cdots\rho_{s_pt}, \qquad (r>s_1>s_2>\ldots>s_p>t)$$

and write dashes over *any* l of them, the product is the same as

$$\rho_{r,s_1}\rho_{s_1s_2}\cdots\rho_{s_{p-l}s_{p-l+1}}\rho'_{s_{p-l+1}s_{p-l+2}}\rho'_{s_{p-l+2}s_{p-l+3}}\cdots\rho'_{s_pt}.$$

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§ 35. Consider now the functions

$$u(z)=\frac{\omega}{2\pi\iota}\left\{\frac{\eta}{\omega}z-\zeta(z)\right\},\quad v(z)=\frac{\omega'}{2\pi\iota}\left\{\zeta(z)-\frac{\eta'}{\omega'}z\right\},$$

where ζ is the usual Weierstrassian ζ function and $\eta=\zeta(\omega)$, $\eta'=\zeta(\omega')$. If we adopt the usual convention that the imaginary part of $\left(\frac{\omega'}{\omega}\right)$ is positive, we have

$$u(z+\omega)=u(z),\qquad v(z+\omega)=v(z)+1,$$
$$u(z+\omega')=u(z)+1,\quad v(z+\omega')=v(z).$$

Let us write

$$\Delta F=\kappa^{-1}F(z+\omega)-F(z),$$
$$\Delta' F=\kappa'^{-1}F(z+\omega')-F(z):$$

then if $\phi(z)$ is a doubly-periodic function of the second species whose multipliers are (κ, κ'), we have

$$\Delta(\phi u)=\kappa^{-1}\phi(z+\omega)\,u(z+\omega)-\phi(z)\,u(z)=0,$$
$$\Delta'(\phi u)=\kappa'^{-1}\phi(z+\omega')\,u(z+\omega')-\phi(z)\,u(z)=\phi(z),$$

and similarly $\qquad \Delta(\phi v)=\phi(z),\quad \Delta'(\phi v)=0.$

More generally if we write

$$u^{(r)}=\frac{u(u-1)(u-2)\ldots(u-r+1)}{r!},$$

$$v^{(r)}=\frac{v(v-1)(v-2)\ldots(v-r+1)}{r!},$$

where $u=u(z)$, $v=v(z)$, we easily prove that

$$\Delta(\phi u^{(r)})=0,\qquad \Delta'(\phi u^{(r)})=\phi u^{(r-1)},$$
$$\Delta(\phi v^{(r)})=\phi v^{(r-1)},\quad \Delta'(\phi v^{(r)})=0.$$

§ 36. Referring now to the results of § 27, and adopting the notation there used, it appears that the first set are doubly-periodic functions of the second species localised in Φ and having multipliers (κ, κ'). The second set satisfy

$$\Delta(w_2)=\frac{1}{\kappa}\rho_{21}w_1(z),\quad \Delta'(w_2)=\frac{1}{\kappa'}\rho'_{21}w_1(z).$$

Consequently (§ 35)

$$\Delta\left(w_2-\frac{u}{\kappa'}\rho'_{21}w_1-\frac{v}{\kappa}\rho_{21}w_1\right)=0,$$

$$\Delta'\left(w_2-\frac{u}{\kappa'}\rho'_{21}w_1-\frac{v}{\kappa}\rho_{21}w_1\right)=0,$$

i.e. the functions

$$w_2(z)-\frac{u}{\kappa'}\rho'_{21}w_1(z)-\frac{v}{\kappa}\rho_{21}w_1(z)$$

are doubly-periodic functions of the second species, localised in Φ and having multipliers (κ, κ'), say they are $W_2(z)$. It appears therefore that the integrals of the second set can be represented by

$$W_2(z)+\frac{u}{\kappa'}\rho'_{21}w_1(z)+\frac{v}{\kappa}\rho_{21}w_1(z),$$

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or if we replace w_1 by W_1 in order to have a more symmetrical notation, we have

$$w_2(z) = W_2(z) + \frac{u}{\kappa'}\rho'_{21} W_1(z) + \frac{v}{\kappa}\rho_{21} W_1(z),$$

there being, of course, σ_2 integrals of this kind.

Again, for integrals of the third set, we have

$$\begin{aligned}\Delta w_3 &= \frac{1}{\kappa}\rho_{32} w_2(z) + \frac{1}{\kappa}\rho_{31} w_1(z)\\ &= \frac{1}{\kappa}\rho_{32}\left(W_2(z) + \frac{u}{\kappa'}\rho'_{21} W_1(z) + \frac{v}{\kappa}\rho_{21} W_1(z)\right) + \frac{1}{\kappa}\rho_{31} W_1(z)\\ &= \frac{1}{\kappa}\{\rho_{32} W_2(z) + \rho_{31} W_1(z)\} + \frac{u}{\kappa\kappa'}\rho_{32}\rho'_{21} W_1(z) + \frac{v}{\kappa^2}\rho_{22}\rho_{21} W_1(z);\end{aligned}$$

similarly $$\Delta'(w_3) = \frac{1}{\kappa'}\{\rho'_{32} W_2(z) + \rho'_{31} W_1(z)\} + \frac{u}{\kappa'^2}\rho'_{32}\rho'_{21} W_1(z) + \frac{v}{\kappa\kappa'}\rho'_{32}\rho_{21}.$$

From the first of these it is easy to see that

$$\begin{aligned}\Delta\Big(w_3 - \frac{v}{\kappa}\rho_{32} W_2 - \frac{u}{\kappa'}\rho'_{32} W_2 - \frac{v}{\kappa}\rho_{31} W_1 - \frac{u}{\kappa'}\rho'_{31} W_1\\ - \frac{u^{(2)}}{\kappa'^2}\rho'_{32}\rho'_{21} W_1 - \frac{uv}{\kappa\kappa'}\rho_{32}\rho'_{21} W_1 - \frac{v^{(2)}}{\kappa^2}\rho_{32}\rho_{21} W_1\Big)\\ = 0,\end{aligned}$$

and similarly (since $\rho_{32}\rho'_{21} = \rho'_{32}\rho_{21}$)

$$\Delta'\left(w_3 - \frac{v}{\kappa}\rho_{32} W_2 - \ldots\right) = 0.$$

We deduce from this that

$$\begin{aligned}w_3(z) = W_3(z) + \frac{v}{\kappa}\rho_{32} W_2(z) + \frac{u}{\kappa'}\rho'_{32} W_2(z) + \frac{v}{\kappa}\rho_{31} W_1(z) + \frac{u}{\kappa'}\rho'_{31} W_1(z)\\ + \frac{u^{(2)}}{\kappa'^2}\rho'_{32}\rho'_{21} W_1(z) + \frac{uv}{\kappa\kappa'}\rho_{32}\rho'_{21} W_1(z) + \frac{v^{(2)}}{\kappa^2}\rho_{32}\rho_{21} W_1(z),\end{aligned}$$

where $W_3(z)$ are σ_3 localised doubly-periodic functions (multipliers κ, κ').

§ 37. Generally we say that we can form an expression for $w_r(z)$ by the following rule:

Take any product $$\rho_{rs_1}\rho_{s_1s_2}\rho_{s_2s_3}\cdots\rho_{s_{p-1},t},$$

where $$r > s_1 > s_2 > \ldots > s_{p-1} > t\ (p \leqslant r-1).$$

Write dashes over all but the first x of them. Then the symbolic expression for the integrals of the rth set is the sum of all possible terms

$$\frac{u^{(p-x)}v^{(x)}}{(\kappa')^{p-x}(\kappa)^x}\rho_{rs_1}\rho_{s_1s_2}\cdots\rho_{s_{x-1}s_x}\rho'_{s_xs_{x+1}}\cdots\rho'_{s_{p-1},t} W_t(z).$$

where $W_t(z)$ are σ_t functions which are doubly-periodic of the second species localised in Φ and having multipliers (κ, κ').

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The sum just mentioned will be denoted by

$$\underset{r}{S}\frac{u^{(p-x)}\,v^{(x)}}{(\kappa')^{p-x}(\kappa)^{x}}\rho_{rs_1}\rho_{s_1s_2}\cdots\rho'_{s_xs_{x+1}}\cdots\rho'_{s_{p-1},t}W_t(z):$$

it includes a term $W_r(z)$ corresponding to $p=x=0$.

To prove this we first observe that the results of § 36 shew it to be true in the cases $r=1, 2, 3$. Assume the formulae true for $w_1(z)$, $w_2(z)$, ... $w_{r-1}(z)$, and consider the effect of the operation Δ on the functions $F(z)$ which have for their symbolic expression the symbol above written with the term $W_r(z)$ omitted: evidently

$$\Delta(F)=\frac{1}{\kappa}\left\{\text{sum of all possible terms like }\frac{u^{(p-x)}\,v^{(x-1)}}{(\kappa')^{p-x}\,\kappa^{x-1}}\rho_{rs_1}\rho_{s_1s_2}\cdots\rho'_{s_xs_{x+1}}\cdots\rho_{s_{p-1},t}W_t(z)\right\}.$$

Noticing (§ 35) that the terms for which $x=0$ disappear when operated on by Δ, we see, by collecting together all terms in which ρ_{rs_1} is the first matrix of the product, that

$$\Delta(F)=\frac{1}{\kappa}\left\{\sum_{s_1=1}^{r-1}\rho_{rs_1}\left(\underset{s_1}{S}\frac{u^{(q-y)}\,v^{(y)}}{(\kappa')^{(q-y)}\,\kappa^{y}}\rho_{s_1s_2}\rho_{s_2s_3}\cdots\rho'_{s_xs_{x+1}}\cdots\rho'_{s_{p-1},t}W_t(z)\right\}\right.,$$

where $q\,(=p-1)$ is the total number of ρ's in the matrix product in the bracket ( ), which corresponds to the general term above written, and $y\,(=x-1)$ is the number of them undashed.

It follows that
$$\Delta(F)=\frac{1}{\kappa}\left\{\sum_{s_1=1}^{r-1}\rho_{rs_1}w_{s_1}(z)\right\},$$

and therefore that

$$\Delta(w_r-F)=0 \quad\text{.....(xxii).}$$

Again, we see that

$$\Delta'(F)=\frac{1}{\kappa'}\left\{\text{sum of all possible terms like }\frac{u^{(p-x-1)}\,v^{(x)}}{(\kappa')^{p-x-1}\,\kappa^{x}}\rho_{rs_1}\cdots\rho'_{s_xs_{x+1}}\cdots\rho'_{s_{p-1},t}W_t(z)\right\},$$

where of course the terms corresponding to products of matrices which contain no undashed letters disappear (§ 35). In virtue of the remark made at the end of § 34

$$\Delta'(F)=\frac{1}{\kappa'}\left\{\text{sum of all possible terms like }\frac{u^{(p-x-1)}\,v^{(x)}}{(\kappa')^{p-x-1}\,\kappa^{x}}\rho'_{rs_1}\rho_{s_1s_2}\cdots\rho_{s_xs_{x+1}}\rho'_{s_{x+1}s_{x+2}}\cdots\rho'_{s_{p-1},t}W_t(z)\right\},$$

where now there are $p-x-1$ dashed letters at the tail of the matrix product. Collecting all terms for which ρ'_{rs_1} is the first term of this product, we see that

$$\Delta'(F)=\frac{1}{\kappa'}\left\{\sum_{s_1=1}^{r-1}\rho'_{rs_1}\left(\underset{s_1}{S}\frac{u^{(q-x)}\,v^{(x)}}{(\kappa')^{q-x}\,\kappa^{x}}\rho_{s_1s_2}\cdots\rho_{s_xs_{x+1}}\rho'_{s_{x+1}s_{x+2}}\cdots\rho'_{s_{p-1},t}W_t(z)\right)\right\},$$

where $q\,(=p-1)$ is the number of ρ's in the matrix product of the term in the bracket ( ) corresponding to the general term above written, and x is the number of them undashed. Hence

$$\Delta'(F)=\frac{1}{\kappa'}\left\{\sum_{s_1=1}^{r-1}\rho'_{rs_1}w_{s_1}(z)\right\},$$

and therefore

$$\Delta'(w_r-F)=0 \quad\text{.....(xxiii).}$$

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From (xxii) and (xxiii) it is clear that the functions $w_r(z) - F(z)$ are doubly-periodic functions of the second species, say $W_r(z)$. Then

$$w_r(z) = F(z) + W_r(z),$$

which proves the formula to be true for $w_r(z)$. But we have seen that it is true for $r = 1, 2, 3$; therefore it is true for $r = 4$; and successively for $r = 5, 6, \ldots$. The rule given above therefore suffices for the determination of the integral belonging to the various sets.

§ 38. We proceed to make an important deduction from the analytical forms at which we arrived in the preceding paragraph. It is embodied in the following:

Theorem. The analytical expressions for the rth set of integrals belonging to the group associated with the pair $(\kappa\kappa')$ are *all precisely of degree* $r-1$ in $u(z)$ and $v(z)$.

In order to prove this it will be necessary to shew that none of the functions

$$\sum_{x=0}^{r-1} \frac{u^{(r-x-1)}\, v^{(x)}}{(\kappa')^{r-x-1}\,(\kappa)^x}\, \rho_{r,r-1}\, \rho_{r-1,r-2} \cdots \rho_{r-x+1,r-x}\, \rho'_{r-x,r-x-1} \cdots \rho'_{21}\, W_1(z)$$

are evanescent, i.e. that there exists at least one matrix

$$\rho_{r,r-1}\, \rho_{r-1,r-2} \cdots \rho_{r-x+1,r-x}\, \rho'_{r-x,r-x-1} \cdots \rho'_{21}$$

such that if q is an integer $\leqslant \sigma_r$ the elements of its qth row do not all vanish.

We saw (§ 27) that there exists no non-evanescent linear combinations of the functions $\rho_{s,s-1} W_1(z)$, such that it and the same combination of the functions $\rho'_{s,s-1} W_1(z)$ *both* vanish. Since the functions $W_1(z)$ are linearly independent, this amounts to saying that if m_s is any matrix having σ_s columns, which is such that the elements of its qth row do not all vanish, then the elements of the qth row of one at least of the products $m_s\, \rho_{s,s-1}$, $m_s\, \rho'_{s,s-1}$ has the same property.

Referring again to § 27 we see that at least one of $\rho_{r,r-1}$, $\rho'_{r,r-1}$ is such that the elements of its qth row are not all zero: denote this one (or either, if both have the property) by $\rho_{r,r-1}$ so that $\bar{\rho}_{r,r-1}$ is $\rho_{r,r-1}$ or $\rho'_{r,r-1}$ as the case may be. An application of what has been said above shews that one at least of

$$\bar{\rho}_{r,r-1}\, \rho_{r-1,r-2},\ \bar{\rho}_{r,r-1}\, \rho'_{r-1,r-2}$$

is such that the elements of its qth row do not all vanish; we can denote this by

$$\bar{\rho}_{r,r-1}\, \bar{\rho}_{r-1,r-2},$$

where $\bar{\rho}_{r-1,r-2}$ is either $\rho_{r-1,r-2}$ or $\rho'_{r-1,r-2}$. Proceeding in this way we get a product

$$\bar{\rho}_{r,r-1}\, \bar{\rho}_{r-1,r-2}\, \bar{\rho}_{r-2,r-3} \cdots \bar{\rho}_{21},$$

such that the elements of its qth row do not all vanish. This product is equal to

$$\rho_{r,r-1}\, \rho_{r-1,r-2} \cdots \rho_{r-x+1,r-x}\, \rho'_{r-x-1} \cdots \rho'_{21},$$

where x is the number of the $\bar{\rho}$'s undashed (§ 34), and so the theorem is established.

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§ 39. Thus taking any generalised Picard equation whose solutions are uniform in a doubly-periodic region Φ we can say, as a result of our investigation:—

Corresponding to an associated pair of roots (κ, κ') (multiplicity n) of the fundamental system of period equations we can find n monogenic functions, whose principal branches are linearly independent polynomials in

$$u\left(=\frac{\omega}{2\pi\iota}\left\{\frac{\eta}{\omega}z-\zeta(z)\right\}\right) \text{ and } v\left(=\frac{\omega'}{2\pi\iota}\left\{\zeta(z)-\frac{\eta' z}{\omega'}\right\}\right),$$

with coefficients which are doubly-periodic functions of the second species localised in Φ and having multipliers (κ, κ').

These polynomials divide into s sets where s is the number of invariants of the pair (κ, κ'), the rth of the sets consisting of polynomials (equal in number to the rth invariant of (κ, κ')) which are *all* of degree $r-1$ in u and v.

In the particular case of ordinary Picard equations, Φ may be taken to be the whole plane with the exception of the singularities of the coefficients of the equation; and what has been said of principal branches applies to the complete function, the coefficients of the polynomials in u and v being now single-valued doubly-periodic functions of the second species.

Still dealing with ordinary Picard equations, if we make the further limitation that its integrals are all regular at the various singularities of the coefficients, which are isolated so far as regards the finite part of the plane, then the coefficients of the polynomials in u and v will be the meromorphic doubly-periodic functions of the second species usually considered in treatises on elliptic functions.

II.

§ 1. It will be observed that in the previous section, when we were considering the monogenic functions which were solutions of generalised Picard equations we had to confine ourselves to particular branches—principal branches as we called them. The remaining branches (possibly infinite in number) when regarded as localised in Φ will certainly be expressible as linear functions of some or all of these principal branches, but that is about all we can say in general.

Difficulties of this nature seem to shew that it is only by restricting the nature of the solutions that we will be able to attain definite knowledge of *all* branches of any given solution, and if this is true of a limited class of equations it will be *à fortiori* true of the general class. The present section is devoted to the consideration of a class of solutions which, though very particular when looked at from a general point of view, includes the solutions of the ordinary Picard equation and of that of Halphen. As a preparation for this it will be well to resume for a little while the general considerations of (I, §§ 1—4).

§ 2. Suppose then that we take the equation of I, § 1, which is not necessarily assumed to be of the ordinary Picard type. Using the notation of the paragraph referred to, we saw that if we continued any power series $P_r(z-z_0)$ to the point $(z_0+\omega)$ along a definite path (γ) we were able to deduce from the resulting series one $Q_r(z-z_0)$ which was a solution of the differential equation. The method we employed is obviously applicable if, instead of ω, we take any

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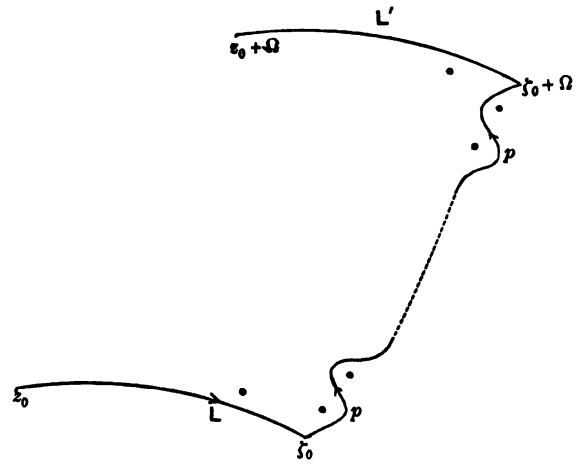
other period Ω $(= n\omega + n'\omega')$ where n and n' are any integers positive or negative. It follows, therefore, that corresponding to each point $z_0 + n\omega + n'\omega'$ we obtain one or more power series* in $(z - z_0)$ which are solutions of the equation. From our general theory we know that not more than m of these are linearly independent; so there will be a certain minimum number c of them such that all the others can be expressed as linear functions of these c. Any such c series—say $w_1(z - z_0)$, $w_2(z - z_0)$, ... $w_c(z - z_0)$—must obviously be linearly independent: we call them a *basic* system at the point z_0 for the monogenic function f_r whose initial element is $P_r(z - z_0)$.

§ 3. Suppose now that we take any other point ζ_0 which is not a singularity of f_r. Instead of considering the elements of f_r at the points $z_0 + n\omega + n'\omega'$ as above, let us study the elements of the function at the points $\zeta_0 + n\omega + n'\omega'$.

Let L be a definite path from z_0 to ζ_0 which passes through no singularity of the coefficients, and let the series $w_1(z - z_0)$, $w_2(z - z_0)$, ... $w_c(z - z_0)$ continued along this path by a standard chain become $\bar{\omega}_1(z - \zeta_0)$, $\bar{\omega}_2(z - \zeta_0)$, ... $\bar{\omega}_c(z - \zeta_0)$: it is not difficult to see that the latter series are a basic system for the point ζ_0. In the first place, since we can find periods $\Omega_1, \Omega_2 \ldots \Omega_c$ such that $w_1(z - (z_0 + \Omega_1))$, $w_2(z - (z_0 + \Omega_2))$, ... $w_c(z - (z_0 + \Omega_c))$ are elements of f_r, it is clear that $\bar{\omega}_1(z - (\zeta_0 + \Omega_1))$, $\bar{\omega}_2(z - (\zeta_0 + \Omega_2))$, ... $\bar{\omega}_c(z - (\zeta_0 + \Omega_c))$ are also elements of it. Again, let $\Pi(z - \zeta_0)$ be any element of f_r whose base is ζ_0, and p be any path between ζ_0



and $\zeta_0 + \Omega$, Ω being any period $n\omega + n'\omega'$. When the element $\Pi(z - \zeta_0)$ is continued along L from ζ_0 to z_0 we get a power series in $(z - z_0)$, say $P(z - z_0)$. Imagine that this last series is continued along the path LpL' where L' is the path congruent with L joining $z_0 + \Omega$ and $\zeta_0 + \Omega$: the resulting power series $P'(z - (z_0 + \Omega))$ is equal to

$$m_1 w_1(z - (z_0 + \Omega)) + m_2 w_2(z - (z_0 + \Omega)) + \ldots + m_c w_c(z - (z_0 + \Omega)),$$

by hypothesis (§ 2) the m's being certain constants.

Now if $\Pi'(z - (\zeta_0 + \Omega))$ is the power series, we obtain by continuing $\Pi(z - \zeta_0)$ from ζ_0 to $\zeta_0 + \Omega$ by a standard chain along p, $P'(z - (z_0 + \Omega))$ when continued along L' to $(\zeta_0 + \Omega)$ will become $\Pi'(z - (\zeta_0 + \Omega))$; also the series $w_r(z - (z_0 + \Omega))$ will become $\bar{\omega}_r(z - (\zeta_0 + \Omega))$ when continued along the same path. Hence

$$\Pi'(z - (\zeta_0 + \Omega)) = m_1 \bar{\omega}_1(z - (\zeta_0 + \Omega)) + \ldots + m_c \bar{\omega}_c(z - (\zeta_0 + \Omega)).$$

* We shall describe these, in future, as the Q series for z_0.

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If therefore γ be the number for the point ζ_0 corresponding to c for the point z_0, we have

$$\gamma \leqslant c.$$

Mutatis mutandis we prove

$$c \leqslant \gamma.$$

It follows from this that the number c is the same whatever base-point is taken: it may be defined to be the *content* of the solution (monogenic function) f_r. After what has been said above it will be evident that when any solution of content "c" is given we can find, at any base z_0, c linearly independent power series which are solutions of the equation.

§ 4. We now prove the following:

Theorem. Every solution of a linear differential equation (order m) whose coefficients are uniform doubly-periodic functions, is a solution of an equation of the same type whose order is "c" the content of that solution.

Let us adopt the notation of the preceding paragraphs so that

$$w_1(z-z_0),\ w_2(z-z_0)\ldots w_c(z-z_0)$$

are a basic system at the point z_0 for f_r. It is easily seen that, as every element of f_r at z_0 is of the form

$$l_1 w_1(z-z_0)+l_2 w_2(z-z_0)+\ldots+l_c w_c(z-z_0),$$

they all satisfy the equation

$$\begin{vmatrix} \frac{d^c w}{dz^c} & \frac{d^c w_1}{dz^c} & \frac{d^c w_2}{dz^c} & \cdots\cdots & \frac{d^c w_c}{dz^c} \\ \frac{d^{c-1} w}{dz^{c-1}} & \frac{d^{c-1} w_1}{dz^{c-1}} & \cdots\cdots & & \frac{d^{c-1} w_c}{dz^{c-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d^{c-r} w}{dz^{c-r}} & \frac{d^{c-r} w_1}{dz^{c-r}} & \cdots\cdots & & \frac{d^{c-r} w_c}{dz_c} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ w & w_1 & \cdots\cdots & & w_c \end{vmatrix} = 0 \quad\text{.....(i).}$$

The coefficient of $\frac{d^c w}{dz^c}$ on the left-hand side of the equation is the determinant

$$D_0(z-z_0) = \begin{vmatrix} \frac{d^{c-1} w_1}{dz^{c-1}} & \frac{d^{c-1} w_2}{dz^{c-1}} & \cdots\cdots & \frac{d^{c-1} w_c}{dz^{c-1}} \\ \frac{d^{c-2} w_1}{dz^{c-2}} & \frac{d^{c-2} w_2}{dz^{c-2}} & \cdots\cdots & \frac{d^{c-2} w_c}{dz^{c-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^{c-r} w_1}{dz^{c-r}} & \frac{d^{c-r} w_2}{dz^{c-r}} & \cdots\cdots & \frac{d^{c-r} w_c}{dz^{c-r}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ w_1 & w_2 & \cdots\cdots & w_c \end{vmatrix},$$

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whilst that of $\frac{d^{c-r}w}{dz^{c-r}}$ is $-D_r(z-z_0)$, where $D_r(z-z_0)$ is what $D_0(z-z_0)$ becomes when we replace the elements of its rth row by $\frac{d^c w_1}{dz^c}, \ldots \frac{d^c w_c}{dz^c}$ respectively.

The functions $D_0(z-z_0), D_1(z-z_0), \ldots D_c(z-z_0)$ are all power series in $(z-z_0)$, and so each of them may be regarded as the initial element of a monogenic function: suppose these are $\delta_0, \delta_1, \ldots \delta_c$ respectively. Since the singularities of the functions f_r are some or all of the points of the set S (I, § 1), those of the functions $\delta_0, \delta_1, \ldots \delta_c$ will be contained in this set.

§ 5. Consider what happens when we continue the function element $D_s(z-z_0)$ round a closed path (p) passing through no point of the set (S). The series $w_r(z-(z_0+\Omega_r))$ when continued round the path (P_r), which passes through $z_0+\Omega_r$ and is congruent with p, becomes equal to a linear function of $w_1(z-(z_0+\Omega_r)), w_2(z-(z_0+\Omega_r)), \ldots w_c(z-(z_0+\Omega_r))$. Hence the series $w_r(z-z_0)$ continued round (p) becomes the same linear function of $w_1(z-z_0), w_2(z-z_0), \ldots w_c(z-z_0)$, say

$$a_{r1}w_1(z-z_0)+a_{r2}w_2(z-z_0)+\ldots+w_{rc}w_c(z-z_0).$$

It follows that when $D_s(z-z_0)$ is so treated it will become

$$\begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1c} \\ a_{21} & a_{22} & \ldots\ldots & a_{2c} \\ a_{31} & a_{32} & \ldots\ldots & a_{3c} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{c1} & a_{c2} & \ldots\ldots & a_{cc} \end{vmatrix} D_s(z-z_0), \qquad (s=0, 1, 2, \ldots c)$$

i.e. a mere multiple of itself.

Taking in particular the function δ_0, all its function elements at a given base (cf. § 9, below), are mere multiples of one another: the zeros of all these elements will therefore be the same points. Hence, as the number of zeros of a power series in any region contained *within* its circle of convergence is finite, the zeros of δ_0 can only have the set (S) or a component of it for their first derived set. The set (S') of the zeros of δ_0 is therefore reducible.

Let us next consider what happens when we continue $D_s(z-z_0)$ to the point $(z_0+\Omega)$ along a path, passing through no point of the set (S). When we treat $w_r(z-z_0)$ in this manner it becomes, as is easily seen,

$$m_{r1}w_1(z-(z_0+\Omega))+m_{r2}w_2(z-(z_0+\Omega))+\ldots+m_{rc}w_c(z-(z_0+\Omega)),$$

where the m's are certain constants. Thus $D_s(z-z_0)$ will become

$$\begin{vmatrix} m_{11} & m_{12} & \ldots\ldots & m_{1c} \\ m_{21} & m_{22} & \ldots\ldots & m_{2c} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ m_{c1} & m_{c2} & \ldots\ldots & m_{cc} \end{vmatrix} D_s(z-(z_0+\Omega)).$$

This shews in particular that the set (S'), like the set (S), is doubly-periodic periods $(\omega\omega')$.

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Now suppose that ζ_0 is any point which does not belong to either of the sets S, S'; and that when $D_s(z-z_0)$ is continued to ζ_0 along a certain path L, passing through no singularity of δ_s it becomes $\Delta_s(z-\zeta_0)$. The quotient

$$\frac{\Delta_s(z-\zeta_0)}{\Delta_0(z-\zeta_0)}$$

will be a power series (say $Q_s(z-\zeta_0)$) with a definite radius of convergence ρ_s for each value of s. Clearly $Q_s(z-\zeta_0)$ can be continued over the whole plane, save for possible obstacles which belong to one or other of the sets S, S', which, as we have seen, are both reducible. In other words we obtain by this process a monogenic function $q_s(z)$ whose singularities consist of some or all of the points belonging to the reducible set $(S+S')$.

§ 6. It is easy to see that each of the functions $q_s(z)$ is a uniform doubly-periodic function. That it is uniform follows from the fact that $D_s(z-z_0)$, and consequently (§ 9 below), $\Delta_s(z-\zeta_0)$, when continued round a closed path, returns as a multiple of itself, the said multiple being independent of s (§ 5). For a corresponding reason it appears that $Q_s(z-\zeta_0)$ when continued to the point $(\zeta_0+\Omega)$ becomes $Q_s(z-(\zeta_0+\Omega))$, so that $q_s(z)$ is doubly-periodic (periods $\omega\omega'$).

We proceed to prove that f_r is a solution of the differential equation

$$\frac{d^c w}{dz^c} = q_1(z)\frac{d^{c-1}w}{dz^{c-1}} + q_2(z)\frac{d^{c-2}w}{dz^{c-2}} + \ldots + q_c(z).w \quad \text{.....(ii).}$$

Using the notation of § 3, suppose that $\bar{\omega}_1(z-\zeta_0), \bar{\omega}_2(z-\zeta_0), \ldots \bar{\omega}_c(z-\zeta_0)$ is the basic system at ζ_0, deduced by continuation from $w_1(z-\zeta_0), w_2(z-\zeta_0), \ldots w_c(z-\zeta_0)$ along L. Since every element of f_r whose base is (z_0) satisfies (i), every element whose base is ζ_0 satisfies

$$\begin{vmatrix} \frac{d^c w}{dz^c} & \frac{d^c\bar{\omega}_1}{dz^c} & \ldots\ldots & \frac{d^c\bar{\omega}_c}{dz^c} \\ \frac{d^{c-1}w}{dz^{c-1}} & \frac{d^{c-1}\bar{\omega}_1}{dz^{c-1}} & \ldots\ldots & \frac{d^{c-1}\bar{\omega}_c}{dz^{c-1}} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ w & \bar{\omega}_1 & \ldots\ldots & \bar{\omega}_c \end{vmatrix} = 0,$$

which is
$$\Delta_0(z-\zeta_0)\frac{d^c w}{dz^c} = \Delta_1(z-\zeta_0)\frac{d^{c-1}w}{dz^{c-1}} + \ldots + \Delta_c(z-\zeta_0).w,$$

or
$$\frac{d^c w}{dz^c} = Q_1(z-\zeta_0)\frac{d^{c-1}w}{dz^{c-1}} + \ldots + Q_c(z-\zeta_0).w,$$

provided that $|z-\zeta_0| < \rho$ the least of the numbers $\rho_1, \rho_2 \ldots \rho_s$. It follows at once by known theorems that f_r* is a solution of (ii).

* The reader will perceive that not only f_r but *every* function, which has an element at z_0 expressible as a linear function of the basic system at z_0, is a solution of (ii).

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§ 7. From the theorem of §§ 4—6 we may deduce an interesting conclusion as to the nature of possible uniform solutions of *any* equation whose coefficients are uniform doubly-periodic functions.

Suppose that we have a uniform solution f_r of such an equation. If "c" is its content, we see from the theorem mentioned that this solution satisfies an equation of the same type of order c. It is not difficult to see that all integrals of this latter equation are uniform, i.e. that it is of the ordinary Picard type. Adopting the above notation (§ 3) we observe that as f_r is uniform each of the series $w_r(z-(z_0+\Omega_r))$ continued round any closed path returns into itself. Hence the series $w_r(z-z_0)$ when so treated have the same properties. In other words, each of the series $w_r(z-z_0)$ can be regarded as the initial element of a uniform monogenic function, say $g_r(z)$. As the series $w_r(z-z_0)$ are linearly independent, the functions

$$g_1(z), g_2(z), \dots g_c(z),$$

are also linearly independent, i.e. are a fundamental system of solutions of the equation satisfied by f_r. The latter equation is therefore of the ordinary Picard type.

We conclude* that the only possible uniform solutions of a linear equation whose coefficients are uniform doubly-periodic functions (periods $\omega\omega'$) are linear functions of polynomials in

$$u\left(=\frac{\omega}{2\pi\iota}\left\{\frac{\eta}{\omega}z-\zeta(z)\right\}\right) \text{ and } v\left(=\frac{\omega'}{2\pi\iota}\left\{\zeta(z)-\frac{\eta' z}{\omega'}\right\}\right),$$

whose coefficients are all uniform doubly-periodic functions of the second species having the same multipliers.

Another interesting fact—and this is a generalisation of Picard's theorem† in a different direction from that of I, § 16—is that if any equation of this type has a uniform integral at all, then at least one of its solutions is a uniform doubly-periodic function of the second species. If we please, we may state this result in terms of "content," and say that every equation which has a uniform solution has at least one whose content is unity.

§ 8. The solutions to which we referred in § 1 as being the object of our investigations are called *exclusive*. In order to fully understand what is meant by this term, consider any ordinary linear differential equation of order m, say

$$\frac{d^m w}{dz^m}=t_1(z)\frac{d^{m-1}w}{dz^{m-1}}+\dots+t_r(z)\frac{d^{m-r}w}{dz^{m-r}}+\dots+t_r(z),$$

where the t's are any uniform functions having only a reducible set of singularities: they need not necessarily be doubly-periodic functions.

As in I, § 1, if z_0 is any base point, which is not one of the singularities of the coefficients, we can find m linearly independent power series in $(z-z_0)$, say

$$P_1(z-z_0),\ P_2(z-z_0), \dots P_m(z-z_0),$$

which satisfy the differential equation, and have a radius of convergence $\nless |c-z_0|$ where

* *Vide* I, § 39.

† It would perhaps be more correct to call this generalisation Picard's theorem, for it is embodied in the note to *Comptes Rendus* (t. xc. pp. 128—31). In the account in his *Traité d'Analyse* Picard assumes *all* the integrals of the equation uniform (t. III. p. 407).

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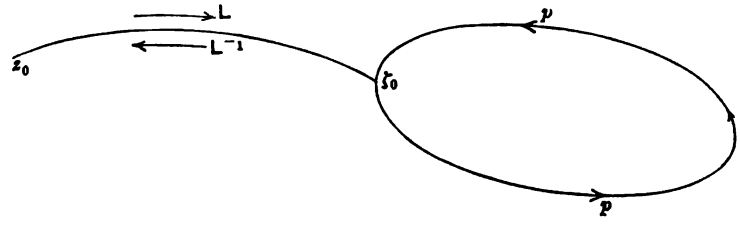
c is that singularity of the coefficients nearest to z_0. Any one of these series, $P_r(z-z_0)$, may be regarded as the initial element of a certain monogenic function f_r, and since the t's are as described, the singularities of each of these functions will be a reducible set of points.

Suppose now that, starting from z_0 we continue the series $P_r(z-z_0)$, round any closed path, back to this base, and that the resulting power series is $P'(z-z_0)$. We shall have

$$P_r'(z-z_0) = l_1 P_1(z-z_0) + l_2 P_2(z-z_0) + \ldots + l_m P_m(z-z_0),$$

where $l_1, l_2 \ldots l_m$ are certain constants, which in general depend on the closed path. If it happens that not all the numbers $l_1, l_2, \ldots l_{r-1}, l_{r+1}, \ldots l_m$ are zero, $P_r'(z-z_0)$ will be linearly independent of $P_r(z-z_0)$, and so we shall have a new solution of the equation. But should all these numbers vanish, however, the closed path is chosen, we speak of the monogenic function f_r as an *exclusive** solution of the equation.

§ 9. It will be noticed that thus far we have considered a definite element of f_r: in order to justify the application of the adjective to the complete function it will be necessary to shew that all its elements possess the same property. This is not difficult. For suppose that $\Pi_r(z-\zeta_0)$ is an element of f_r deduced by the continuation of $P_r(z-\zeta_0)$ along a definite path L. When we continue $\Pi_r(z-\zeta_0)$ round a closed path (p) we obtain a new power series in $(z-\zeta_0)$, say $\Pi_r'(z-\zeta_0)$. Denoting by L^{-1} the path L described in the opposite direction to the above, the path LpL^{-1} may be regarded as a closed path returning to z_0. The series $P_r(z-z_0)$ continued along this will return to z_0 again as $P_r'(z-z_0) = l_r P_r(z-z_0)$ where l is a certain constant. But it is clear that the power



series $P_r'(z-z_0)$ continued along the path L reaches ζ_0 as $l_r \Pi_r(z-\zeta_0)$: so if we continue $P_r(z-z_0)$ along L to the point ζ_0 and then round the path p back again to ζ_0 the resulting power series is $l_r \Pi_r(z-\zeta_0)$. In other words $\Pi_r(z-\zeta_0)$ continued round p back to ζ_0 reaches this point again as a series $\Pi_r'(z-\zeta_0)$ equal to $l_r \Pi_r(z-\zeta_0)$. This proves what we require.

§ 10. Let us consider those equations with uniform doubly-periodic coefficients which are such that *every* solution is exclusive. For brevity we call these *Halphen equations*, because it turns out that the class of equations defined in this way is that which Halphen considers†, when we make the additional restrictions, (1) that the number of singularities of any coefficient contained in a given period-parallelogram is finite, and (2) that the integrals are all regular.

* The reader will observe that *uniform* solutions are exclusive solutions.

† *Mém. des Sav. Étrang.* t. XXVIII. (1882), No. 1, p. 60 seq. *Functions Elliptiques*, II. Partie (1888), Chap. XIII.

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Theorem. Any quotient* of two solutions of a Halphen equation is a uniform function in the whole plane, and its singularities are a reducible set of points.

Suppose that f_1 and f_2 are any two solutions. We may confine ourselves to the case in which the elements of these at a given base are not mere multiples of one another, for then the theorem is obvious.

Just as in § 5 for the case of δ_0 we may prove that the zeros of f_1 are a reducible set, say S'. Suppose that ζ_0 is a point which belongs neither to S' nor to S, the set of singularities of the coefficients of the equation. Take elements $\Pi_1(z-\zeta_0)$, $\Pi_2(z-\zeta_0)$ of f_1 and f_2 respectively at the base ζ_0. Then

$$\frac{\Pi_2(z-\zeta_0)}{\Pi_1(z-\zeta_0)}$$

can, within a certain circle whose centre is ζ_0 and radius a definite number ρ, be expressed as a power series, say $Q(z-\zeta_0)$. We wish to shew that the quotient q whose initial element is $Q(z-\zeta_0)$ is uniform.

When $\Pi_1(z-\zeta_0)$ is continued round any closed path (p) it becomes, say $l_1\Pi_1(z-\zeta_0)$ by hypothesis: similarly $\Pi_2(z-\zeta_0)$ becomes $l_2\Pi_2(z-\zeta_0)$. Again

$$\alpha_1\Pi_1(z-\zeta_0)+\alpha_2\Pi_2(z-\zeta_0)\,(\alpha_1\neq 0,\ \alpha_2\neq 0),$$

may be regarded as the initial element of a solution of the equation; and so when continued round (p) it will become

$$k\,(\alpha_1\Pi_1(z-\zeta_0)+\alpha_2\Pi_2(z-\zeta_0)),\ \text{say}.$$

But it must also become

$$\alpha_1 l_1\,\Pi_1(z-\zeta_0)+\alpha_2 l_2\,\Pi_2(z-\zeta_0).$$

Since $\alpha_1\neq 0$, $\alpha_2\neq 0$ and the series $\Pi_1(z-\zeta_0)$, $\Pi_2(z-\zeta_0)$ are linearly independent, we must have

$$l_1=k=l_2.$$

It follows at once (cf. § 6 above), that $Q(z-\zeta_0)$ continued round (p) returns to itself, i.e. q is uniform: moreover as the only possible singularities of q are points of the set $S+S'$, they must be a reducible set. The theorem is thus established.

§ 11. The class of equations which were referred to in the preceding paragraph as having been studied by Halphen, consists of those which have the properties, (1) their coefficients are elliptic functions, (2) their integrals are all regular at a singularity of the coefficients, and (3) any quotient of two integrals is a uniform function.

From the theorem of § 10 it appears that any equation which has the properties (1) and (2), and the further one, that all its integrals are exclusive, belongs to the class just mentioned. In order to justify the statement at the commencement of § 10, we must establish the converse.

* Two monogenic functions f_1 and f_2 may have an infinite number of quotients, each being obtained from power series equal to $\frac{\Pi_1(z-\zeta_0)}{\Pi_2(z-\zeta_0)}$, where Π_1 and Π_2 are elements of f_1 and f_2 respectively at a base ζ_0 which is not a zero of $\Pi_2(z-\zeta_0)$.

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Suppose then that conditions (1), (2) and (3) hold; and that $c_1 \dots c_t$ are the singularities of the coefficients contained in a period-parallelogram. From (3) it appears that the roots of the indicial equation* at the singularity c_r must be

$$\nu_r,\ \nu_r + \iota_1,\ \nu_r + \iota_2, \dots \nu_r + \iota_{m-1},$$

where $\iota_1, \iota_2 \dots \iota_{m-1}$ are integers positive or negative; and further, none of the integrals which belong to these indices can contain logarithms. Consequently any integral can be expressed by

$$(z - c_r)^{\nu_r} \times \text{a uniform function}$$

in the neighbourhood of c_r, i.e. if $k_r = e^{2\pi\nu_r}$, each element of a solution, when continued along a loop round c_r, returns as k_r times itself. As the same argument applies to the other singularities, it appears that all solutions are exclusive. The statement referred to is therefore justified.

§ 12. We proceed to give some idea of the analytic form of the integrals of a Halphen equation which is of the generalised Picard type. We will shew below (§ 16) that every Halphen equation can be reduced to one of this type, so that it will be useful to study its solutions carefully.

Adopting the notation of I, § 1, let the equation be

$$\frac{d^m w}{dz^m} = p_1(z)\frac{d^{m-1}w}{dz^{m-1}} + \dots + p_r(z)\frac{d^{m-r}w}{dz^{m-r}} + \dots + p_m(z) \quad \text{.....(iii).}$$

If we substitute $w = v\ e^{\frac{1}{m}\int_{z_0}^{z} p_1(z)\,dz}$ the equation becomes

$$\frac{d^m v}{dz^m} = \Pi_2(z)\frac{d^{m-2}v}{dz^{m-2}} + \dots + \Pi_r(z)\frac{d^{m-r}v}{dz^{m-r}} + \dots + \Pi_m(z) \quad \text{.....(iv),}$$

where the coefficient of $\frac{d^{m-1}v}{dz^{m-1}}$ vanishes and

$$\Pi_2(z) = \frac{(m-1)}{2m}\{p_1(z)\}^2 + p_2(z) - \frac{m-1}{2m}p'_1(z);$$

and generally $\Pi_r(z)$ is a polynomial in $p_1(z), p_2(z), \dots p_m(z)$ and the differential coefficients of $p_1(z)$. Clearly the functions $\Pi(z)$ are single-valued and doubly-periodic and, as the only possible singularities of a differential coefficient of $p_1(z)$ are those of the function, they have for their singularities a reducible set of points. The coefficients of the equation (iv) are therefore uniform doubly-periodic functions.

Again if (iii) is of the generalised Picard type, (iv) is also of that type, for just as in the usual theory of elliptic functions it may be proved that

$$\int p_1(z)\,dz$$

taken round a path enclosing Θ (I, § 5) is zero.

Moreover if (iii) is a Halphen equation there is no difficulty in seeing that (iv) is also one.

* Forsyth, *Theory of Differential Equations* (1902), Vol. IV. p. 85.

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§ 13. Consider the equation (iv), it being supposed that (iii) is a Halphen equation: for the purposes of the present paragraph we need not assume the latter to be of generalised Picard type.

Let σ be the set of points which are singularities of the coefficients of (iv) and let z_0 be any point not belonging to it. We can find m linearly independent power series $S_1(z-z_0)$, $S_2(z-z_0)$, ... $S_m(z-z_0)$ which satisfy the equation (iv). When they are continued round a closed path p which does not pass through any point of the set σ and returns to z_0 they become a new set $S_1'(z-z_0)$, ... $S_m'(z-z_0)$ which are all the same constant multiple (h) of the corresponding series of the original system (§ 10 above). Thus the determinant

$$\Delta(z-z_0)=\begin{vmatrix} \dfrac{d^{m-1}S_1(z-z_0)}{dz^{m-1}} & \dfrac{d^{m-1}S_2(z-z_0)}{dz^{m-1}} & \cdots\cdots & \dfrac{d^{m-1}S_m(z-z_0)}{dz^{m-1}} \\ \dfrac{d^{m-2}S_1(z-z_0)}{dz^{m-2}} & \cdots\cdots & & \dfrac{d^{m-2}S_m(z-z_0)}{dz^{m-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ S_1(z-z_0) & S_2(z-z_0) & \cdots\cdots & S_m(z-z_0) \end{vmatrix},$$

when continued round (p) will become the same determinant, save that dashed letters S_r replace undashed ones, i.e. it becomes $h^m\,\Delta(z-z_0)$. But it is known* that this determinant $\Delta(z-z_0)$ is a mere constant, and so when continued round (p) returns to z_0 unchanged. It follows that $h^m=1$.

Take now any function $\{S_r(z-z_0)\}^m$.

By well-known theorems this is a power series in $(z-z_0)$ say $T(z-z_0)$ and can, therefore, be regarded as the initial element of a monogenic function F_r. The only possible singularities of this latter are those of the function f_r whose initial element is $S_r(z-z_0)$; they will therefore belong to the set σ, and so must be a reducible set. We easily prove that F_r is single-valued. For when $T(z-z_0)$ is continued round the closed path (p), above mentioned, it returns to z_0 as

$$\{S_r'(z-z_0)\}^m,$$

i.e. as $$h^m\{S_r(z-z_0)\}^m,$$

i.e. as $$T(z-z_0):$$

which proves what we require.

From this we deduce the interesting fact that *every solution of a Halphen equation of the form* (iii), § 12, *has for its analytical expression*

$$e^{\frac{1}{m}\int_{z_0}^{z} p_1(z)\,dz}\times\{F_r\}^{\frac{1}{m}},$$

where F_r is a function which is uniform in the whole plane and has for its singularities a reducible set of points.

* Forsyth, *Theory of Differential Equations* (1902), Vol. IV. p. 40.

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§ 14. It is not difficult to obtain a more precise idea of the nature of the functions F_r.

Suppose in the first instance that (iii), § 12, besides being a Halphen equation is of the generalised Picard type and that Φ is the periodic region of the equation.

From the theorem of § 10 it is clear that if we can obtain a monogenic function such that a quotient of it and any one of the solutions of equation (iv) is uniform, then the quotient of this function and any other solution has the same property.

Now if (k, k') are any associated pair of roots of a fundamental system of period equations corresponding to solutions of (iv) localised in Φ, it appears from I, § 16, that there exists an integral (f) of the equation whose principal branch say $\bar{f}(z)$ is doubly-periodic of the second species with multipliers (k, k'). Consequently $F(z) = \{f\}^m$ must have a corresponding property and therefore as it is uniform* must be a uniform doubly-periodic function of the second species with multipliers (k^m, k'^m). Consider the quotient

$$\frac{F(z)}{\{G(z)\}^m} \quad \text{(v)},$$

where

$$G(z) = e^{az} \frac{\sigma(z-c)}{\sigma(z-b)},$$

and "c" is one of the points belonging to the set of singularities of the coefficients of (iv), whilst a and b are so chosen that

$$a\omega + \eta(b-c) = \log k,$$
$$a\omega' + \eta'(b-c) = \log k'.$$

It is easily seen that

$$G(z+\omega) = k\,G(z), \quad G(z+\omega') = k'\,G(z).$$

Hence the quotient (v) will be a uniform doubly-periodic function of the first kind, say $H(z)$. As

$$\left(\frac{F(z)}{H(z)}\right)^{1/m} = \frac{f}{(H(z))^{1/m}}$$

is the uniform function G it follows, from what has been already said, that every solution of (iv) is of the form

$$\{H(z)\}^{1/m} K(z),$$

where $H(z)$ is a uniform doubly-periodic function whose singularities consist of some or all of those of the coefficients of (iv), and $K(z)$ is uniform in the whole plane and has also a reducible set of singularities.

§ 15. There still remains the question of the form of the functions $K(z)$. In order to obtain this let us refer to the results of I, §§ 36, 37. We saw there that if (κ, κ')† were an associated pair of roots of the fundamental system of period equations, whose invariants were $\sigma_1, \sigma_2, \ldots \sigma_s$, we had σ_1 monogenic functions, say $f_{11}, f_{12}, \ldots f_{1\sigma_1}$, whose principal branches $w_{11}(z), \ldots w_{1\sigma_1}(z)$ are doubly-periodic functions of the second species (multipliers κ, κ') localised in Φ. From the results § 14 we see that the quotients

$$\frac{f_1}{\{H(z)\}^{1/m}}$$

* § 13 above.

† The reader will observe that this pair are not necessarily the same as (k, k') used in the previous paragraph. The latter were employed in order to deduce the existence of $H(z)$.

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are uniform functions. Recalling that $H^{1/m} = \frac{f}{G}$ and that f is exclusive, it appears that these are uniform doubly-periodic functions of the second species (multipliers κ, κ'); say they are $K_1(z)$. Then these σ_1 monogenic functions will have for their analytic expressions

$$\{H(z)\}^{1/m} K_1(z).$$

Again we had σ_2 integrals $f_{21}, f_{22}, \ldots f_{2\sigma_2}$, whose principal branches $w_{21}(z), w_{22}(z), \ldots w_{2\sigma_2}(z)$, have their expressions given by

$$w_2(z) = W_2(z) + \frac{u}{\kappa'} \rho'_{21} W_1(z) + \frac{v}{\kappa} \rho_{21} W_1(z),$$

where the functions $W_1(z)$ are the same as $w_1(z)$ and $W_2(z)$ have the same properties as $W_1(z)$. Now suppose that $h(z)$ is a branch of $\{H(z)\}^{1/m}$ localised in Φ. We have

$$\frac{w_2(z)}{h(z)} - \frac{u}{\kappa'} \rho'_{21} \frac{W_1(z)}{h(z)} - \frac{v}{\kappa} \rho_{21} \frac{W_1(z)}{h(z)} = \frac{W_2(z)}{h(z)}.$$

Since $w_2(z)$ and $W_1(z)$ are solutions of the equation it is clear that when the localised functions which are the left-hand members of this equality are continued out of Φ, they will be uniform functions having a reducible set of singularities: denote these functions by $K_2(z)$. Then as

$$W_2(z+\omega) = \kappa W_2(z),\ W_2(z+\omega') = \kappa' W_2(z),\ h(z+\omega) = h(z),\ h(z+\omega') = h(z),$$

it appears that they must be uniform doubly-periodic functions of the second species multipliers (κ, κ'). It follows that we have

$$f_2 = \{H(z)\}^{1/m} \left[K_2(z) + \frac{u}{\kappa'} \rho'_{21} K_1(z) + \frac{v}{\kappa} \rho_{21} K_1(z) \right].$$

Proceeding in this manner we arrive at the general result that the rth set of monogenic functions are given by

$$f_r = \{H(z)\}^{1/m} \times \left[S_r \frac{u^{(p-x)}\, v^{(x)}}{(\kappa')^{p-x} (\kappa)^x} \rho_{rs_1} \rho_{s_1 s_2} \cdots \rho_{s_{x-1} s_x} \rho'_{s_x s_{x+1}} \cdots \rho'_{s_{p-1}, t} K_t(z) \right],$$

the functions corresponding to the expression in the square brackets being all polynomials in u and v of degree $(r-1)$ (I, § 38) whose coefficients are uniform doubly-periodic functions of the second species with multipliers (κ, κ').

These results apply of course to an equation of the form (iv) § 12: they are however applicable at once to one of the form (iii) provided that we multiply each solution by

$$e^{\frac{1}{m}\int_{z_0}^{z} p_1(z)\, dz}.$$

§ 16. As we mentioned in § 12 it is possible to reduce every Halphen equation to one of generalised Picard type. For it appears from what was said in § 13 that any element of a solution of the equation continued round a closed path (p) enclosing Θ returns multiplied by

$$h\ e^{\frac{1}{m}\int_p p_1(z)\, dz},$$

the integral being taken round p. We pointed out at the end of § 12 that this latter

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is zero, so the multiplier is merely h*, an mth root of unity. Suppose that instead of taking the sides of a period parallelogram to be $\omega\omega'$ we take them to be say $m\omega$ and ω'. If we proceed as in I, § 5, we obtain a new region $\bar{\Theta}$ instead of Θ. The effect of continuing any element of a solution round a path $\bar{p}$ enclosing $\bar{\Theta}$ will be to multiply it by h^m since $\bar{p}$ encloses m regions Θ. As $h^m = 1$ it follows that if we regard the periods of the coefficients as $m\omega$ and ω' the equation is one of generalised Picard type. The results of § 15 may therefore be applied: but it must not be forgotten that we now have ω replaced everywhere by $m\omega$.

§ 17. Our consideration of exclusive integrals of an equation has so far been confined to the case in which *all* of them have this property. There remains for us the discussion of the most general possible exclusive integral an equation with uniform doubly-periodic coefficients can have.

To this effect suppose that we are given an exclusive solution (f_r) of (iii). Let c be its content and $w_1(z-z_0), \ldots w_c(z-z_0)$ a basic system at the point z_0. We can always select the system so that one of its members is an element of f_r. For any element of f_r at z_0 is known to be of the form

$$l_1 w_1(z-z_0) + l_2 w_2(z-z_0) + \ldots + l_c w_c(z-z_0),$$

where the l's are not all zero. Consequently if l_s does not vanish it will be sufficient to replace $w_s(z-z_0)$ by this element. Imagine this done so that one of the series (say $w_1(z-z_0)$) is an element of f_r. From the way in which a basic system was defined (§ 2) it is known that there exist periods $\Omega_2 \ldots \Omega_c$ such that the series $w_r(z-(z_0+\Omega_r))$ are also elements of the function (f_r). The hypothesis that (f_r) is exclusive involves (§ 9) that each of the series $w_r(z-(z_0+\Omega_r))$ continued round a closed path returns as a multiple of itself. Hence each of the series $w_2(z-z_0), \ldots w_c(z-z_0)$ will have the same property. In other words they may be regarded as the initial elements of $c-1$ monogenic functions $g_2, \ldots g_c$ which are exclusive. From the theorem of §§ 4—6 we know that these together with f_r are solutions of an equation (ii) § 6 of order c. But though these functions are all exclusive it by no means follows that *every* solution of the equation (ii) has this property, *i.e.* it is not necessarily a Halphen equation.

§ 18. In order to proceed further it will be necessary to make an analysis of the basic system $w_1(z-z_0), \ldots w_c(z-z_0)$.

Take any element of the system say $w_{s_1}(z-z_0)$ and consider the various Q-series (in $(z-z_0)$) which are solutions of the equation (iii) arrived at as in § 2 from the elements of f_r at the points $z_0 + n\omega + n'\omega'$. There may be other elements of the basic system $w_{s_2}(z-z_0), \ldots w_{s_t}(z-z_0)$ such that

$$l_1 w_{s_1}(z-z_0) + l_2 w_{s_2}(z-z_0) + \ldots + l_t w_{s_t}(z-z_0) \qquad (l_1 \neq 0,\ l_2 \neq 0, \ldots l_t \neq 0)$$

is one of these. If so we group them together with $w_{s_1}(z-z_0)$. Imagine this done for all those Q-series whose expression in terms of the basic system involves $w_{s_1}(z-z_0)$: we thus

* If h is not a special mth root (Burnside and Panton, *Theory of Equations* (1899), Vol. I. p. 95) it will be possible to substitute a sub-multiple of m instead of m itself throughout what follows. The argument we use is applicable to *every* case.

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arrive at a group say G_{11} of the members of our basic system. Next take each element of this group other than $w_{s_1}(z-z_0)$ and proceed with it exactly as we did with the latter, adding any new members of the basic system thus arrived at to the group G_{11}: we thus obtain a group G_{12} say. The members of this group which do not belong to G_{11} we treat in similar fashion: and so on until finally we obtain a group G_1 such that by this process we cannot add to it any other member of the basic system. We call such a group *transitive*: it has the property that if $w_s(z-z_0)$ and $w_t(z-z_0)$ are members of it we can find a chain of members of the group $w_{p_1}(z-z_0)$, $w_{p_2}(z-z_0), \ldots w_{p_{r-1}}(z-z_0)$, such that there exists a Q-series, the first of which being expressed in terms of the basic system has the coefficients of $w_s(z-z_0)$ and $w_{p_1}(z-z_0)$ non-evanescent, the second those of $w_{p_1}(z-z_0)$ and $w_{p_2}(z-z_0)$ and so on, the last having the coefficients of $w_{p_{r-1}}(z-z_0)$ and $w_t(z-z_0)$ non-evanescent. It is convenient to speak of the series $w_{p_1}(z-z_0), \ldots w_{p_{r-1}}(z-z_0)$ as a *chain* connecting $w_s(z-z_0)$ and $w_t(z-z_0)$.

The transitive group G_1 may or may not contain all the elements

$$w_1(z-z_0), \ldots w_c(z-z_0).$$

If it does not we take one of them not belonging to G_1 and treat it as we treated $w_{s_1}(z-z_0)$: we thus obtain another transitive group G_2 which obviously cannot contain any member of the group G_1. Should it happen that there is still a member of the basic system which belongs neither to G_1 nor to G_2 we take it and generate another group G_3, and so on; until finally every member of the system is assigned to a definite group.

§ 19. Suppose that the transitive groups arrived at in the manner of the preceding paragraph are

$$G_1, G_2, \ldots G_\tau. \qquad (\tau \leqslant c)$$

It is not difficult to see that EVERY *linear combination of the members of any one of these groups may be regarded as the initial element of a monogenic function which is exclusive.*

To establish this it will be sufficient to shew that when we continue any two members of a group (say G_1) round a closed path (p) returning to z_0, they each become the *same* multiple of themselves. Moreover, as any two of these members can be connected by a chain (§ 18) our problem simplifies still further into that of proving the property just mentioned to belong to two consecutive members of such a chain. The argument now follows much the same lines as that of § 10 above. For suppose that $w_{p_1}(z-z_0)$, $w_{p_2}(z-z_0)$ are the consecutive members. Then we know that there exists a Q-series of the form

$$k_1 w_{p_1}(z-z_0) + k_2 w_{p_2}(z-z_0) + \ldots \quad \text{.....(vi)},$$

where $k_1 \neq 0$, $k_2 \neq 0$, and there may of course be other members of the group G_1 involved besides those written. In § 17 we saw that all members of the basic system when continued round a closed path (p) return as mere multiples of themselves: but obviously the argument there used applies not merely to these, but to all Q-series in general. Let us assume therefore that for any path (p) these multiples are μ_1, μ_2, &c. for

$$w_{p_1}(z-z_0), \; w_{p_2}(z-z_0), \; \&c.$$

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and μ for (vi). Then when (vi) is continued round (p) it becomes

$$\mu\,(k_1\,w_{p_1}(z-z_0)+k_2\,w_{p_2}(z-z_0)+\ldots):$$

but it also becomes $$k_1\mu_1 w_{p_1}(z-z_0)+k_2\mu_2 w_{p_2}(z-z_0)+\ldots.$$

Since no linear relation can exist among the members of a basic system (§ 2) it follows that the coefficients of the various w's in these two expressions must be identical, i.e. among other possible relations we must have

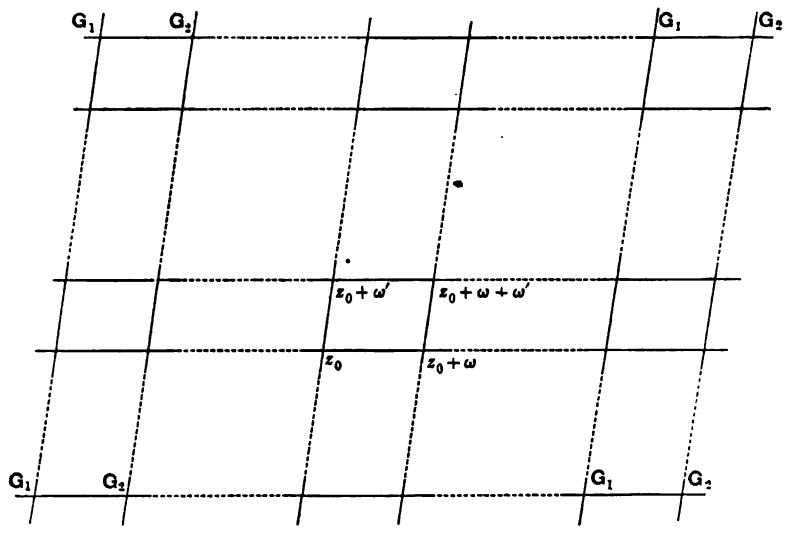
$$\mu_1=\mu=\mu_2,$$

as $k_1 \neq 0$, $k_2 \neq 0$. We have therefore proved what we desired.

§ 20. Take a net-work of period parallelograms whose corners are the points

$$z_0+n\omega+n'\omega',$$

where n and n' are any integers positive or negative. It will be convenient to say that the line in which the points $z_0+n\omega+n'\omega'$ lie for a fixed value of n' and a variable n is horizontal: while that in which they lie for a fixed n and a variable n' is vertical.



After what has been said above it will be clear that the various Q-series corresponding to the corners of the net-work are each expressible as linear functions of members of one of the groups $G_1, G_2, \ldots G_\tau$. In the case before us all of these series at a given corner are mere multiples of one another, and will therefore all be expressible in terms of the members of one group. Let us mark the letter of the corresponding group at each corner of the net-work, and consider the possible arrangements of these group-letters. In the first place, let us consider those points (marked G_1 of course) which correspond to members of the basic system belonging to G_1. We proceed to prove that those letters which are on the right of them (in the same horizontal line) must be all the same. Obviously we can confine ourselves to the case of two consecutive members of a chain. Suppose then that $w_{p_1}(z-z_0)$, $w_{p_2}(z-z_0)$ (§ 19) are such, being

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Q-series corresponding to $z_0+\Omega_{p_1}$, $z_0+\Omega_{p_2}$ respectively; moreover, let (vi) (which we denote by $w(z-z_0)$) be a Q-series at $z_0+\Omega$. Imagine a path Γ drawn from z_0 to $z_0+\omega$ so as not to pass through any singularity of the coefficients of the equation; and let γ, γ_1 and γ_2 be paths congruent to Γ joining

$$z_0+\Omega \text{ with } z_0+\Omega+\omega,$$

$$z_0+\Omega_{p_1} \text{ with } z_0+\Omega_{p_1}+\omega,$$

and $$z_0+\Omega_{p_2} \text{ with } z_0+\Omega_{p_2}+\omega \text{ respectively.}$$

The series $$w_{p_1}(z-(z_0+\Omega_{p_1}))$$

when continued along γ_1 to the point $(z_0+\Omega_{p_1}+\omega)$ will become a power series in

$$z-(z_0+\Omega_{p_1}+\omega), \text{ say } \vec{w}_{p_1}(z-(z_0+\Omega_{p_1}+\omega)):$$

clearly the series $$\vec{w}_{p_1}(z-z_0)$$

is a Q-series corresponding to the point

$$(z_1+\Omega_{p_1}+\omega).$$

In a similar way we obtain from the series

$$w_{p_2}(z-(z_0+\Omega_{p_2})),\ w(z-(z_0+\Omega))$$

two new Q-series $$\vec{w}_{p_2}(z-z_0) \text{ and } \vec{w}(z-z_0).$$

Then since $$w(z-z_0)=k_1w_{p_1}(z-z_0)+k_2w_{p_2}(z-z_0)+\dots,$$

we shall have $$\vec{w}(z-z_0)=k_1\vec{w}_{p_1}(z-z_0)+k_2\vec{w}_{p_2}w_{p_2}(z-z_0)+\dots.$$

The left-hand side of this equality being a Q-series will be expressible as a linear function of the members of *one* transitive group. Hence if the group-letters at the points

$$(z_0+\Omega_{p_1}+\omega) \text{ and } (z_0+\Omega_{p_2}+\omega)$$

be *different*, the series $$\vec{w}_{p_1}(z-z_0),\ \vec{w}_{p_2}(z-z_0)$$

will be linear functions of members of two different groups, and so there must be at least one non-evanescent linear relation among the series

$$\vec{w}_{p_1}(z-z_0),\ \vec{w}_{p_2}(z-z_0),\ \dots.$$

But such a relation is impossible, for it would involve the same relation between the series

$$\vec{w}_{p_1}(z-(z_1+\omega)),\ \vec{w}_{p_1}(z-(z_0+\omega)),\ \dots$$

which by continuation along Γ to z_0 is easily seen to lead to the same relation between the series

$$w_{p_1}(z-z_0),\ w_{p_2}(z-z_0),$$

which are known to be linearly independent (§ 2). Our assumption that the group-letters at

$$z_0+\Omega_{p_1}+\omega,\ z_0+\Omega_{p_2}+\omega$$

are different is therefore false. We conclude that the group-letters to the right of the points corresponding to consecutive members of a chain, and hence of all members of a given group (say G_1) are the same (say G_2). Moreover, as the Q-series, corresponding to every point marked with the letter G_1, are linear functions of members of G_1 we see that *everywhere we meet with the letter G_1, that immediately on its right is G_2. Mutatis mutandis*

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we prove corresponding results in regard to the group-letters to the left of G_1, and also above and below it (in the same vertical line).

§ 21. There is another property of these group-letters which we shall require. It is that if a letter G_1 occurs at a particular corner of the net-work, then it must occur again in the horizontal and vertical lines through the point. For consider such a corner and the horizontal line through it. As we pass along this line in a particular direction, say to the right, we are bound to meet some letter (G_s) twice; for there are only a finite number (τ) of group-letters. Suppose that G_1 occurs at $z_0 + n\omega + n'\omega'$ and G_s at the points

$$z_0 + n_1\omega + n'\omega', \quad z_0 + n_2\omega + n'\omega'. \qquad (n < n_1 < n_2)$$

Evidently then if we pass from the first letter G_s to the left we meet with the letter G_1 at the $(n_1 - n)$th corner. As this property must be independent of the point at which G_s is marked (§ 20), it follows that the point

$$z_0 + (n_2 + n - n_1)\,\omega + n'\omega'$$

must be marked G_1, and so as $n_2 > n_1$, G_1 must recur. In a similar way we prove that G_1 must be met again as we pass either to the left of, or above or below

$$(z_0 + n\omega + n'\omega').$$

§ 22. Suppose now that the series $w_1(z - z_0)$, which is an element of f_r (§ 17), is a member of the group G_1 which has say γ_1 constituents. The letter G_1 will of course be that marked at the point z_0. Let $z_0 + \nu\omega$ be the first point on the horizontal line through z_0 and to the right of it at which the letter G_1 recurs. Similarly let $z_0 + \nu'\omega'$ be the first point in the vertical line through z_0 and above it at which the same letter is marked. Then it is obvious that if we consider the points $z_0 + n\nu\omega + n'\nu'\omega'$ where n and n' are any integers positive or negative, the Q-series corresponding to them are all expressible as linear functions of members of the group G_1. In other words, if we consider the coefficients of the equation (iii) as having periods $\nu\omega$, $\nu'\omega'$ instead of ω, ω', f_r is a solution of it, such that its basic system* consists of γ ($\leqslant \gamma_1$) series which are each linear functions of the γ_1 members of the group G_1. Hence it follows from §§ 4—6 above that f_r is a solution of a linear equation of order γ, say

$$\frac{d^\gamma w}{dz^\gamma} = \chi_1(z)\frac{d^{\gamma-1}w}{dz^{\gamma-1}} + \chi_2(z)\frac{d^{\gamma-2}w}{dz^{\gamma-2}} + \ldots + \chi_\gamma(z)\,w \quad \text{.....(vii)},$$

where the χ's are uniform doubly-periodic functions (periods $\nu\omega$, $\nu'\omega'$).

Every element of a solution of (vii) at z_0 is obviously a linear function of members of this new basic system, and so of members of the group G_1. As we saw in § 19 that any function which had this property was exclusive, it follows that all solutions of the equation (vii) are exclusive, i.e. it is a Halphen equation (§ 10). We have proved therefore that *every exclusive solution of an equation* (iii) *must be a solution of a Halphen equation* (vii), *the periods of whose coefficients are multiples of those of the original one.* In §§ 10—16 we discussed these equations at length, so the reader will have no difficulty in perceiving the analytic form of these solutions. He is merely cautioned not to forget the change of periods to which we have alluded.

* It does not necessarily follow that this system and the group G_1 are identical.

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ON THE SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS HAVING DOUBLY-PERIODIC COEFFICIENTS

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